

A DATA-BASED DAMPING MODELING TECHNIQUE*

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Damping mechanisms exist in all vibration systems, but their nature is little understood and there is no systematic method for modeling general damping. This paper describes a novel damping modeling method (the Method of Energy Approximation, or MEA). This method is novel because it is a *unique* damping modeling method without assumed damping linearity; it is based on experimental data instead of physical principles; hence it is applicable to vibration systems of various materials and configurations; and it is suitable for vibration system transient control. Among the three quantities essential to an understanding of the dynamics of a vibration system, mass, stiffness, and damping, the last is the most complex and least understood. Therefore, with recent technology advances in such areas as composite materials and smart materials, the need for a good damping modeling method is more urgent than ever.

KEYWORDS: *Nonlinear vibration; damping; data-based modeling*

1. INTRODUCTION

We present a novel method for modeling the damping mechanism of a vibration system. Among the three quantities essential to an understanding of the dynamics of any mechanical structure, mass, stiffness, and damping, the last is the most complex and least understood. On the other hand, damping is the most critical factor in ensuring the robustness and stability margin of a structure. Therefore, with recent technology advances in composite materials and smart structures, the need for a good damping modeling method is more urgent than ever. For example, if the damping characteristics of state-of-the-art helicopter rotors fabricated from advanced polymeric fibrous composite materials are not well understood, then design must be a worst-case scenario, which often results in a state of over-design. A consequence of the over-design philosophy is inferior performance, because of the attendant cascading effects associated with the heavier design.

Material damping is generally a complex function of frequency, temperature, type of deformation, amplitude, and structural geometry. Current popular (linear) treatments of

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damping in structural dynamics are not physically motivated and do not reflect the above dependence.

Furthermore, a flexible structure passive damping (including internal material damping and external, e.g., air damping) depends not only on the microscopic structure of the materials but also on the bonding and spatial ordering of the different materials bonded together. While sophisticated computer-based techniques enable us to make highly precise calculations of the mass and stiffness properties of flexible structures, the method for passive damping modeling is very primitive and very limited. For example, the primary sources of damping for flexible structures could fit into three categories (see Ashley [1]).

1. Material damping due to internal friction;
2. Damping at joints and interconnections;
3. Artificially introduced damping (dashpots).

Each type of damping, in turn, depends on many factors. For example, the damping at joints and interconnections depends on the following factors:

1. *Types of interconnection*: It is well recognized that structures with pinned or bolted joints have significantly higher damping rates than identical structures with welded or tightly clamped joints;
2. *Joint loads*: Joint damping is dependent on joint loads, and hence gravity will influence damping measurements.
3. *Macroslippage*: Beards and Williams [2] reported that significant damping can be obtained when joints are allowed to undergo rotational slippage. Simple models such as Coulomb friction with macroslip predict the damping to be dependent on the normal loads across the interface of a joint. This would infer that large load should increase damping rates. However, if joints allow some macroslip, this large load would prevent or reduce the amount of slippage that would occur and thus reduce damping. Therefore, a simple model such as Coulomb friction damping cannot describe the damping at joints allowing macroslippage;
4. *Microslippage*: Because of surface imperfections, joint interface contact pressure is not uniformly distributed. This allows localized slippage while the overall joint remains “locked”—microslip. For example, when material damping measurements are made using a cantilever beam specimen, a prime concern is how the specimen is clamped to the “wall” such that macroslip contributions are minimized.

In summary, due to its complex nature, it is impractical to construct working models of damping mechanisms based on certain physical principles, even if they exist. Furthermore, due to the rapid advances of material science in the last two decades, various materials with high strength/weight ratios have become available. Most of these modern materials exhibit nonlinear damping behavior. Therefore, a unified and systematic approach to damping modeling without a linearity assumption is highly desirable.

2. REVIEW OF PREVIOUS RESEARCH

Several methods for incorporating material damping into structural models have been used and continue to be used within the engineering community. These methods include viscous damping, frequency-dependent viscous damping, hysteretic damping, complex

modulus, structural damping, viscoelasticity, hereditary integrals (memory functions), and modal damping [3]. Each has some utility, but each suffers from one flaw or another. Even if some potentially accurate models exist (e.g., viscoelasticity), they are not widely used in the engineering community, perhaps because of a lack of physical motivation, or because such models are difficult to use.

Several more recent and more commonly used methods are as follows:

- The most common method available so far for modeling passive damping is, under the assumption of a linear damping rate, which is questionable, the complex modulus approach [4]–[7]. This approach is applicable only to the analysis of steady-state response situations. A critical issue in smart structures is to control transient responses, and it would be highly desirable to have a systematic damping modeling method which is suitable for these. A poor damping model may lead to the design of an unstable or poorly performing closed loop system.
- Golla, Hughes, and McTavish (GHM), of the University of Toronto, have developed a time-domain finite element formulation of viscoelastic material damping [8], [9]. Their work was motivated by some physical evidence, but was guided by the observation that experimental results, often recorded in the frequency domain, are often of little use in time-domain models. The results reported in [10] resemble to those of GHM in some ways, for example, in the introduction of additional “dissipation coordinates.” However, the results differ in other ways: no attempt is made to provide a physical interpretation of the GHM dissipation coordinates as thermodynamic field variables; the GHM model is restricted to consideration of what is termed “microstructural damping.” The GHM technique can be successfully used to fit a portion of an experimentally determined curve of damping versus frequency, and standard structural analysis tools can be used to solve the resulting equations. This method, as well as all other methods, assumes that the damping mechanism is linear, and thus cannot deal with nonlinear damping case.
- D. J. Segalman, of Sandia National Laboratory, has addressed the calculation of stiffness and damping matrices for structures made from linearly viscoelastic materials [11]. This is basically a perturbation technique: the perturbation solution for a slightly viscoelastic structure is required to match the corresponding solution for a slightly damped structure. Segalman works exclusively in the time domain and avoids introducing additional coordinates, although the resulting stiffness and damping matrices are generally unsymmetric. How the assumption of “small viscoelasticity” limits the utility of the approach is still unknown.
- Torvik and Bagley, of the Air Force Institute of Technology, have also developed a technique of material damping modeling [12], [13]. The core of their approach is the use of fractional time derivatives in material constitutive equations. Their development was motivated by the observation that the frequency dependence observed in real materials is often weaker than the dependence predicted by first-order viscoelastic models. With four- and five-parameter models, they have been able to accurately represent the elastic and dissipative behavior of over one hundred materials over frequency ranges as broad as eight decades. For most viscoelastic polymeric materials they have examined, the parameter representing the order of differentiation is in the range of $1/2$ to $2/3$. The application of the general fractional derivative approach to time-domain analysis, however, is cumbersome, and is an area of continuing research.

