

ON THE STRUCTURE OF THE DEFLAGRATION FOR THE GENERALIZED REACTION-RATE MODEL

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The structure of the deflagration is examined by means of an asymptotic analysis of the physical-plane boundary-value problem, with Lewis-Semenov number unity, in the limit of the activation-temperature ratio, $\beta = T_a/T_b$, greater than order unity, for the generalized reaction-rate-model case of (1) the heat-addition-temperature ratio, $\alpha = (T_b - T_u)/T_u$, of order unity [where T_a , T_b , and T_u are the activation, adiabatic-flame (and/or burned-gas), and unburned-gas temperatures, respectively]; and (2) the exponent, a , which characterizes the pre-exponential thermal dependence of the reaction-rate term, unity. This examination indicates that the deflagration has a four-region structure. To obtain a uniformly valid solution of the problem, in addition to the (classical) upstream diffusion-convection and downstream diffusion-reaction regions, a far-upstream (or cold-boundary) region and a far-downstream (or hot-boundary) region must be introduced.

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1. STATEMENT OF THE PROBLEM

For the direct, first-order, one-step, irreversible, exothermic, unimolecular chemical reaction $R \rightarrow P$, the steady, one-dimensional, low-Mach-number, isobaric, laminar deflagration, for Lewis-Semenov number unity, is modeled by the following (nondimensional) boundary-value problem (cf. Bush and Fendell [1], Fendell [2]) in the domain $(-\infty < \xi < \infty)$:

$$\frac{d\tau}{d\xi} = \tau - \varepsilon, \quad (1.1a)$$

$$\frac{d\varepsilon}{d\xi} = \Lambda(1 - \tau)\tau^a \exp\{-\beta(1 - \tau)/(\alpha^{-1} + \tau)\}; \quad (1.1b)$$

$$\tau, \varepsilon \rightarrow 0 \text{ as } \xi \rightarrow -\infty, \quad (1.2a)$$

$$\tau, \varepsilon \rightarrow 1 \text{ as } \xi \rightarrow \infty. \quad (1.2b)$$

In the above, (1) ξ is the (modified) spatial coordinate; (2) τ is the normalized temperature, and ε is the normalized stoichiometrically adjusted mass-flux fraction of the product; (3) β is the activation-temperature ratio, T_a/T_b , greater than order unity, and α is the heat-addition-temperature ratio, $(T_b - T_u)/T_u$, of order unity, where T_a , T_b , and T_u are the activation, adiabatic-flame (and/or burned-gas), and unburned-gas temperatures, respectively; (4) a is the exponent that characterizes the pre-exponential thermal dependence of the reaction-rate term, of order unity; and (5) Λ is the normalized Damköhler number, greater than order unity (see Bush and Fendell [1] for further details and references).

The boundary-value problem of (1.1) and (1.2), for the reaction-rate model defined by $\alpha^{-1} = 0$ and $a = 0$, in the limit of $\beta \rightarrow \infty$, has been studied analytically in phase space (Bush and Fendell [1]) and in physical space (Bush [3]). For this case, the exponential factor has been modified, so that the right-hand side of (1.1b) vanishes as $\xi \rightarrow -\infty$, just as boundedness requires that the left-hand side vanish in this limit. Such a compatible choice to resolve the cold-boundary difficulty was adopted earlier by Friedman and Burke [4].

This boundary-value problem has also been studied analytically in phase space for the reaction-rate model defined by $\alpha \sim \mathcal{O}(1)$, and $a = 1$ (Bush and Fendell [1]) and for the one defined by $\alpha \sim \mathcal{O}(1)$, and $a > 0$ (Fendell [2]), in the limit of $\beta \rightarrow \infty$. In each case, it is the pre-exponential factor, rather than the exponential factor, that reduces the right-hand side of (1.1b) to zero at the cold boundary—and resolves the cold-boundary difficulty.

In what follows, by means of an asymptotic analysis in physical space of the generalized (reaction-rate) model boundary-value problem of (1.1) and (1.2), for $\beta \rightarrow \infty$, with $\alpha \sim \mathcal{O}(1)$, and $a = 1$, it is shown that a four-region structure must be introduced for the deflagration in order to obtain uniformly valid solutions from the cold boundary to the hot boundary.

