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On the Solution of the Differential Equation Occurring in the Problem of Heat Convection in Laminar Flow through a Tube with Slip—Flow

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A technique is developed for evaluation of eigenvalues in the solution of the differential equation

$$d^2y/dr^2 + (1/r)dy/dr + \lambda^2(\beta - r^2)y = 0$$

which occurs in the problem of heat convection in laminar flow through a circular tube with slip-flow ($\beta > 1$). A series solution requires the expansions of coefficients involving extremely large numbers. No work has been reported in the case of $\beta > 1$, because of its computational complexity in the evaluation of the eigenvalues. In this paper, a matrix was constructed and a computational algorithm was obtained to calculate the first four eigenvalues. Also, an asymptotic formula was developed to generate the full spectrum of eigenvalues. The computational results for various values of β were obtained.

1. INTRODUCTION

In the solution of the problem of forced convection heat transfer in a circular tube in laminar flow, the following ordinary differential equation

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is obtained after applying the separation-of-variables technique to the governing partial differential equation [1]:

$$d^2y/dr^2 + (1/r)dy/dr + \lambda^2(\beta - r^2)y = 0 \tag{1.1}$$

The interval for the solution of Eq. (1.1) is $0 \leq r \leq 1$, and the boundary conditions are:

$$y(0) = 1 \tag{1.2}$$

$$y(1) = 0 \tag{1.3}$$

The first boundary condition arises from the physical condition that the temperature must be finite at the centerline of the tube, and the second boundary condition arises from the assumed boundary condition that the temperature at the tube-wall is uniform.

The solution of Eq. (1.1) that satisfies the condition (1.2) may be found by the method of Frobenius to be:

$$y(r, \lambda) = \sum_{k=0}^{\infty} a_k r^{2k} \tag{1.4}$$

where the coefficients have the following recursion relationship:

$$4k^2 a_k + \lambda^2[a_{k-1} - a_{k-2}] = 0 \tag{1.5}$$

The first two coefficients have the values:

$$a_0 = 1, \quad a_1 = -\lambda^2\beta/4$$

To satisfy condition (1.3), the parameter λ must take on an infinite set of values (the eigenvalues for the Graetz Problem). The eigenvalues are evaluated from the following relationship:

$$y_n(1, \lambda_n) = \sum_{k=0}^{\infty} a_k(\lambda) = \sum_{k=0}^{\infty} d_k \lambda^{2k} = 1 + d_1\lambda^2 + d_2\lambda^4 + d_3\lambda^6 + \ldots = 0 \tag{1.6}$$

where d_k is the coefficient of λ^{2k} by rearranging the terms of a_k.

One of the procedures used by previous investigators to determine the eigenvalues for the case with $\beta = 1$ was to evaluate the coefficient a_k to some relative large value of k, and then to evaluate the coefficients d_k. The eigenvalues were then found as the solution of the resulting polynomial equation obtained by truncating Eq. (1.6) at the appropriate term. This technique has definite computational disadvantages, particularly in evaluating the eigenvalues beyond the first two or three. The algebraic complexity of the coefficients a_k increases rapidly as k increases. Approximately, 5n terms are needed in Eq. (1.6) to determine the accurate value of λ_n; i.e., 20 terms would be required to determine λ_4. For the case with $\beta = 1$, Graetz found the first two eigenvalues only in 1883[2]. Later, Abramowitz found the first five eigenvalues by using a rapid converging series in 1953[3]. Sellars et al. obtained an asymptotic formula for the larger eigenvalues in 1956[4] by using the WKB approximation [5] and had generated the full spectrum of the eigenvalues.

For the case with $\beta > 1$, no work has been reported. The purpose of this paper is to formulate a computationally effective technique for the calculation of eigenvalues for the case with $\beta > 1$. An asymptotic formula for $\beta > 1$ has been developed for large eigenvalues. Unfortunately, for $\beta > 1$, the asymptotic formula will not give approximations with error less than 1 percent until $n \geq 5$. However, a computational algorithm has been specially developed for evaluating the first four eigenvalues. The combination of the two formulas yields a full spectrum of eigenvalues for $\beta > 1$.

2. ASYMPTOTIC FORMULA FOR LARGE EIGENVALUES

In this section an asymptotic formula for the case with $\beta > 1$ is developed by using the WKB approximation.

Let

$$y = e^{g(r)} \tag{2.1}$$

where $g(r)$ satisfies the following differential equation

$$g'' + g'^2 + \frac{1}{r} g' + \lambda^2(\beta - r^2) = 0 \tag{2.2}$$

An asymptotic solution is sought in the form

$$g = \lambda g_0 + g_1 + \lambda^{-1} g_2 + \ldots + \lambda^{-(n-1)} g_n + \ldots \tag{2.3}$$

So we have

$$g' = \lambda g_0' + g_1' + \lambda^{-1} g_2' + \ldots + \lambda^{-(n-1)} g_n' + \ldots$$

$$g'' = \lambda g_0'' + g_1'' + \lambda^{-1} g_2'' + \ldots + \lambda^{-(n-1)} g_n'' + \ldots$$

Substituting g' and g'' in Eq. (2.3)

$$(\lambda g_0'' + g_1'' + \lambda^{-1} g_2'' + \ldots) + \{[\lambda^2 (g_0')^2 + (g_1')^2 + \lambda^{-2}(g_2')^2 + \ldots]$$
$$+ 2(\lambda g_0' g_1' + g_0' g_2' + \lambda^{-1} g_0' g_3' + \ldots) + \lambda^{-1} g_1' g_2' + \ldots\}$$
$$+ \frac{1}{r} \{\lambda g_0 + g_1 + \lambda^{-1} g_2 + \ldots\} + \lambda^2(\beta - r^2) = 0$$