- To deal with the constrained-layer damping problem, Parekh and Harris, of Anamet Lab., Inc., have developed an automated procedure to derive modal damping values. The procedure uses the NASTRAN finite element program with DMAP modifications to derive modal loss factors using a Modal Strain Energy (MSE) approach. The frequency-dependent properties of the constrained viscoelastic layer are taken into account in an iterative solution. The Ritz procedure, a specialized Lanczos method for eigenvalue extraction, is used in the procedure together with standard NASTRAN super-element techniques to increase eigenvalue solution efficiency.

It is generally felt that the Modal Strain Energy (MSE) approach using commercially available finite element programs is the most computationally efficient method for analyzing constrained-layer damping problems. One of the major problems confronting MSE, however, is the frequency-dependent material properties of the viscoelastic layer.

- There has been also significant amount of research activities studying the distributed parameter models of flexible structures. To formulate internal passive damping, strictly proportional and asymptotically proportional damping operators have been reported in A. V. Balakrishnan [14], [15], as well as by G. Chen and D. Russell [16], and by S. Chen and R. Triggiani [17]. F. Huang [18], [19], studied the spectral properties of the systems in the form

$$\ddot{x}(t) + B\dot{x}(t) + Ax(t) = 0$$

where B is a closed linear operator related in various ways to A^α with $1/2 \leq \alpha \leq 1$. Some fundamental results for the holomorphic property and the exponential stability of the semigroups associated with these systems were obtained. The above model is the abstract form of a distributed parameter model (partial differential equation model), where A is the stiffness partial differential operator with its well-defined domain.

The strictly proportional damping operator is essentially the square root of the stiffness operator A . In this case, the eigenvalues have the proportionality property

$$\frac{\Re(\lambda_n)}{\Im(\lambda_n)} = -\frac{\xi}{\sqrt{1-\xi^2}}$$

The drawback of strictly proportional damping is that the damping operator contains a nonlocal feature (a combination of integral, as well as differential, operator), which is unnatural if we consider that internal passive damping is due to the structure's material itself. However, if strict proportionality is relaxed to asymptotic proportionality, that is,

$$\lim_{n \rightarrow \infty} \frac{\Re(\lambda_n)}{\Im(\lambda_n)} = \text{constant}$$

then the nonlocal feature can be avoided.

A. V. Balakrishnan, based upon his theory on the fractional power of closed linear operators [20], explicitly calculated the strictly proportional and asymptotic proportional damping operators for the beam bending model [15], in which one end of

the beam is clamped and the other end has an end-body attached to it. In [14], the strictly proportional damping operator is given explicitly for beam torsion model.

- Other relevant work on the engineering aspects of damping modeling focuses primarily on the development of experimental techniques and measurement of damping in various materials [21], [22].

All of the above methods share the same limitation: a linear damping property must be assumed. But whether the damping considered is linear or not remains to be answered in the first place. The method we have developed is the only one which does not assume damping linearity. This method provides a systematic way of modeling damping without having to know whether it is linear or nonlinear.

3. METHOD OF ENERGY APPROXIMATION

We will present a new method of damping modeling called the Method of Energy Approximation (MEA). In order to make the presentation concise and clear, the idea of our method is explained in a question-and-answer format.

We explain the method through a single-DOF model. Suppose that experimental data corresponding to the free response of the following single-DOF model with unknown damping $D(x, \dot{x})$ are available:

$$\ddot{x}(t) + D(x, \dot{x}) + \omega_0^2 x = 0 \quad (1)$$

We propose to determine the damping model of a smart structure from its (transient) free response data to obtain a repeatable model, because, due to our lack of knowledge and the complexity, it would be unrealistic to derive the damping model for a smart structure based on physical principles, even if they exist. By using experimental data instead of physical principles, this method can be applied to structures made of different materials and with different geometries.

Specifically, the experimental data we need to collect are the peak amplitudes of a smart structure's transient free response. Let these be denoted by $\{a(t_n), n = 1, 2, \dots, N\}$, i.e., $\pm a(t)$ stands for the envelope of the decaying vibration and $\{t_n, n = 1, 2, \dots, N\}$ are the peak times. Our objective is to obtain a (generally nonlinear) damping model based on these data so that the damped vibration model can reproduce the data.

We are looking for a model of the special form (energy type damping) $D_0\left(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}\right) \dot{x}$, for which we only need to determine a single variable function, $D_0(E)$. In other words, we are looking for the damping model only among a smaller class of damping models, energy type damping models, hence the name Method of Energy Approximation.

Naturally, the following question arises:

Question # 1: How can you restrict attention to only the smaller class of energy type damping models? What is the relationship between a general damping model and the energy type damping models?

We have found the following fact [23]: For any (generally nonlinear) damping model $D(x, \dot{x})$, there is one and only one (for the uniqueness studies, see [24], Section 3)

corresponding energy type damping model $D_0(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}) \bar{x}$, where the function $D_0(\cdot)$ is determined by $D(x, \dot{x})$ through the following relation:

$$D_0(E) = \frac{1}{\pi\sqrt{2E}} \int_0^{2\pi} D\left(\frac{\sqrt{2E}}{\omega_0} \sin \psi, \sqrt{2E} \cos \psi\right) \cos \psi d\psi \quad (2)$$

where $E = (\omega_0^2 x^2 + \dot{x}^2)/2$, the total energy. We claim that the corresponding energy type damping model $D_0(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}) \bar{x}$ is an “excellent” approximation of the original damping model $D(x, \dot{x})$. That is, the original damping model and its corresponding energy damping model are “practically indistinguishable”.

Immediately, two more questions arise:

Question # 2: Where does relation (2) come from? What motivates such a relation?

Question #3: How can we show that the energy type damping model $D_0(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}) \bar{x}$ determined by (2) is such an “excellent” approximation of the original damping model $D(x, \dot{x})$ that they are “practically indistinguishable”? What do we mean by “excellent” and “practically indistinguishable”?

Question # 3 is more fundamental, but let us answer Question # 2 first.

The reasoning behind relation (2) is as follows: In the case of weak damping, we can assume that the variation of the vibration amplitude is very small within each cycle (period), or, approximately, we assume that the vibration amplitude stays at the same level within each cycle.