2. ASYMPTOTIC ANALYSIS

The model deflagration boundary-value problem under consideration (i.e., $\beta \rightarrow \infty$, $\alpha \sim \mathcal{O}(1)$, and $a = 1$) requires the analysis of four principal regions: (1) a relatively thin downstream region, near the hot boundary, where $(1 - \tau) \sim \mathcal{O}(\beta^{-1})$; (2) a relatively thicker upstream region, near the cold boundary, where $\tau \sim \mathcal{O}(1)$; (3) a far-upstream region (the thickness of which is comparable to that of the upstream region), nearer to the cold boundary, where $\tau \sim \mathcal{O}(\beta^{-1})$; and (4) a far-downstream region (the thickness of which is comparable to those of the upstream regions), nearer to the hot boundary, where $(1 - \tau) \sim \mathcal{O}(\beta^{-1} \exp(-\beta))$.

2.1 The Downstream Region

Based upon previously presented results (Bush [3]), the appropriate independent and dependent variables for the (classical) downstream region are

$$\zeta(\xi; \beta) = \beta\xi; \quad (2.1)$$

$$\tau(\xi; \beta) = 1 - \beta^{-1} G(\zeta; \beta): G(\zeta; \beta) \cong G_0(\zeta) + \beta^{-1} G_1(\zeta) + \dots, \tag{2.2a}$$

$$\varepsilon(\xi; \beta) = E(\zeta; \beta): E(\zeta; \beta) \cong E_0(\zeta) + \beta^{-1} E_1(\zeta) + \dots. \tag{2.2b}$$

Throughout the flowfield, the eigenvalue, Λ , has the representation

$$\Lambda(\beta) = \beta^2 \Lambda^*(\beta): \Lambda^*(\beta) \cong \Lambda_0 + \beta^{-1} \Lambda_1 + \dots. \tag{2.3}$$

In terms of the downstream variables, (1.1) can be written as

$$\frac{dG}{d\zeta} = - \{ (1 - E) - \beta^{-1} G \}, \tag{2.4a}$$

$$\frac{dE}{d\zeta} = \Lambda^* G (1 - \beta^{-1} G) \exp \left\{ - \frac{G}{(K - \beta^{-1} G)} \right\}, \tag{2.4b}$$

with $K = (1 + \alpha^{-1})$. Introduction of the variables of (2.1) and (2.2) and the representation of (2.3) into (2.4) yields, for the domain $(-\infty < \zeta < \infty)$,

$$\frac{dG_0}{d\zeta} = - (1 - E_0), \dots, \tag{2.5a}$$

$$\frac{dE_0}{d\zeta} = \Lambda_0 G_0 \exp \left\{ - \frac{G_0}{K} \right\}, \dots. \tag{2.5b}$$

From (1.2b), the downstream boundary conditions for these equations are taken to be

$$G_0 \rightarrow 0, \dots, E_0 \rightarrow 1, \dots \text{ as } \zeta \rightarrow \infty. \tag{2.6a}$$

In anticipation of the downstream-region/upstream-region matching (see Bush [3]), the upstream boundary conditions for these equations are taken to be

$$G_0 \rightarrow \infty, \dots, E_0 \rightarrow 0, \dots \text{ as } \zeta \rightarrow -\infty. \tag{2.6b}$$

As an intermediate step, the leading-order boundary-value problem, from (2.5) and (2.6), may be written in phase-plane form, i.e., with $G_{k0} = (G_0/K)$,

$$\frac{d(1 - E_0)}{dG_{k0}} = \frac{(2\Lambda_0 K^2) G_{k0} \exp(-G_{k0})}{2(1 - E_0)}; \tag{2.7a}$$

$$(1 - E_0) \rightarrow 0 \text{ as } G_{k0} \rightarrow 0, (1 - E_0) \rightarrow 1 \text{ as } G_{k0} \rightarrow \infty. \tag{2.7b}$$

The solution, $E_0(G_{k0})$, of (2.7) is

$$E_0 = 1 - [1 - (1 + G_{k0}) \exp(-G_{k0})]^{1/2}, \tag{2.8}$$

for

$$\Lambda_0 = \frac{1}{2K^2} = \frac{1}{2(1 + \alpha^{-1})^2}. \tag{2.9}$$

This is the form for Λ_0 given in [1] and [2].