Comparing the coefficients of the same order of λ, we have

$$\lambda^2: \; g_0' = \pm i \sqrt{\beta - r^2} \tag{2.4}$$

$$\lambda: \quad g_0'' + 2 g_0' g_1' + \frac{1}{r} g_0' = 0$$

therefore,

$$g_1' = -\frac{1}{2} \left(\frac{1}{r} + \frac{g_0''}{g_0'} \right) \tag{2.5}$$

And since

$$g_0'' = (g_0')' = \frac{\mp ir}{\sqrt{\beta - r^2}} = \frac{r}{g_0'}$$

therefore,

$$g_1' = -\frac{1}{2}\left[\frac{1}{r} + \frac{r}{(g_0')^2}\right] \tag{2.6}$$

By integrating Eq. (2.6), we obtain

$$g_1 = -ln\sqrt{g_0' r} \tag{2.7}$$

When λ is large, the remaining terms in Eq. (2.3) are neglected. Substitutions of Eqs. (2.4) and (2.7) in Eq. (2.1) yield

$$\begin{aligned} y &= e^{g(r)} = e^{\lambda g_0 + g_1} \\ &= e^{\lambda(\pm i\int_0^r \sqrt{\beta-\xi^2}d\xi + C) + (-ln\sqrt{g_0' r})} \\ &= e^{\lambda C} e^{\lambda(\pm i\int_0^r \sqrt{\beta-\xi^2}d\xi)} e^{(-ln\sqrt{g_0' r})} \\ &= \frac{Ae^{i\lambda\int_0^r \sqrt{\beta-\xi^2}d\xi} + Be^{-i\lambda\int_0^r \sqrt{\beta-\xi^2}d\xi}}{\sqrt{g_0' r}} \end{aligned}$$

or

$$y = \frac{Ae^{i\lambda\int_0^r \sqrt{\beta-\xi^2}d\xi} + Be^{-i\lambda\int_0^r \sqrt{\beta-\xi^2}d\xi}}{\sqrt{r}\,(\beta-r^2)^{1/4}} \tag{2.8}$$

where A and B are complex constants.

Eq. (2.8) is the WKB approximation and is valid for $0 < r \leq 1$ and large n. Now the coefficients A and B must be determined so that Eq. (2.8) will correspond to the regular Eq. (1.1), where r is small. For small r, Eq. (2.8) can be approximated by

$$R = \frac{Ae^{i\lambda\sqrt{\beta}r} + Be^{-i\lambda\sqrt{\beta}r}}{\sqrt{r}\,(\beta - r^2)^{1/4}} \tag{2.9}$$

When r is small, $\lambda^2(\beta - r^2) \to \lambda^2\beta$, the classical solution to the Eq. (1.1) is given by $J_0(\lambda r\sqrt{\beta})$. On the other hand, for large λr (r is small), the asymptotic expression for $J_0(\lambda r\sqrt{\beta})$ is

$$J_0(\lambda r\sqrt{\beta}) = \sqrt{\frac{2}{\pi\lambda r\sqrt{\beta}}}\, cos(\lambda r\sqrt{\beta} - \frac{\pi}{4}) \tag{2.10}$$

and thus for small r but large λr, asymptotically, we have

$$\sqrt{\frac{2}{\pi\lambda r\sqrt{\beta}}}\, cos(\lambda r\sqrt{\beta} - \frac{\pi}{4}) = \frac{Ae^{i\lambda\sqrt{\beta}r} + Be^{-i\lambda\sqrt{\beta}r}}{\sqrt{r}\,(\beta - r^2)^{1/4}} \tag{2.11}$$

Therefore,

$$A = (\beta - r^2)^{1/4}\sqrt{\frac{2}{\pi\lambda r\sqrt{\beta}}}\, e^{-i\pi/4} \tag{2.12}$$

$$B = (\beta - r^2)^{1/4}\sqrt{\frac{2}{\pi\lambda r\sqrt{\beta}}}\, e^{i\pi/4} \tag{2.13}$$

and for $0 < r \leq 1$

$$y(r) = \sqrt{\frac{2}{\pi\lambda r\sqrt{\beta}}}\, cos(\lambda \int_0^r \sqrt{\beta - \xi^2}\, d\xi - \frac{\pi}{4}) \tag{2.14}$$

The boundary condition $y(1) = 0$ leads to

$$cos(\lambda \int_0^1 \sqrt{\beta - \xi^2}\, d\xi - \frac{\pi}{4}) = 0$$

and thus an asymptotic formula of the eigenvalues for large n is given by

$$\lambda_n(\beta) = \frac{(4n-1)\pi}{4}\,[\int_0^1 \sqrt{\beta - \xi^2}\, d\xi]^{-1} \qquad for\ n \to \infty \tag{2.15}$$

Let

$$I(\beta) = \int_0^1 \sqrt{\beta - \xi^2}\, d\xi \tag{2.16}$$

The numerical result is shown in Table I. This asymptotic formula will not give satisfying approximations until $n \geq 5$. This will be discussed later.

3. COMPUTATIONAL ALGORITHM FOR SMALL EIGENVALUES

In order to use Eq. (1.6) to evaluate the small eigenvalues, a method for the evaluation of the coefficients d_k is developed in this section. This method involves the generation of a matrix and works out for cases $\beta = 1$ and $\beta > 1$ respectively.