Then, within each cycle, the system response can be approximated by

$$x(t) = a(t)/\omega_0 \sin \omega_0 t$$

and, consequently,

$$\dot{x}(t) \approx a(t) \cos \omega_0 t$$

by noticing that the amplitude $a(t)$ is slowly varying. It is assumed to be constant within each cycle. Then it is easy to verify that

$$\sqrt{2E} = a(t)$$

Then,

$$x(t) = \frac{\sqrt{2E}}{\omega_0} \sin \omega_0 t$$

$$\dot{x}(t) = \sqrt{2E} \cos \omega_0 t$$

Next, for the “true” but unknown damping model $D(x, \dot{x})$, our objective is to choose damping model of the energy form, $D_0(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}) \bar{x}$, where $D_0(\cdot)$ is a function of energy.

The critical question now is how to determine the function $D_0(E)$ in an optimal and unique fashion. We propose to choose $D_0(E)$ such that the following error is minimized over each cycle,

$$\min_{D_0(\cdot)} \|D(x, \dot{x}) - D_0\left(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}\right)\dot{x}\|_{L^2[t, t + \frac{2\pi}{\omega_0}]}$$

where the above norm is an L^2 norm over precisely one cycle.

Then, explicitly, the above minimization becomes

$$\min_{D_0(\cdot)} \int_0^{2\pi/\omega_0} [D\left(\frac{\sqrt{2E}}{\omega_0} \sin \omega_0 t, \sqrt{2E} \cos \omega_0 t\right) - D_0(E) \sqrt{2E} \cos \omega_0 t]^2 dt$$

Making the substitution $\psi = \omega_0 t$, the above minimization is equivalent to

$$\min_{D_0(\cdot)} \int_0^{2\pi} [D\left(\frac{\sqrt{2E}}{\omega_0} \sin \psi, \sqrt{2E} \cos \psi\right) - D_0(E) \sqrt{2E} \cos \psi]^2 d\psi$$

This is a quadratic minimization with respect to $D_0(E)$, and the minimal $D_0(E)$ is exactly given by (2), thus revealing how (2) is obtained.

Next, in order to answer Question # 3, let us briefly review the so-called Krylov-Bogoliubov approximation.

The Krylov-Bogoliubov approximation is a method of computing the approximate solution of a single-DOF nonlinear oscillation with small damping coefficient $\epsilon > 0$:

$$\ddot{x}(t) + \epsilon D(x, \dot{x}) + \omega_0^2 x(t) = 0$$

Let the solution sought be of the form

$$x(t) = a(t) \cos(\psi_0 t + \phi(t))$$

where $a(t)$, the amplitude, and $\phi(t)$, the phase shift, are slow varying functions of t . By perturbation and averaging, Krylov and Bogoliubov [·] established the systems of ordinary differential equations of $a(t)$ and $\phi(t)$ for $0 \leq t < 1/\epsilon$.

Simulation studies [26], [27], using the Runge-Kutta method indicate that the Krylov-Bogoliubov approximation provides a remarkably good approximation of a nonlinear damping model for all reasonable damping constants. Naturally, we desire that

$$\ddot{x} + \epsilon D(x, \dot{x}) + \omega_0^2 x = 0 \tag{3}$$

and that

$$\ddot{x} + \epsilon D_0\left(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}\right)\dot{x} + \omega_0^2 x = 0 \tag{4}$$

would have the same Krylov-Bogoliubov approximation, that is, they would have “practically indistinguishable” responses. Fortunately, this is indeed the case [23].

FACT 1 For a nonlinear damping model $D(x, \dot{x})$ and its corresponding energy damping model defined by (2), both (3) and (4) have exactly the same Krylov-Bogoliubov approximation.

The proof of this conclusion has been given in [23] and will not be repeated here.

Let us look at another supporting fact, again, in terms of the Krylov-Bogoliubov approximation. To use the corresponding energy type damping model, $D_0(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}) \bar{x}$, to approximate the original damping model, $D(x, \dot{x})$, is equivalent to dropping off the difference $D(x, \dot{x}) - D_0(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}) \bar{x}$ in the following decomposition

$$D(x, \dot{x}) = D_0(\frac{\omega_0^2 x^2 + \dot{x}^2}{2})\dot{x} + [D(x, \dot{x}) - D_0(\frac{\omega_0^2 x^2 + \dot{x}^2}{2})\dot{x}] \quad (5)$$

This is, in fact, an orthogonal decomposition of $D(x, \dot{x})$ on energy type damping space and its orthogonal complement (the transform defined by (2) is a projection operator with norm 1). Interesting details in this regard can be found in [24]. One may naturally ask:

what is the contribution of the damping difference $D(x, \dot{x}) - D_0(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}) \bar{x}$, which has been dropped off in the approximation? Of course, ideally, its contribution should be negligible. This is, again, fortunately the case.

FACT 2 The Krylov-Bogoliubov approximation of a vibration with the difference damping

$$\ddot{x} + D(x, \dot{x}) - D_0(\frac{\omega_0^2 x^2 + \dot{x}^2}{2})\dot{x} + \omega_0^2 x = 0$$

is exactly the same as the response of zero damping vibration

$$\ddot{x} + \omega_0^2 x = 0$$

This conclusion will be very clear after reading Section Four, “On the Possibility of Other Choices of Nonlinear Damping.”

These two facts clearly indicate the following: in the orthogonal decomposition (5), the energy type damping component contains all the “effective damping” which determines the decay of the vibration amplitude, while the difference part makes almost no contribution to the decay of the vibration amplitude.

In order to gain some quantitative intuitions, the following two numerical examples are in order.

EXAMPLE 1 Consider a nonlinear damping model of the form, which can be used to model damping forces resulting from the internal friction of metals [28],

$$D(x, \dot{x}) = 2\xi\omega_0(1 + \epsilon x^2)\dot{x}$$

By (2), we can have the corresponding energy type damping

$$D_0(E)\dot{x} = 2\xi\omega_2(1 + 3/2\epsilon E)\dot{x}$$

Hence, the difference is given by

$$D(x, \dot{x}) - D_0(E)\dot{x} = \frac{\xi\epsilon}{2} \omega_0(\dot{x}^2 - 3\omega_0^2 x^2)\dot{x}$$

The vibration with the above difference does exhibit a sinusoidal behavior (Figure 1), which confirms the prediction of Fact 2. Its phase plane plot has no visible deviation from a closed curve (circle) at the scale shown.