Once $E_0(G_{k0})$ is known, $\zeta_k = (\zeta/K) = \zeta_k(G_{k0})$ is determined from

$$\frac{d\zeta_k}{dG_{k0}} = - \frac{1}{\{1 - E_0(G_{k0})\}}; \zeta_k = - \int_{G_{k0}^o}^{G_{k0}} \frac{dt}{\{1 - E_0(t)\}}, \tag{2.10}$$

for $\zeta_k \rightarrow 0$ as $G_{k0} \rightarrow G_{k0}^o = \text{const.}$ (to be determined). From (2.10), at the “downstream edge” of this region,

$$\begin{aligned} \zeta_k &\sim \sqrt{2}[-\log G_{k0} + \log G_{k0}^b - \frac{1}{3} G_{k0} + \dots] \rightarrow \infty \text{ as } G_{k0} \rightarrow 0: \\ G_{k0} &\sim G_{k0}^b \exp(-\zeta_k / \sqrt{2})(1 + \dots) \rightarrow 0 \text{ as } \zeta_k \rightarrow \infty, \end{aligned} \tag{2.11a}$$

where

$$G_{k0}^b = G_{k0}^o \exp \left\{ \frac{1}{\sqrt{2}} \int_0^{G_{k0}^o} \left[\frac{1}{\{1 - E_0(t)\}} - \frac{\sqrt{2}}{t} \right] dt \right\}. \tag{2.11b}$$

At the “upstream edge” of this region,

$$\begin{aligned} \zeta_k &\sim - (G_{k0} - G_{k0}^u) + \frac{1}{2} (G_{k0} + 2) \exp(-G_{k0})(1 + \dots) \rightarrow -\infty \text{ as } G_{k0} \rightarrow \infty: \\ G_{k0} &\sim ((-\zeta_k) + G_{k0}^u)(1 + \dots) \rightarrow \infty \text{ as } \zeta_k \rightarrow -\infty, \end{aligned} \tag{2.12a}$$

where

$$G_{k0}^u = G_{k0}^o - \int_{G_{k0}^o}^{\infty} \left[\frac{1}{\{1 - E_0(t)\}} - 1 \right] dt. \tag{2.12b}$$

With the downstream and upstream behaviors for $G_{k0}(\zeta_k)$ determined, the corresponding behaviors for $E_0(\zeta_k)$ are found to be

$$E_0 \sim 1 - (G_{k0}^b / \sqrt{2}) \exp(-\zeta_k / \sqrt{2})(1 + \dots) \rightarrow 1 \text{ as } \zeta_k \rightarrow \infty; \tag{2.13}$$

$$E_0 \sim \frac{1}{2} \exp(-G_{k0}^u) \exp(-(-\zeta_k)) [(-\zeta_k) + (G_{k0}^u + 1)] \\ (1 + \dots) \rightarrow 0 \text{ as } \zeta_k \rightarrow -\infty. \tag{2.14}$$

Since, for this region, higher-order approximations are not pursued in this paper, from (2.11) and (2.13), τ and ε , as $\zeta_k \rightarrow \infty$, are given by

$$\tau = 1 - \beta^{-1} [KG_{k0}^b \exp(-\zeta_k / \sqrt{2})(1 + \dots)][1 + \mathcal{O}(\beta^{-1})]; \tag{2.15}$$

$$\varepsilon = [1 - (G_{k0}^b / \sqrt{2}) \exp(-\zeta_k / \sqrt{2})(1 + \dots)][1 + \mathcal{O}(\beta^{-1})]. \tag{2.16}$$

In [3], where higher-order approximations for this region are pursued, it is found that the solutions for the downstream region, considered, are not uniformly valid as the hot boundary is approached ($\zeta_k \rightarrow \infty$). This (near) downstream region must be supplemented by a far-downstream region. Details of this far-downstream region are presented in Sec. 2.4.