(3.1) Expansion of a_k

The coefficients a_k can be determined from Eq. (1.5) for the case with $\beta = 1$ first as follows:

$$a_0 = 1$$

$$a_1 = -\lambda^2/[2^{(2)(1)}(1!)^2]$$

$$a_2 = [\lambda^2 2^2 + \lambda^4]/[2^{(2)(2)}(2!)^2]$$

$$a_3 = -[\lambda^4(2^2 + 4^2) + \lambda^6]/[2^{(2)(3)}(3!)^2$$

$$a_4 = [\lambda^4 2^2 6^2 + \lambda^6(2^2 + 4^2 + 6^2) + \lambda^8]/[2^{(2)(4)}(4!)^2]$$

TABLE I The first five eigenvalue for different β by asymptotic formula Eq. (2.15)

K_n	β	$I(\beta)$	$\lambda_1(\beta)$	$\lambda_2(\beta)$	$\lambda_3(\beta)$	$\lambda_4(\beta)$	$\lambda_5(\beta)$
(0.00)	(1.00)	(0.78521)	(3.001)	(7.002)	(11.003)	(15.004)	(19.005)
0.02	1.08	0.8408	2.802	6.539	10.275	14.012	17.748
0.04	1.16	0.89037	2.646	6.175	9.703	13.232	16.760
0.06	1.24	0.93641	2.516	5.871	9.226	12.581	15.936
0.08	1.32	0.9798	2.405	5.611	8.818	12.024	15.230
0.10	1.40	1.02103	2.308	5.385	8.462	11.538	14.615
0.12	1.48	1.06044	2.222	5.184	8.147	11.110	14.072

$$a_5 = -\{\lambda^6[2^2 6^2 + 8^2(2^2 + 4^2)] + \lambda^8(2^2 + 4^2 + 6^2 + 8^2) + \lambda^{10}\}/[2^{(2)(5)}(5!)^2]$$

$$a_6 = \{\lambda^8 2^2 6^2 10^2 + \lambda^8[2^2 6^2 + 8^2(2^2 + 4^2) + 10^2(2^2 + 4^2 + 6^2)] + \lambda^{10}(2^2 + 4^2 + \ldots + 10^2) + \lambda^{12}\}/[2^{(2)(6)}(6!)^2]$$

$$a_7 = -\{\lambda^8[2^2 6^2 10^2 + 12^2[2^2 6^2 + 8^2(2^2 + 4^2)]] + \lambda^{10}[2^2 6^2 + 8^2(2^2 + 4^2) + 10^2(2^2 + 4^2 + 6^2) + 12^2(2^2 + 4^2 + 6^2 + 8^2)] + \lambda^{12}(2^2 + 4^2 + \ldots + 12^2) + \lambda^{14}\}/[2^{(2)(7)}(7!)^2]$$

$$a_8 = \{\lambda^8 2^2 6^2 10^2 14^2 + \lambda^{10}[2^2 6^2 10^2 + 12^2[2^2 6^2 + 8^2(2^2 + 4^2)] + 14^2[2^2 6^2 + 8^2(2^2 + 4^2) + 10^2(2^2 + 4^2 + 6^2)]] + \lambda^{12}[2^2 6^2 + 8^2(2^2 + 4^2) + 10^2(2^2 + 4^2 + 6^2) + 12^2(2^2 + 4^2 + 6^2 + 8^2) + 14^2(2^2 + 4^2 + \ldots + 10^2)] + \lambda^{14}(2^2 + 4^2 + \ldots + 14^2) + \lambda^{16}\}/[2^{(2)(8)}(8!)^2] \quad (3.1)$$

(3.2) Matrix of *A*

The coefficients in Eqs. (3.1) that multiply λ_n to a given power of $2k$ may be collected and arranged in a matrix form, as shown below.

j	1	2	3	4	5	6	7	...
i	(λ^2)	(λ^4)	(λ^6)	(λ^8)	(λ^{10})	(λ^{12})	(λ^{14})	...
1	$-\frac{1}{4}$	0	0	0	0	0	0	...
2	$\frac{4}{64}$	$\frac{1}{64}$	0	0	0	0	0	...
3	0	$\frac{-20}{2304}$	$\frac{-1}{2304}$	0	0	0	0	...
4	0	$\frac{144}{147456}$	$\frac{56}{147456}$	$\frac{1}{147456}$	0	0	0	...
5	0	0	$\frac{-1424}{14745600}$	$\frac{-120}{14745600}$	$\frac{-1}{14745600}$	0	0	...
6	0	0	$\frac{14400}{2123366400}$	$\frac{7024}{2123366400}$	$\frac{220}{2123366400}$	$\frac{1}{2123366400}$	0	...
7	0	0	0	$\frac{-219456}{416579814400}$	$\frac{-24304}{416579814400}$	$\frac{-364}{416579814400}$	$\frac{-1}{416579814400}$	...
8	0	0	0	$\frac{2822400}{106542032486400}$	$\frac{1596160}{106542032486400}$	$\frac{67424}{106542032486400}$	$\frac{560}{106542032486400}$	
...	...	...	...	...	...	...	...	...