EXAMPLE 2 Consider Coulomb friction damping

$$D(x, \dot{x}) = \xi \text{sign}(\dot{x})$$

Calculation from (2) gives the corresponding energy type damping

$$D_0(E)\dot{x} = \frac{4\xi}{\pi\sqrt{\omega_0^2 x^2 + \dot{x}^2}} \dot{x}$$

The difference damping is given by

$$D(x, \dot{x}) - D_0(E)\dot{x} = \xi\left(1 - \frac{4}{\pi} \frac{|\dot{x}|}{\sqrt{\omega_0^2 x^2 + \dot{x}^2}}\right)\text{sign}(\dot{x})$$

The vibration with the above difference damping does exhibit sinusoidal behavior for all reasonable damping (Figure 2). In other words, this confirms the above prediction that the vibration with the difference damping is virtually the same as zero damping vibration—sinusoidal.

Of course, the above facts, although very important, indicate only the closeness of the *free* motions of the original damping model and the corresponding energy type damping model. In order to further validate the MEA, other supporting evidence concerning forced motions must be sought. In addition, the frequencies of these forcing inputs should span the entire spectrum of prime interest, which suggests white noise input as an ideal candidate of forcing inputs because the white noise spectrum covers the entire frequency range.

Let $\{\sigma n(t), t \geq 0\}$ be a white noise process with spectral density σ^2 over the entire frequency range. We investigate the relationship between the two stochastic processes generated by the following damped vibration systems, respectively,

$$\ddot{x} + \epsilon D(x, \dot{x}) + \omega_0^2 x = \sigma n(t) \tag{6}$$

$$\ddot{x} + \epsilon D_0\left(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}\right)\dot{x} + \omega_0^2 x = \sigma n(t) \tag{7}$$

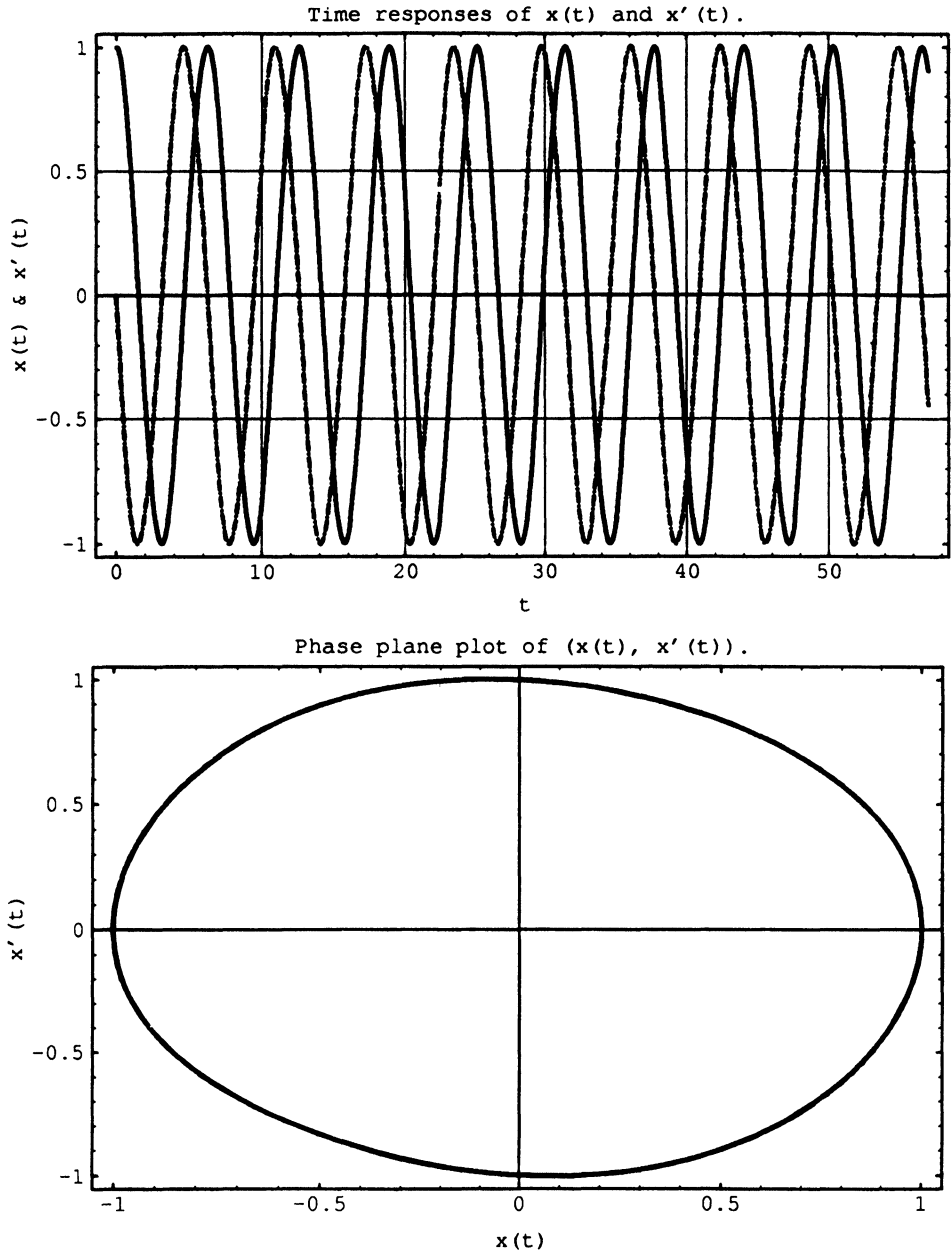


Figure 1 Time responses and phase plane plot of the vibration with difference damping (Example 1).

We are interested in the quantity $E x^2(t)$, stationary variance, a central problem in the nonlinear random vibration area. Our question is: for any nonlinear damping model $D(x, \dot{x})$, if we replace it by its corresponding energy type damping model, what will be the difference between the corresponding stationary variances?