From (2.12) and (2.14), with $G_{k0}^u = -1$, τ and ε , as $\zeta_k \rightarrow -\infty$, are given by

$$\tau = 1 - \beta^{-1} [K((- \zeta_k) - 1)(1 + \dots)][1 + \mathcal{O}(\beta^{-1})]; \tag{2.17}$$

$$\varepsilon = \left[\frac{1}{2} e(-\zeta_k) \exp(-(-\zeta_k))(1 + \dots)\right][1 + \mathcal{O}(\beta^{-1})]. \tag{2.18}$$

2.2 THE UPSTREAM REGION

For the (classical) upstream region, the independent variable is ξ , with $-\infty < \xi < 0$, and the appropriate dependent variables are

$$\tau(\xi; \beta) = F(\xi; \beta): \quad F(\xi; \beta) \cong F_0(\xi) + \beta^{-1} F_1(\xi) + \dots, \tag{2.19a}$$

$$\varepsilon(\xi; \beta) = \beta J(\xi; \beta) \exp\{-\beta H(\xi; \beta)\} :$$

$$J(\xi; \beta) \cong J_0(\xi) + \beta^{-1} J_1(\xi) + \dots,$$

$$H(\xi; \beta) = \frac{(1 - F(\xi; \beta))}{[K - (1 - F(\xi; \beta))]}$$

$$\cong H_0(\xi) + \beta^{-1} H_1(\xi) + \dots$$

$$\begin{aligned}
 &= \frac{(1 - F_0(\xi))}{[K - (1 - F_0(\xi))]} - \beta^{-1} \frac{KF_1(\xi)}{[K - (1 - F_0(\xi))]^2} \\
 &+ \dots .
 \end{aligned}
 \tag{2.19b}$$

The representation for the eigenvalue, Λ , is now

$$\Lambda(\beta) = \beta^2 \Lambda^*(\beta): \Lambda^*(\beta) \cong \Lambda_0 + \beta^{-1} \Lambda_1 + \dots = \frac{1}{2K^2} + \dots .
 \tag{2.20}$$

In terms of the upstream variables, (1.1) can be written as

$$\frac{dF}{d\xi} = F - \beta J \exp \left\{ -\beta \frac{(1 - F)}{[K - (1 - F)]} \right\},
 \tag{2.21a}$$

$$\frac{KJ}{[K - (1 - F)]^2} \frac{dF}{d\xi} + \beta^{-1} \frac{dJ}{d\xi} = \Lambda^*(1 - F)F.
 \tag{2.21b}$$

Introduction of the variables of (2.19) and the representation of (2.20) into (2.21) yields, for the domain ($-\infty < \xi < 0$),

$$\frac{dF_0}{d\xi} = F_0, \frac{dF_1}{d\xi} = F_1, \dots,
 \tag{2.22a}$$

$$\frac{KJ_0}{[K - (1 - F_0)]^2} \frac{dF_0}{d\xi} = \frac{2}{K^2} (1 - F_0)F_0, \dots .
 \tag{2.22b}$$

Directly, from (2.22a), it is found that the temperature-function solution, $F(\xi; \beta)$, has the asymptotic representation

$$F \cong A_0 \exp(\xi) + \beta^{-1} A_1 \exp(\xi) + \dots .
 \tag{2.23}$$

For $A_0 = 1, A_1 = K, \dots$, such that $F_0(\xi) = \exp(\xi), F_1(\xi) = K \exp(\xi) = KF_0(\xi), \dots$,

$$F \sim [1 - (-\xi) + \dots][1 - \beta^{-1}(-K) + \dots] \rightarrow 1 \text{ as } \xi \rightarrow 0_-;
 \tag{2.24a}$$

$$F \sim \exp(\xi)[1 + \beta^{-1}K + \dots] \rightarrow 0 \text{ as } \xi \rightarrow -\infty.
 \tag{2.24b}$$

Based upon (2.19b) and the results of (2.23) and (2.24), $H(F_0(\xi); \beta)$ can be expressed as

$$H \cong \frac{(1 - F_0)}{K - (1 - F_0)} - \beta^{-1} \frac{K^2 F_0}{[K - (1 - F_0)]^2} + \dots \tag{2.25}$$

With the representations for F and/or H determined, from (2.22b), the solution for the leading-order mass-flux function, $J_0(F_0(\xi))$, is found to be

$$J_0 = \frac{(1 - F_0)[K - (1 - F_0)]^2}{2K^3} \tag{2.26}$$

Solutions for the higher-order mass-flux functions are not pursued here.