It may be represented by a matrix ***A*** as follows:

$$\mathbf{A}_{ixk} = \begin{bmatrix} a_{1,1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_{2,1} & a_{2,2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{3,2} & a_{3,3} & 0 & 0 & 0 & \cdots \\ 0 & a_{4,2} & a_{4,3} & a_{4,4} & 0 & 0 & \cdots \\ 0 & 0 & a_{5,3} & a_{5,4} & a_{5,5} & 0 & \cdots \\ 0 & 0 & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{3.2}$$

It can be seen that the coefficients along the diagonal of the matrix $\boldsymbol{A}$ are coefficients of Bessel function $J_0(\lambda r)$.

From the matrix and the expansions a_k we observe that

$$a_{1,1} = \frac{(-1)^1}{2^{(2)(1)}(1!)^2}$$

$$a_{2,2} = \frac{(-1)^2}{2^{(2)(2)}(2!)^2}$$

$$a_{2,1} = a_{2,2}\, 2^2 = a_{2,2}\, 2^{(2)(1)} 1^2 = a_{2,2}\, 2^{(2)(1)} \sum_{s_1=1}^{1} (s_1)^2$$

$$a_{3,3} = \frac{(-1)^3}{2^{(2)(3)}(3!)^2}$$

$$a_{3,2} = a_{3,3}\,(2^2 + 4^2) = a_{3,3}\, 2^{(2)(3-2)}(1^2 + 2^2) = a_{3,3}\, 2^{(2)(1)} \sum_{s_1=1}^{2} (s_1)^2$$

$$a_{4,4} = \frac{(-1)^4}{2^{(2)(4)}(4!)^2}$$

$$a_{4,3} = a_{4,4}\,(2^2 + 4^2 + 6^2) = a_{4,4}\, 2^{(2)(4-3)}(1^2 + 2^2 + 3^2)$$

$$= a_{4,4}\, 2^{(2)(1)}(1^2 + 2^2 + 3^2) = a_{4,4}\, 2^{(2)(1)} \sum_{s_1=1}^{3} (s_1)^2$$

$$a_{4,2} = a_{4,4}\,(2^2 6^2) = a_{4,4}\,2^{(2)(4-2)}(3^2 1^2)$$

$$= a_{4,4}\,2^{(2)(2)}\,[\sum_{s_2=1}^{2}(s_2+1)^2[\sum_{s_1=1}^{s_2-1}(s_1)^2]]$$

where we define the summation for upper limit of zero as follows:

$$\sum_{s_1=1}^{0}(s_1)^2 = 0$$

From these observations we can deduce a formulation to calculate $a_{i,k}$ directly related to the index without the expansions of each term a_k. Thus, the diagonal elements are given by:

$$a_{i,i} = \frac{(-1)^i}{2^{2i}(i!)^2} \tag{3.3}$$

and for $i < k$

$$a_{i,k} = a_{i,i}\,2^{2\Delta}\sum_{s_\Delta=\Delta}^{k}(s_\Delta+\Delta-1)^2[\sum_{s_{\Delta-1}=\Delta-1}^{s_\Delta-1}(s_{\Delta-1}+\Delta-2)^2$$

$$[...[\sum_{s_2=2}^{s_3-1}(s_2+1)^2[\sum_{s_1=1}^{s_2-1}(s_1{}^2]]...]] \tag{3.4}$$

where $\Delta = i - k$, and then the coefficients in eigenfunction can be written as

$$d_k = \sum_{i=k}^{2k} a_{i,k} \tag{3.5}$$

Example 1.

$$a_{5,3} = a_{5,5}\,2^{2(5-3)}[\sum_{s_2=2}^{3}(s_2+1)^2[\sum_{s_1=1}^{s_2-1}s_1{}^2]]$$

$$= \frac{(-1)^5}{2^{2(5)}(5!)^2}\,2^4[3^2\sum_{s_1=1}^{2-1}s_1{}^2 + 4^2\sum_{s_1=1}^{3-1}s_1{}^2]$$

$$= \frac{-1}{14{,}745{,}600} 2^4[3^2 1^2 + 4^2(1^2 + 2^2)]$$

$$= \frac{-1{,}424}{14{,}745{,}600}$$

Example 2.

$$a_{8,5} = a_{8,8}\, 2^{2(8-5)} \sum_{s_3=2}^{5} (s_3 + 2)^2 [\sum_{s_2=2}^{s_3-1} (s_2 + 1)^2 [\sum_{s_1=1}^{s_2-1} s_1{}^2]]$$

$$= \frac{(-1)^8}{2^{2(8)}(8!)^2} 2^6[5^2 \sum_{s_2=2}^{3-1} (s_2 + 1)^2 [\sum_{s_1=1}^{s_2-1} s_1{}^2] + 6^2 \sum_{s_2=2}^{4-1} (s_2 + 1)^2 [\sum_{s_1=1}^{s_2-1} s_1{}^2]$$

$$+ 7^2 \sum_{s_2=2}^{5-1} (s_2 + 1)^2 [\sum_{s_1=1}^{s_2-1} s_1{}^2]]$$

$$= \frac{1}{2^{10}(8!)^2} 2^6\{5^2 3^2 1^2 + 6^2[3^2 1^2 + 4^2(1^2 + 2^2)] + 7^2[3^2 1^2 + 4^2(1^2$$

$$+ 2^2)] + 5^2(1^2 + 2^2 + 3^2)]\}$$

$$= \frac{1{,}596{,}160}{106{,}542{,}032{,}486{,}400}$$

These two examples illustrate that the formulation (3.4) is effective and correct.