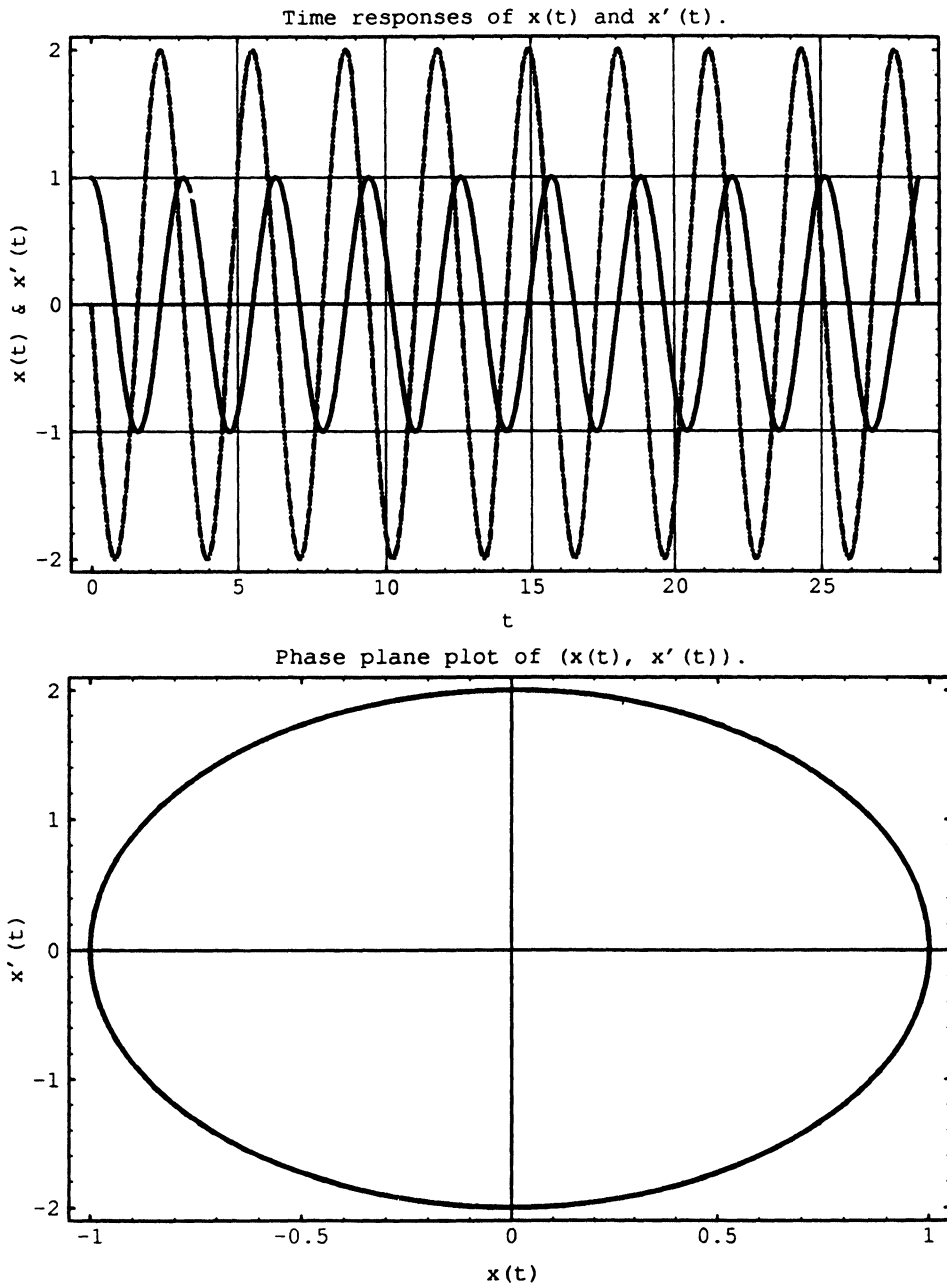


Figure 2 Time responses and phase plane plot of the vibration with difference damping (Example 2).

To answer this question, we first reformulate our problem as follows: Consider the following "artificial" damped vibration model

$$\ddot{x} + (\epsilon - \lambda)D_0(x, \dot{x}) + \lambda D(x, \dot{x}) + \omega_0^2 x = \sigma n(t) \quad (8)$$

where $0 \leq \lambda \leq \epsilon$. The motivation behind invoking this model (8) is that when $\lambda = \epsilon$, (8) reduces to the original damping model. When $\lambda = 0$, (8) reduces to the model with the corresponding energy type damping. In other words, (8) relates the two models by using the parameter λ .

If we agree to use the subscript λ to denote the quantities corresponding to (8) with parameter λ , such as $x_\lambda(t)$, then our problem can be restated as “what is the difference between $\mathbf{E}x_\epsilon^2(t)$ and $\mathbf{E}x_0^2(t)$?”

Let us define the function

$$V(\lambda) = \mathbf{E}x_\lambda^2(t) \quad \lambda \in [0, \epsilon] \quad (9)$$

Then $V(0)$ is the stationary variance corresponding to the original damping model, and $V(\epsilon)$ is the corresponding energy damping model. Therefore, it is important to investigate the behavior of $V(\lambda)$ over $[0, \epsilon]$, in particular the difference $V(\lambda) - V(0)$.

Suppose that $V(\lambda)$ is smooth, then one can write

$$V(\lambda) = V(0) + V'(0)\lambda + \frac{V''(\bar{\lambda})}{2!} \lambda^2$$

where $\bar{\lambda}$ belongs to $[0, \lambda]$. Naturally, we want $V'(0) = 0$, so that the difference $V(\lambda) - V(0)$ is of the order $O(\lambda^2)$. Fortunately, this is indeed the case, which again supports the MEA.

FACT 3 For the function $V(\lambda)$ defined in (9), there holds

$$V(\lambda) - V(0) = O(\lambda^2)$$

Based upon the above investigation, we have realized that to find a damping model is essentially to find its corresponding energy type damping model, that is, to determine the function $D_0(\cdot)$. Therefore, our last question is

Question # 4: How do you find the function $D_0(\cdot)$ from experimental data, hence to obtain the damping model $D_0(\frac{\omega_0^2}{2}x^2 + \dot{x}^2/2)x$?

Very often, nonlinear damping model is not available and what is available is only the experimental data of the free response amplitude of a nonlinearly damped vibration system. In other words, we have the amplitude data $a(t_n)$ at time t_n , $n = 1, 2, \dots, N$, where $\{t_n, n = 1, 2, \dots, N\}$ are (some of) the peak times. In general, peak times are not evenly spaced due to the nonlinear nature of the system. Our objective is to obtain the nonlinear damping model based upon the data available so that the nonlinear damping model obtained can reproduce the data.

We look for a nonlinear damping model of the form

$$D_0\left(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}\right)\dot{x}$$

that is, to determine $D_0(E)$.