From the solutions of (2.23)–(2.26), for the upstream region, τ and ε can be expressed as

$$\tau = F_0[1 + \beta^{-1} K + O(\beta^{-2})]; \tag{2.27}$$

$$\begin{aligned} \varepsilon = \beta \frac{(1 - F_0)[K - (1 - F_0)]^2}{2K^3} \exp \left\{ -\beta \frac{(1 - F_0)}{[K - (1 - F_0)]} \right. \\ \left. + \frac{K^2 F_0}{[K - (1 - F_0)]^2} \right\} [1 + O(\beta^{-1})]. \end{aligned} \tag{2.28}$$

These upstream solutions for τ and ε , (2.27) and (2.28), as $(-\xi) = \varphi(\beta)\chi \rightarrow 0$, for χ fixed, $\beta \rightarrow \infty$, match to the downstream solutions for τ and ε , (2.17) and (2.18), as $(-\zeta) = \beta\varphi(\beta)\chi \rightarrow \infty$ for χ fixed, $\beta \rightarrow \infty$.

Upstream, for $F_0 \rightarrow 0$, $\beta \rightarrow \infty$, such that $\beta F_0 = f_0 \sim O(1)$, (2.27) and (2.28) yield

$$\tau = (\alpha\beta)^{-1} (\alpha f_0)[1 + (\alpha\beta)^{-1} (1 + \alpha) + O((\alpha\beta)^{-2})]; \tag{2.29}$$

$$\varepsilon = [(\alpha\beta) \exp\{-(\alpha\beta)\}] \left[\frac{\exp\{(1 + \alpha)(\alpha f_0)\}}{2(1 + \alpha)^3} \right] [1 + O((\alpha\beta)^{-1})]. \tag{2.30}$$

Note that $(\alpha\beta)^{-1} \rightarrow 0$ (algebraically), and that $(\alpha\beta) \exp\{-(\alpha\beta)\} \rightarrow 0$ (exponentially), as $\beta \rightarrow \infty$. Here, it is recalled that this analysis is for $\alpha \sim O(1)$. With $F_0 = \exp(\xi)$, it follows that

$$f_0 = \beta \exp(\xi) = \exp(\xi_u) \exp(\xi) = \exp(\xi_u + \xi), \text{ with } \xi_u = \log \beta. \tag{2.31}$$

Thus, $\beta F_0 = f_0 \sim O(1)$ for $(\xi_u + \xi) = \eta \sim O(1)$, i.e., $\beta F_0 = f_0 = \exp(\eta) \sim O(1)$. In turn, (2.29) and (2.30) can be written as

$$\tau = (\alpha\beta)^{-1} [\alpha \exp(\eta)][1 + (\alpha\beta)^{-1} (1 + \alpha) + O((\alpha\beta)^{-2})]; \quad (2.32)$$

$$\varepsilon = [(\alpha\beta) \exp\{-(\alpha\beta)\}] \left[\frac{\exp\{(1 + \alpha)[\alpha \exp(\eta)]\}}{2(1 + \alpha)^3} \right] [1 + O((\alpha\beta)^{-1})]. \quad (2.33)$$

From (2.32), it is seen that the temperature function, τ , of $O((\alpha\beta)^{-1})$, goes to zero (exponentially), as required from (1.2a), as the cold upstream boundary is approached (i.e., $\eta \rightarrow -\infty$). However, from (2.33), it is seen that the mass-flux function, ε , of $O((\alpha\beta) \exp\{-(\alpha\beta)\})$, goes to a finite value as the upstream boundary is approached—in contradiction of (1.2a). In Sec. 2.3, a far-upstream region, nearer to the upstream boundary, is introduced, the solutions of which resolve the “cold-boundary difficulty” suggested by (2.33).

2.3 The Far-Upstream Region

For the far-upstream region, based on (2.29)–(2.33), it is taken that the appropriate independent and dependent variables are

$$\eta(\xi; \beta) = \xi_u(\beta) + \xi, \quad \text{with } \xi_u(\beta) = \log \beta; \quad (2.34)$$

$$\tau(\xi; \beta) = \beta^{-1} \phi(\eta; \beta) = (\alpha\beta)^{-1} (\alpha\phi(\eta; \beta));$$

$$\alpha\phi(\eta; \beta) = \Phi(\eta; \beta) \cong \Phi_0(\eta) + (\alpha\beta)^{-1} (\alpha\Phi_1(\eta)) + \dots, \quad (2.35a)$$

$$\varepsilon(\xi; \beta) = (\alpha\beta) \exp\{-(\alpha\beta)\} \Psi(\eta; \beta): \Psi(\eta; \beta) \cong \Psi_0(\eta) + (\alpha\beta)^{-1} (\alpha\Psi_1(\eta)) + \dots. \quad (2.35b)$$