For the case with $\beta > 1$ the corresponding expansions of a_k are as follows:

$$a_0 = 1$$

$$a_1 = -\lambda^2\beta/[2^{(2)(1)}(1!)^2]$$

$$a_2 = [\lambda^2 2^2 + \lambda^4\beta^2]/[2^{(2)(2)}(2!)^2]$$

$$a_3 = -[\lambda^4\beta(2^2 + 4^2) + \lambda^6\beta^3]/[2^{(2)(3)}(3!)^2$$

$$a_4 = [\lambda^4 2^2 6^2 + \lambda^6\beta^2(2^2 + 4^2 + 6^2) + \lambda^8\beta^4]/[2^{(2)(4)}(4!)^2]$$

$$a_5 = -\{\lambda^6\beta[2^2 6^2 + 8^2(2^2 + 4^2)] + \lambda^8\beta^3(2^2 + 4^2 + 6^2 + 8^2) + \lambda^{10}\beta^5\}/[2^{(2)(5)}(5!)^2]$$

$$a_6 = \{\lambda^6 2^2 6^2 10^2 + \lambda^8\beta^2[2^2 6^2 + 8^2(2^2 + 4^2) + 10^2(2^2 + 4^2 + 6^2)] + \lambda^{10}\beta^4(2^2 + 4^2 + \ldots + 10^2) + \lambda^{12}\beta^6\}/[2^{(2)(6)}(6!)^2]$$

$$a_7 = -\{\lambda^8\beta[2^2 6^2 10^2 + 12^2[2^2 6^2 + 8^2(2^2 + 4^2)] + \lambda^{10}\beta^3[2^2 6^2 + 8^2(2^2 + 4^2) + 10^2(2^2 + 4^2 + 6^2) + 12^2(2^2 + 4^2 + 6^2 + 8^2)] + \lambda^{12}\beta^5(2^2 + 4^2 + \ldots + 12^2) + \lambda^{14}\beta^7\}/[2^{(2)(7)}(7!)^2]$$

$$a_8 = \{\lambda^8 2^2 6^2 10^2 14^2 + \lambda^{10}\beta^2[2^2 6^2 10^2 + 12^2[2^2 6^2 + 8^2(2^2 + 4^2)] + 14^2[2^2 6^2 + 8^2(2^2 + 4^2) + 10^2(2^2 + 4^2 + 6^2)]] + \lambda^{12}\beta^4[2^2 6^2 + 8^2(2^2 + 4^2) + 10^2(2^2 + 4^2 + 6^2) + 12^2(2^2 + 4^2 + 6^2 + 8^2) + 14^2(2^2 + 4^2 + \ldots + 10^2)] + \lambda^{14}\beta^6(2^2 + 4^2 + \ldots + 14^2) + \lambda^{16}\beta^8\}/[2^{(2)(8)}(8!)^2] \qquad (3.1)'$$

rewritten in matrix form as

$$\mathbf{A}_{ixk} = \begin{bmatrix} a_{1,1}\beta & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_{2,1} & a_{2,2}\beta^2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{3,2}\beta & a_{3,3}\beta^3 & 0 & 0 & 0 & \cdots \\ 0 & a_{4,2} & a_{4,3}\beta^2 & a_{4,4}\beta^4 & 0 & 0 & \cdots \\ 0 & 0 & a_{5,3}\beta & a_{5,4}\beta^3 & a_{5,5}\beta^5 & 0 & \cdots \\ 0 & 0 & a_{6,3} & a_{6,4}\beta^2 & a_{6,5}\beta^4 & a_{6,6}\beta^6 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \qquad (3.2)'$$

Let

$$a_{i,k}' = a_{i,k}\beta^{i-2\Delta} \qquad (3.6)$$

then

$$d_k' = \sum_{i=k}^{2k} a_{i,k}' \tag{3.7}$$

4. NUMERICAL RESULTS

The asymptotic formula and computational algorithm presented in the previous sections have been applied numerically to the evaluation of the eigenvalues of the Graetz Problem in slip-flow.

(4.1) The first four eigenvalues calculated by the computational algorithm

Since the sensitive coefficients $a_{i,k}$ in Eq. (3.2) are becoming very small for large k because of the factorial $(k!)^2$ in the denominators, but they are essential for evaluating the eigenvalues, a scaling factor g was introduced in the calculation, that is,

$$\begin{aligned} y(\lambda') &= \sum_{k=0}^{\infty} \lambda'^{2k} d'_k = 1 + \lambda'^2 d'_1 + \lambda'^4 d'_2 + \lambda'^6 d'_3 + \ldots \\ &= \sum_{k=0}^{\infty} (\lambda' g)^{2k} \frac{d'_k}{g^{2k}} = 1 + (\lambda' g)^2 \frac{d'_1}{g^2} + (\lambda' g)^4 \frac{d'^2}{g^4} + (\lambda' g)^6 \frac{d'_3}{g^6} + \ldots \\ &= \sum_{k=0}^{\infty} \lambda^{2k} d_k = 1 + \lambda^2 d_1 + \lambda^4 d_2 + \lambda^6 d_3 + \ldots = 0 \end{aligned} \tag{4.1}$$

Thus $\lambda = \lambda' g$ and $d'k = g^{2k} d_k$.

In order to assess the accuracy of the computation of Eq. (3.4), a comparison of the first ten d_k using Mathcad 5.0 was made, as shown in Table II. In the table, the data in the second column are the computational coefficients d_k (with $g = 10$) using Eq. (3.4); the data in the fourth column are the exact coefficients d_k calculated by expansions using the symbolic processor in Mathcad 5.0; the third column gives the equivalent decimal values of the fourth column.