Krylov-Bogoliubov approximation gives

$$\frac{da(t)}{dt} = -\frac{a(t)}{2} D_0\left(\frac{\omega_0^2 a^2(t)}{2}\right)$$

Equivalently, $D_0(\cdot)$ is given by

$$D_0\left(\frac{\omega_0^2 a^2(t)}{2}\right) = -\frac{2}{a(t)} \frac{da(t)}{dt}$$

We can numerically approximate $da(t)/dt|_{t=t_n}$ by

$$[a(t_n) - a(t_{n-1})]/(t_n - t_{n-1}).$$

If we let

$$c_n = -\frac{2}{a(t_n)} \frac{a(t_n) - a(t_{n-1})}{t_n - t_{n-1}} \quad n = 1, 2, \dots, N \quad (10)$$

then, at each t_n ,

$$D_0\left(\frac{\omega_0^2 a^2(t_n)}{2}\right) = c_n \quad n = 1, 2, \dots, N$$

Notice that $\{c_n, n = 1, 2, \dots, N\}$ can be easily computed from the data available.

Now let

$$f_0(a) = D_0\left(\frac{\omega_0^2 a^2}{2}\right)$$

then we can construct the function $f_0(a)$ based upon the correspondence relation

$$f_0(a(t_n)) = c_n \quad n = 1, 2, \dots, N \quad (11)$$

Then,

$$D_0(x) = f_0\left(\sqrt{\frac{2}{\omega_0^2} x}\right)$$

that is, the damping model is given by

$$f_0(\sqrt{x^2 + (\dot{x}/\omega_0)^2})\dot{x}$$

EXAMPLE 3 Let us consider the special case (as always assumed by other methods): if $\ln a(t)$ is a downward straight line, say,

$$\ln a(t) = -\xi\omega_0 t$$

then,

$$D_0(E) = \text{Const.} = 2\xi\omega_0$$

that is, the damping model is $D_0(E)\dot{x} = 2\xi\omega_0\dot{x}$, the linear damping case.

EXAMPLE 4 This is a simple example, in which we assume that the data available follow the curve

$$a(t) = \frac{1}{(\alpha + \beta t)^\gamma} \quad \alpha, \beta, \gamma > 0$$

Then, we can use the above expression to compute $da(t)/dt$,

$$\begin{aligned} D_0\left(\frac{\omega_0^2 a^2(t)}{2}\right) &= -\frac{2}{a(t)} \frac{da(t)}{dt} \\ &= 2\gamma\beta[a(t)]^{1/\gamma} \\ &= f_0(a) \end{aligned}$$

Therefore,

$$D_0(x) = 2\gamma\beta \left(\frac{2}{\omega_0^2} x\right)^{1/(2\gamma)}$$

that is, the nonlinear damping is given by

$$2\gamma\beta[x^2 + (\dot{x}/\omega_0)^2]^{1/(2\gamma)}$$

We observe that $a(t)$ decays very slowly ($\gamma\beta \ll 1$) implies the damping coefficient $2\gamma\beta \ll 1$. And the damping model $D_0(x, \dot{x})$ is independent of α , which is essentially the initial condition ($a(0) = \alpha^{-\gamma}$).

EXAMPLE 5 The MEA has been successfully applied to the nonlinear damping modeling of the first mode vibration of the SCOLE (Spacecraft Control Laboratory Experiment) problem (Figure 3a) [29], [30]. A simplified version of the SCOLE configuration consists of a uniform Bernoulli beam clamped at one end (the space shuttle end) and equipped with rate sensors, and force actuators (reaction wheels) and control moment generators at the antenna end. The uniform beam supporting the antenna undergoes vibration due to the significant length (approximately 60 meters) and the flexible nature of the supporting truss.

Figure 3b is the experimental data of the first mode vibration after taking the natural logarithm. The initial curvature indicates the nonlinearity of the damping mechanism.

By computing $\{c_n, n = 1, 2, \dots, 50\}$, the $f_0(a(t_n)) = c_n$ relation is then obtained. Then, by trial and error, we found that a function of the form

$$f_0(x) = c_1 + c_2 x^7$$

will approximate the data. By picking any two points among the data, we can further determine

$$c_1 = 0.007 \quad c_2 = 0.076$$

Therefore, we obtain

$$D_0(x) = 0.007 + 0.076 \left(\frac{2}{\omega_0} x\right)^{3.5}$$

and hence, the damping model is given by

$$0.007\dot{x} + 0.076[x^2 + (\dot{x}/\omega_0)^2]^{3.5}\dot{x}$$

The damping model obtained is a combination of linear and nonlinear dampings.

4. ON THE POSSIBILITY OF OTHER CHOICES OF NONLINEAR DAMPING

This section gives a thorough investigation of the relation between energy type damping we have proposed and any other form of damping. A little bit of mathematical rigor is required for this study.

For further investigations, we need to introduce the following notations:

$$D = \{d(x, y) \in C(R^2) \mid d(\cdot, \cdot) \text{ even w.r.t. } x, \text{ odd w.r.t. } y\}$$

$$D_E = \left\{f\left(\frac{\omega_0^2 x^2 + y^2}{2}\right) y \mid f(\cdot) \in C(R^+)\right\}$$

Then D_E is a subspace in D and D is a subspace in $C(R^2)$ —the linear space of all continuous functions on R^2 (the real two dimensional space).

Next, let the operator $\mathbf{K}: D \rightarrow D_E$ be defined by

$$\mathbf{K}d(x, y) = f\left(\frac{\omega_0^2 x^2 + y^2}{2}\right) y$$

where

$$f(E) = \frac{1}{\pi\sqrt{2E}} \int_0^{2\pi} d\left(\frac{\sqrt{2E}}{\omega_0} \sin \psi, \sqrt{2E} \cos \psi\right) \cos \psi d\psi$$