The eigenvalue is given by

$$\Lambda(\beta) = \beta^2 \Lambda^*(\beta) = (\alpha\beta)^2 (\Lambda^*(\beta)/\alpha^2):$$

$$(\Lambda^*(\beta)/\alpha^2) \cong (\Lambda_0/\alpha^2) + (\alpha\beta)^{-1} (\Lambda_1/\alpha) + \dots = \frac{1}{2(1+\alpha)^2} + \dots. \quad (2.36)$$

In terms of the far-upstream variables, (1.1) can be written as

$$\frac{d\Phi}{d\eta} = \Phi - (\alpha\beta)^2 \exp\{-(\alpha\beta)\} \Psi, \quad (2.37a)$$

$$\frac{d\Psi}{d\eta} = \left[\frac{(\Lambda^*/\alpha^2)}{(\Lambda_0/\alpha^2)} \right] \frac{[(1 + \alpha)\Phi][1 - (\alpha\beta)^{-1}\Phi]}{2(1 + \alpha)^3} \exp \left\{ \frac{[(1 + \alpha)\Phi]}{[1 + (\alpha\beta)^{-1}\alpha\Phi]} \right\}. \tag{2.37b}$$

Introduction of the variables of (2.35) and the representation of (2.36) into (2.37) yields, in the domain $(-\infty < \eta < \infty)$,

$$\frac{d\Phi_0}{d\eta} = \Phi_0, \quad \frac{d\Phi_1}{d\eta} = \Phi_1, \dots, \tag{2.38a}$$

$$\frac{d\Psi_0}{d\eta} = \frac{[(1 + \alpha)\Phi_0]}{2(1 + \alpha)^3} \exp\{[(1 + \alpha)\Phi_0]\}, \dots. \tag{2.38b}$$

From (2.38a), in this region, the temperature-function solution, $\phi(\eta; \beta)$, has the asymptotic representation

$$\Phi \cong [a_0 \exp(\eta)] + (\alpha\beta)^{-1} [a_1 \exp(\eta)] + \dots. \tag{2.39a}$$

For $a_0 = \alpha, a_1 = \alpha(1 + \alpha), \dots$, such that $\Phi_0(\eta) = [\alpha \exp(\eta)], \Phi_1(\eta) = (1 + \alpha)[\alpha \exp(\eta)] = (1 + \alpha)\Phi_0(\eta), \dots$, then,

$$\Phi = [\alpha \exp(\eta)][1 + (\alpha\beta)^{-1} (1 + \alpha) + O((\alpha\beta)^{-2})]. \tag{2.39b}$$

Here, it is noted that, upstream, $\Phi_0(\eta) = [\alpha \exp(\eta)] \rightarrow 0$ as $\eta \rightarrow -\infty$, and that, downstream, $\Phi_0(\eta) = [\alpha \exp(\eta)] \rightarrow \infty$ as $\eta \rightarrow \infty$. Substitution of (2.39) into (2.38b) produces the following:

$$\frac{d\Psi_0}{d\eta} = \frac{\{(1 + \alpha)[\alpha \exp(\eta)]\}}{2(1 + \alpha)^3} \exp\{(1 + \alpha)[\alpha \exp(\eta)]\}. \tag{2.40a}$$

The solution of (2.40a) for $\Psi_0(\eta)$, for which $\Psi_0(\eta) \rightarrow 0$ as $\eta \rightarrow -\infty$ and for which $\Psi_0(\eta) \rightarrow \infty$ as $\eta \rightarrow \infty$, is

$$\Psi_0 = \frac{\exp\{(1 + \alpha)[\alpha \exp(\eta)]\} - 1}{2(1 + \alpha)^3}. \tag{2.40b}$$