From the comparison we can see that the coefficient d_k computed by Eq. (3.4) are always a little larger than that by Mathcad but the differences are rather small. Based on the comparison we are convinced that the compu-

TABLE II Comparison of Coefficients d_k ($g = 10$)

k *Eq. (3.4)*	*Simulation by Mathcad Version 5.0*	
	Numeric equivalent	*Symbolic solution*
0 1.0000	1.00000	1
1 −18.7500	−18.750000 E-02	−3/16
2 79.2101	79.210069 E-04	73/9,216
3 −144.043	−144.04297 E-06	−59/409,600
4 145.080	145.07980 E-08	603,793/416,197,814,400
5 −92.6715	−92.67144 E-10	−555,379/59,929,893,273,600
6 40.8619	40.861856 E-12	4,266,870,481/104,421,846,039,920,640,000
7 −13.1812	−13.181156 E-14	−37,217,872,147/282,356,671,691,945,410,560,000
8 3.24941	3.2451315 E-16	41,377,942,693,441/127,507,755,229,335,476,282,327,040,000
9 −0.646315	−0.62971547 E-18	−9,281,940,782,645,851/14,739,896,504,511,181,058,237,005,824E+6

tational results should have enough accuracy for accurate determination of the eigenvalues for the first four eigenvalues.

A comparison of the eigenvalues for the Graetz Problem ($\beta = 1$)[3] obtained in this study with those obtained previously by other researchers is given in Table III. The agreement between the first three values in this study and those obtained by Abramowitz is excellent.

Figure 1 shows the plot of eigenfunction with 25 terms. There are six eigenvalues shown in the plot. The first four ones are correct, but the last two is somewhat inaccurate due to the truncation of the eigenfunction expression. Figure 2 shows the behavior of the eigenfunction vs. the number of coefficient terms. The eigenvalues and the convergency of the eigenfunction are sensitive to the accuracy of the coefficients d_k.

Figure 3 shows the plots of eigenfunction vs. various β (or Knudsen number) for slip-flow. It shows that the eigenvalues decrease as β increases. For $\beta > 1$ the plots appear unstable after the fifth root so that only

TABLE III Comparison with previously known eigenvalues for $\beta = 1$

	Sellar et al. [4]	*Jakob [6]*	*Abramowitz [3]*	*Present paper*
n	λ_n	λ_n	λ_n	λ_n
1	2.667	2.705	2.7043644	2.704
2	6.667	6.66	6.679032	6.679
3	10.667	10.3	10.67338	10.670
4	14.667	14.67[a]	14.67108	14.761
5	18.667		18.66987	17.255

[a]Attributed to Lee, Cherry, and Boelte.

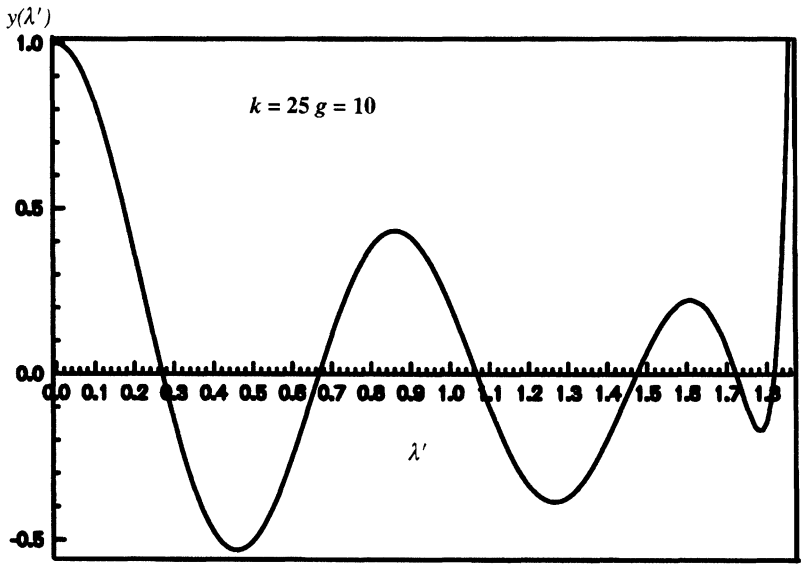

FIGURE 1 The eigenfunction with 25 coefficient terms for $\beta = 1$, $k = 25$, and $g = 10$.

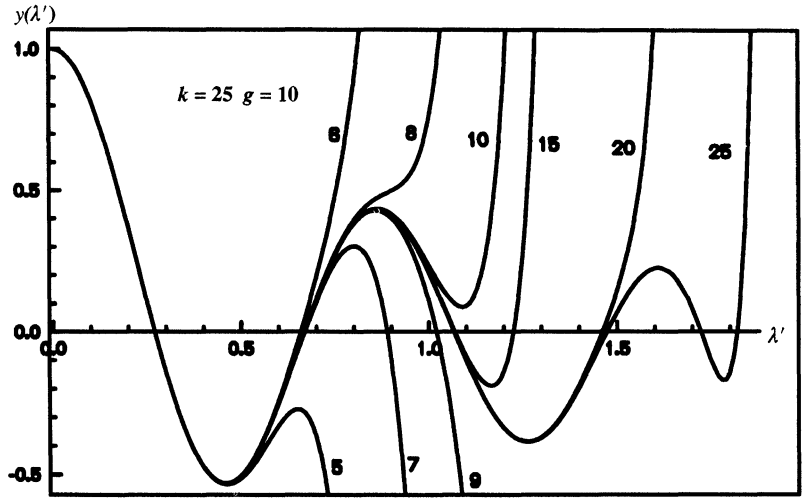

FIGURE 2 Behavior of the eigenfunction vs. the number of coefficient terms for $\beta = 1$, $k = 25$, and $g = 10$.