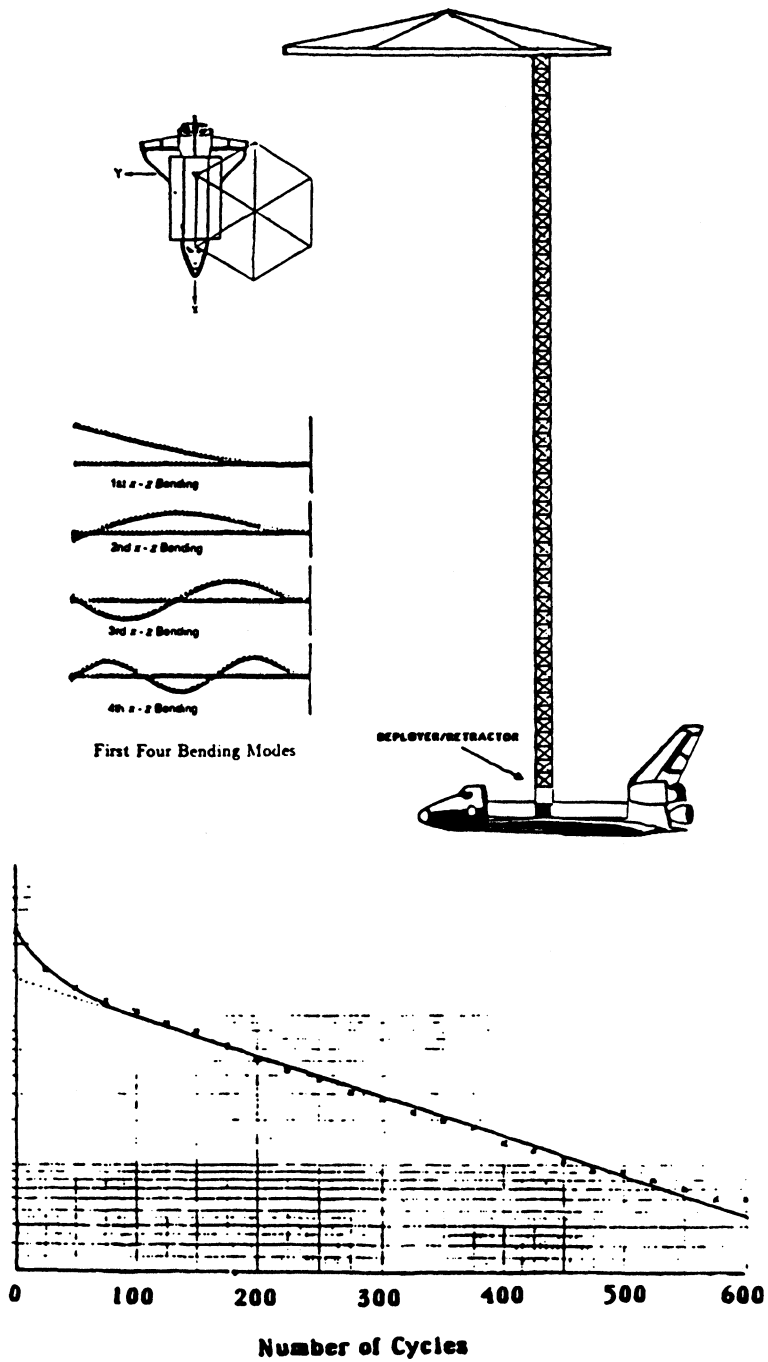


Figure 3 a. Shuttle Orbiter/Antenna Configuration. b. Nonlinear Logarithmic Decay In $a(t)$, where $a(t)$ = amplitude of the free response of the first bending mode. (Data provided by NASA Langley Research Center).

Then, the relation between $D(x, y)$ and $D_0(x, y)$ as in the previous sections can be written as $D_0(x, y) = \mathbf{K}D(x, y)$.

For ease of presentation, we first introduce the following

DEFINITION 1 $D_1(x, y), D_2(x, y) \in D$ are said to be $K - B$ equivalent if the corresponding nonlinearly damped systems

$$\ddot{x}(t) + D_1(x(t), \dot{x}(t)) + \omega_0^2 x(t) = 0$$

$$\ddot{x}(t) + D_2(x(t), \dot{x}(t)) + \omega_0^2 x(t) = 0$$

have the same Krylov-Bogoliubov approximation. We will use the notation

$$D_1(x, y) \stackrel{K-B}{=} D_2(x, y)$$

Obviously, $K - B$ equivalence is commutative, that is, $D_1(x, y) \stackrel{K-B}{=} D_2(x, y)$ and $D_2(x, y) \stackrel{K-B}{=} D_3(x, y)$ imply $D_1(x, y) \stackrel{K-B}{=} D_3(x, y)$.

The objective of this section is to answer the following questions: For any given $D(x, y) \in D$, we already know that $\mathbf{K}D(x, y) = D_0(x, y)$ is $K - B$ equivalent to the $D(x, y)$, that is, the free responses of the nonlinearly damped vibrations corresponding to the dampings $D(x, y)$ and $D_0(x, y)$ have the same Krylov-Bogoliubov approximation. Now the question is whether there is other $\tilde{D}(x, y)$ in D or D_E such that $\tilde{D}(x, y) \stackrel{K-B}{=} D(x, y)$? If yes, what are they?

First, we establish the following necessary and sufficient condition of $K - B$ equivalence.

FACT 4 For any $D_j(x, y) \in D, j = 1, 2,$

$$D_1(x, y) \stackrel{K-B}{=} D_2(x, y)$$

if and only if

$$\mathbf{K}D_1(x, y) = \mathbf{K}D_2(x, y)$$

Proof: “ \Rightarrow ” $D_1(x, y) \stackrel{K-B}{=} D_2(x, y)$ implies

$$\int_0^{2\pi} D_1(a \sin \psi, a\omega_0 \cos \psi) \cos \psi d\psi = \int_0^{2\pi} D_2(a \sin \psi, a\omega_0 \cos \psi) \cos \psi d\psi \quad (12)$$

for all $a \geq 0$. Replacing a by a/ω_0 in (12), we obtain

$$\int_0^{2\pi} D_1(a/\omega_0 \sin \psi, a \cos \psi) \cos \psi d\psi = \int_0^{2\pi} D_2(a/\omega_0 \sin \psi, a \cos \psi) \cos \psi d\psi \quad (13)$$

Let

$$\mathbf{K}D_j(x, y) = \mu_j \left(\frac{\omega_0^2 x^2 + y^2}{2} \right) y, j = 1, 2.$$

Then (13) implies $\mu_1(E) = \mu_2(E)$ for $E \geq 0$, that is,

$$\mathbf{K}D_1(x, y) = \mathbf{K}D_2(x, y)$$

“ \Leftarrow ” $\mu_1 \equiv \mu_2$ implies (12), and further implies

$$D_1(x, y) \stackrel{K-B}{=} D_2(x, y) \quad \blacksquare$$

In order to answer our questions, we desire to show that \mathbf{K} is a projection operator from D to D_E , for which, of course, an appropriate inner product needs to be introduced on D to obtain an inner product space. The motivation for this is that once we can establish that \mathbf{K} is a projector, we can immediately obtain the answers to our questions.