Hence, for this far-upstream region, τ and ε can be expressed as

$$\tau = (\alpha\beta)^{-1} [\alpha \exp(\eta)] [1 + (\alpha\beta)^{-1} (1 + \alpha) + O((\alpha\beta)^{-2})]; \tag{2.41}$$

$$\varepsilon = [(\alpha\beta) \exp\{-(\alpha\beta)\}] \left[\frac{\exp\{(1 + \alpha)[\alpha \exp(\eta)]\} - 1}{2(1 + \alpha)^3} \right] [1 + O((\alpha\beta)^{-1})]. \tag{2.42}$$

The function τ of (2.41) is that of (2.32), and, as such, satisfies its upstream boundary condition ($\eta \rightarrow -\infty$). The function ε of (2.42) now satisfies its upstream boundary condition ($\eta \rightarrow -\infty$)—and the cold-boundary difficulty is resolved. Further, these far-upstream solutions for τ and ε , (2.41) and (2.42), as $\eta \rightarrow \infty$, are seen to match to the upstream ones, (2.27) and (2.28), as $\xi \rightarrow -\infty$.

2.4 The Far-Downstream Region

When higher-order approximations for the downstream region (of Sec. 2.1) are pursued (see [3]), it is determined that, for this region, the solutions for τ and ε are not uniformly valid as $\zeta \rightarrow \infty$ and $\beta \rightarrow \infty$, such that $\beta^{-1} \zeta = \xi$ is of order unity. This nonuniformity indicates that, for the boundary-value problem under consideration, the downstream region should be supplemented by a far-downstream region.

Based on (2.15) and (2.16), as well as the above, the appropriate independent and dependent variables for this far-downstream region are

$$\lambda(\xi; \beta) = \beta^{-1} \zeta(\xi; \beta) = \xi; \tag{2.43}$$

$$\tau(\xi; \beta) = 1 - \beta^{-1} \exp\{-(\beta\lambda/\sqrt{2K})\}u(\lambda; \beta); u(\lambda; \beta) \cong u_0(\lambda) + \beta^{-1} u_1(\lambda) + \dots, \tag{2.44a}$$

$$\varepsilon(\xi; \beta) = 1 - \exp\{-(\beta\lambda/\sqrt{2K})\}v(\lambda; \beta); v(\lambda; \beta) \cong v_0(\lambda) + \beta^{-1}v_1(\lambda) + \dots. \tag{2.44b}$$

Further, the eigenvalue, Λ , is

$$\Lambda(\beta) = \beta^2 \Lambda^*(\beta); \Lambda^*(\beta) \cong \Lambda_0 + \beta^{-1} \Lambda_1 + \dots = \frac{1}{2K^2} + \beta^{-1} \frac{[3 - (I - 1)K]}{K^2}, \dots. \tag{2.45}$$

The determination of the first-order approximation, i.e., $\Lambda_1 = [3 - (I - 1)K]/K^2$, with $(I - 1) = 0.344$, is presented in [1] and [2].

In terms of the far-downstream variables, (1.1) can be written as

$$\left(\frac{1}{\sqrt{2K}} u - \beta^{-1} \frac{du}{d\lambda} \right) = (v - \beta^{-1}u), \tag{2.46a}$$

$$\left(\frac{1}{\sqrt{2K}} v - \beta^{-1} \frac{dv}{d\lambda} \right) = \Lambda^*u[1 - \beta^{-1}u \exp\{-(\beta\lambda/\sqrt{2K})\}] \times \exp \left\{ -\frac{u \exp\{-(\beta\lambda/\sqrt{2K})\}}{[K - \beta^{-1}u \exp\{-(\beta\lambda/\sqrt{2K})\}]} \right\}. \tag{2.46b}$$

Introduction of the expansions of (2.44) and (2.45) into (2.46) produces, for the domain $(0 < \lambda < \infty)$,

$$\left(\frac{1}{\sqrt{2K}}u_0 - v_0\right) = 0, \left(\frac{1}{\sqrt{2K}}u_1 - v_1\right) = \left(\frac{du_0}{d\lambda} - u_0\right), \dots; \tag{2.47a}$$

$$\left(\frac{1}{\sqrt{2K}}u_0 - v_0\right) = 0, \left(\frac{1}{\sqrt{2K}}u_1 - v_1\right) = \sqrt{2K}\left(\frac{dv_0}{d\lambda} + \Lambda_1 u_0\right), \dots \tag{2.47b}$$