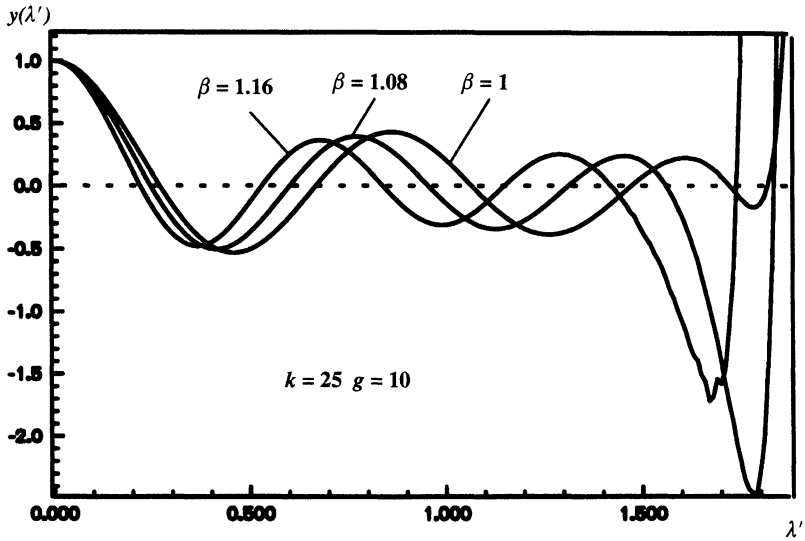

FIGURE 3 The effect of β on the eigenfunction.

the first four values are reliable. The possible cause for the instability is that the truncation errors are magnified by the factor β^i on $a_{i,i}$ in the modified matrix $\boldsymbol{A}$. Values of the coefficients d'_k are listed in Appendix for β from 1 to 1.36 for k ranging from zero to 25, with a scaling factor of $g = 10$. Table IV shows the eigenvalues for slip-flow with different β for the first five values.

Figure 4 shows the plot of eigenfunction (1.4) vs. eigenvalues with β = 1; Figure 5 shows the case with β = 1.08.

TABLE IV Eigenvalues for various β

β	λ_1	λ_2	λ_3	λ_4	λ_5
1.00	2.704	6.679	10.670	14.761	17.255
1.08	2.578	6.320	10.071	13.851	16.576
1.16	2.468	6.013	9.561	13.099	15.836
1.24	2.371	5.747	9.120	12.560	14.646
1.32	2.284	5.513	8.737	11.963	14.573
1.40	2.206	5.305	8.396	11.514	13.938
1.48	2.136	5.119	8.096	11.074	14.273

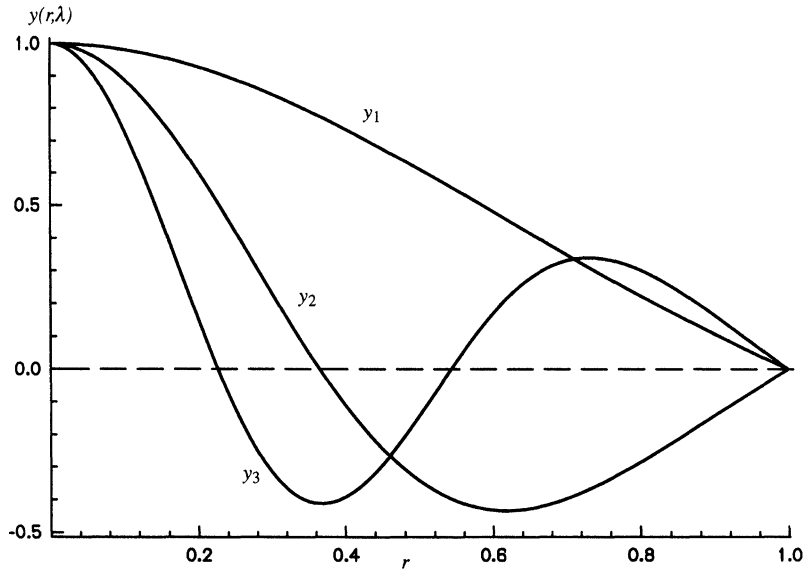

FIGURE 4 Eigenfunction (1.4) with $\beta = 1$ for $n = 4$.

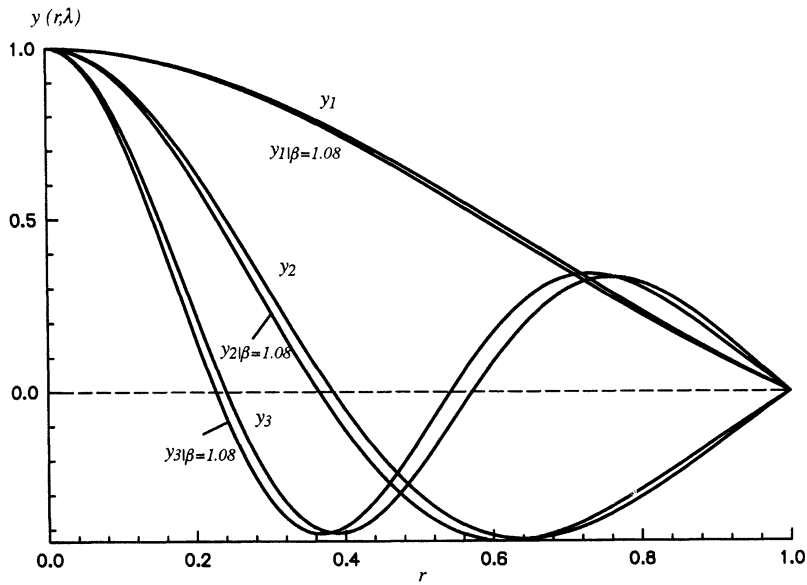

FIGURE 5 Eigenfunction (1.4) with $\beta = 1$ and 1.08 for $n = 4$.