In fact, for any given $D(x, y) \in D$, let $\tilde{D}(x, y)$ be any other nonlinear damping in D such that $D(x, y) = {}^{K-B}D(x, y)$. By *Fact 4*, it is necessary and sufficient for $\tilde{D}(x, y)$ to satisfy

$$\mathbf{K}D(x, y) = \mathbf{K}\tilde{D}(x, y) = D_0(x, y)$$

That is, $\tilde{D}(x, y)$ can be any element in D such that its projection on D_E is $D_0(x, y)$. Therefore, by the fact that \mathbf{K} is a projector, we obtain

$$D(x, y) = D_0(x, y) + \text{any element} \in \mathcal{N}(\mathbf{K})$$

Since it is easy to show that

$$\mathcal{N}(\mathbf{K}) = \{d(x, y) - \mathbf{K}d(x, y) \mid d(x, y) \in D\}$$

we finally conclude that all the $\tilde{D}(x, y)$, which is $\mathbf{K} - B$ equivalent to $D(x, y)$, can be represented by

$$D(x, y) = \mathbf{K}D(x, y) + d(x, y) - \mathbf{K}d(x, y) \text{ for any } d(x, y) \in D \quad (14)$$

In other words, for any given $D(x, y) \in D$, there exist infinitely many $\tilde{D}(x, y) \in D$ such that $D(x, y) = {}^{K-B}D(x, y)$.

However, if we confine our attention to D_E , that is, $\tilde{D}(x, y) \in D_E$, then this $\tilde{D}(x, y)$ is unique.

In fact, by (14), $\tilde{D}(x, y) \in D_E$ implies $d(x, y) \in D_E$ and hence $d(x, y) - \mathbf{K}d(x, y) = 0$. Therefore, for any given $D(x, y) \in D$, there exists unique $\tilde{D}(x, y) \in D_E$ such that $D(x, y) = {}^{K-B}D(x, y)$ and this $\tilde{D}(x, y)$ is given by

$$D(x, y) = \mathbf{K}D(x, y) = D_0(x, y)$$

Next, what is left is to establish that \mathbf{K} is a projection operator. For this purpose, we need to introduce an inner product in the linear space D .

For any given $f(\omega_0^2 x^2 + y^2/2)y \in D_E$, let $(x(t), y(t))$ be the stationary response of

$$\ddot{x} + f\left(\frac{\omega_0^2 x^2 + \dot{x}^2}{2}\right)\dot{x} + \omega_0^2 x = \sigma n(t)$$

with the corresponding stationary probability density $p_f(\omega_0^2 x^2 + y^2/2)$. We define the inner product on D by

$$\begin{aligned} [D_1, D_2] &= \mathbf{E}\{D_1(x(t), y(t))D_2(x(t), y(t))\} \\ &= \iint_{R^2} D_1(x, y)D_2(x, y)p_f\left(\frac{\omega_0^2 x^2 + y^2}{2}\right)dx dy \end{aligned}$$

for any $D_1, D_2 \in D$. We use $[\cdot, \cdot]_f$ to indicate the dependence of the inner product on the choice of $f(\cdot)$.

With the introduction of this inner product, we can now show

FACT 5 $\mathbf{K} D \rightarrow D_E$ is a projection operator.

Proof: It is sufficient to show that

$$\mathbf{K}^* = \mathbf{K}$$

$$\mathbf{K}^2 = \mathbf{K}$$

In fact, for any $D_1, D_2 \in D$, there holds

$$\begin{aligned} [\mathbf{K}D_1, D_2]_f &= [\mathbf{K}D_1, D_2 - \mathbf{K}D_2 + \mathbf{K}D_2]_f \\ &= [\mathbf{K}D_1, \mathbf{K}D_2]_f \\ &= [\mathbf{K}D_1 - D_1 + D_1, \mathbf{K}D_2]_f \\ &= [D_1, \mathbf{K}D_2]_f \end{aligned}$$

that is, $\mathbf{K}^* = \mathbf{K}$.

For any $D \in D$, let

$$\mathbf{K}D(x, y) = \mu\left(\frac{\omega_0^2 x^2 + y^2}{2}\right)y$$

It is easy to show that

$$\mathbf{K}\left[\mu\left(\frac{\omega_0^2 x^2 + y^2}{2}\right)y\right] = \mu\left(\frac{\omega_0^2 x^2 + y^2}{2}\right)y$$

Then,

$$\begin{aligned}
 \mathbf{K}^2 D(x, y) &= \mathbf{K}[\mu(\frac{\omega_0^2 x^2 + y^2}{2})y] \\
 &= \mu(\frac{\omega_0^2 x^2 + y^2}{2})y \\
 &= \mathbf{K}D(x, y)
 \end{aligned}$$

that is, $\mathbf{K}^2 = \mathbf{K}$.

Therefore, using *Fact 4*, one can infer that for any nonlinear damping model $D(x, y) \in D$, the difference $D(x, y) - \mathbf{K}D(x, y)$ is K-B equivalent to 0.

5. CONCLUDING REMARKS

We have presented a novel and systematic method for modeling smart structure damping mechanisms. This method does not assume the linear damping property of a smart structure and the resulting model is based upon experimental data, instead of physical principles. Therefore, this method can be applied to smart structures of different materials and different geometries. Various supporting facts of the MEA have been presented in this paper, which contains very interesting theoretical insights into damping mechanisms.

With the establishment of the MEA in the single-DOF case, the crucial problem is now to extend the MEA to multi-DOF vibrations. This step is crucial, because in practice, smart structures or other vibration systems are seldom single-DOF. The extension of the MEA to multi-DOF cases requires tremendous effort because of the following two facts:

1. Multiple instead of single vibration modes are involved. The damping model should sufficiently describe the vibration amplitude decays for at least all of the most significant modes. The most significant modes of a smart structure, such as a helicopter rotor, include those modes of vibration which can potentially exhibit significant deformation in the operating frequency range, and thus should be controlled by the embedded actuators.
2. Mode coupling exists due to the existence of damping and the initial condition, which is usually a combination of more than one mode. In the process of collecting data, even if one can set the initial condition to be a specific (single) mode, the response may well involve multiple modes due to damping, especially if the damping is not weak enough.

The above two facts are the main reasons for the difficulties which will be encountered in extending the MEA to multi-DOF vibrations.

An initial attempt has been made to extend the MEA. The preliminary research results on multi- but finite-DOF vibrations have been reported in [31]. Research results on continuum models have appeared in [32], [33].

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