The zeroth-order equations of (2.47a) and (2.47b) both yield

$$v_0 = \frac{1}{\sqrt{2K}}u_0. \tag{2.48}$$

In turn, the first-order equations of (2.47a) and (2.47b), in combination with (2.48) yield

$$\frac{du_0}{d\lambda} + \gamma u_0 = 0, \tag{2.49}$$

with $\gamma = 1/2(\sqrt{2K}\Lambda_1 - 1)$. Thus, from (2.48) and (2.49), the zeroth-order far-upstream-region solutions are determined to be

$$u_0 = \sqrt{2K}v_0 = u_0^o \exp(-\gamma\lambda), \tag{2.50}$$

where $u_0^o = KG_{k_0}^b$, in order that the solutions for τ and ε for this region, as $\lambda \rightarrow 0$, match to those, (2.15) and (2.16), for the (near-) downstream region, as $\zeta \rightarrow \infty$. [The determination of the zeroth-order solutions, $u_0(\lambda)$ and $v_0(\lambda)$, requires the consideration of the first-order equations. In the sense that these first-order equations contain convective contributions, this determination of the zeroth-order solutions can be said to involve a diffusion-convection-reaction balance.]

Hence, for this far-downstream region, with $\beta_k = (\beta/\sqrt{2K})$, τ and ε can be expressed as

$$\tau = 1 - \beta_k^{-1} \exp(-\beta_k\lambda)[(G_b^{k_0}/\sqrt{2}) \exp(-\gamma\lambda)][1 + \mathcal{O}(\beta_k^{-1})]; \tag{2.51}$$

$$\varepsilon = 1 - \exp(-\beta_k\lambda)[(G_b^{k_0}/\sqrt{2}) \exp(-\gamma\lambda)][1 + \mathcal{O}(\beta_k^{-1})]. \tag{2.52}$$

The functions τ and ε , of (2.51) and (2.52), respectively, satisfy (1.2b), in that they both go to unity (exponentially) as the hot boundary is approached (i.e., $\lambda \rightarrow \infty$).

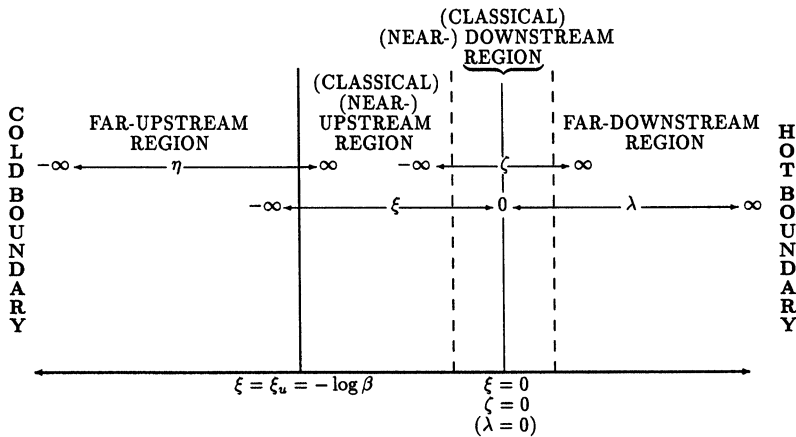


Figure 1 Schematic diagram of the four-region structure of deflagration.

3 RESULTS AND DISCUSSION

The foregoing asymptotic analysis for the generalized reaction-rate model boundary-value problem for the deflagration has revealed that a four-region flame structure is required in order to obtain uniformly valid solutions from the cold boundary to the hot boundary. The details of this structure are shown in Fig. 1.

The (classical) (near-) downstream and (near-) upstream regions, of §2.1 and §2.2, respectively, must be complemented by the far-upstream and far-downstream regions, of §2.3 and §2.4, respectively. The structure of the far-upstream region is dependent on the specific (here, the generalized) reaction-rate model considered; the structure of the far-downstream region is independent of the model considered. For the particular far-upstream region-analysis to hold, in addition to having $\beta \gg O(1)$, it is necessary to have $\log \beta > O(1)$.

It is the analysis of (just) the classical two-region structure (within the four-region structure) that determines the (model-dependent) asymptotic representation for the eigenvalue, Λ . However, it is only by means of an analysis of the four-region structure that it is, in general, possible to incorporate a flame within a complex flow geometry.

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