(4.2) Comparison of the eigenvalues by the asymptotic formula and the computational algorithm

Table V shows the differences between the eigenvalues by Eq. (2.15) and by Eq. (3.4). It shows that for the second values the maximum difference is 3.5 %, for the third values the maximum difference is 2.0 % and for the fourth values the maximum difference is only 1.4 %. That means the maximum differences of the eigenvalues for $n \geq 5$ may be less than 1 %.

5. CONCLUSION

This paper presents a computational technique for evaluation of the eigenvalues of the Graetz Problem in slip-flow, which arises in convection heat transfer in laminar flow in a circular tube. Prior to this work, because of the computational complexity in the techniques used to evaluate the eigenvalues, the previous investigators have discussed the eigenvalues for the case with $\beta = 1$ only. For the case with $\beta > 1$, no work was reported. The formulation and asymptotic formula presented in this work yield a technique that is both computationally effective and relatively simple to apply. Values for the first four eigenvalues for various values of β (> 1) can be calculated by using the computational algorithm and the approximate eigenvalues for $n \geq 5$ can be calculated by using the asymptotic formula.

Acknowledgement

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TABLE V Differences between the eigenvalues by Eq. (2.15) and by Eq. (3.4)

Kn	β	$\Delta_1(\beta)\%$	$\Delta_2(\beta)\%$	$\Delta_3(\beta)\%$	$\Delta_4(\beta)\%$
(0.00)	(1.00)	(11.0)	(4.8)	(3.1)	(1.6)
0.02	1.08	8.7	3.5	2.0	1.4
0.04	1.16	7.2	2.7	1.5	1.0
0.06	1.24	6.1	2.2	1.2	0.2
0.08	1.32	5.3	1.8	0.9	0.5
0.10	1.40	4.6	1.5	0.8	0.2
0.12	1.48	4.0	1.3	0.6	0.3

*$\Delta_n(\beta) = [\lambda_n(\beta) - \lambda_n]/\lambda_n$

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APPENDIX

COEFFICIENT D_K' OF EIGENFUNCTION FOR DIFFERENT β

k	*1.00*	*1.08*	*1.16*	*1.24*	*1.32*
0	1.0000	1.0000	1.0000	1.0000	1.0000
1	−18.7500	−20.7500	−22.7500	−24.7500	−26.7500
2	79.2101	98.2656	119.321	142.377	167.432
3	−144.043	−201.296	−271.689	−356.553	−457.224
4	145.080	229.024	344.257	497.614	696.594
5	−92.6715	−165.560	−277.502	−442.200	−676.202
6	40.8619	82.7280	54.757	272.024	454.598
7	−13.1812	−30.2737	−63.2459	−122.683	−224.130
8	3.24941	8.46654	19.7575	42.3029	84.5014
9	−0.646315	−1.88363	−4.88729	−11.5310	−25.1684
10	0.124692	0.363041	1.00579	2.57510	6.10308
11	−3.58121E-02	−7.81416E-02	−0.196144	−0.505606	−1.25918
12	1.48968E-02	2.45660E-02	4.68932E-02	1.02985E-01	0.243432
13	−5.81835E-03	−9.07529E-03	−1.47620E-02	−2.62552E-02	−5.21826E-02
14	1.82761E-03	2.97430E-03	4.78575E-03	7.83189E-03	1.34943E-02
15	−4.58083E-04	−7.98782E-04	−1.34809E-03	−2.23449E-03	−3.71120E-03
16	9.33666E-05	1.75660E-04	3.16460E-04	5.51241E-04	9.39192E-04
17	−1.57875E-05	−3.21078E-05	−6.20691E-05	−1.15009E-04	−2.05969E-04
18	2.25303E-06	4.95574E-06	1.02964E-05	2.03759E-05	3.86913E-05
19	−2.75302E-07	−6.55009E-07	−1.46339E-06	−3.09708E-06	−6.25517E-06
20	2.91544E-08	7.50298E-08	1.80276E-07	4.08225E-07	8.77967E-07
21	−2.70262E-09	−7.52384E-09	−1.94436E-08	−4.71180E-08	−1.07959E-07
22	2.22204E-10	6.68238E-10	1.85579E-09	4.81032E-09	1.17395E-08
23	−1.57586E-11	−5.16599E-11	−1.55104E-10	−4.31820E-10	−1.12584E-09
24	1.17557E-12	4.04916E-12	1.28004E-11	3.75388E-11	1.03039E-10

COEFFICIENT D_K' OF EIGENFUNCTION FOR DIFFERENT β

k	*1.00*	*1.08*	*1.16*	*1.24*	*1.32*
25	−3.96765E-14	−1.63832E-13	−6.01195E-13	−2.00029E-12	−6.12443E-12