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Nonlinear Unsteady Supersonic Flow Analysis for Slender Bodies of Revolution: Series Solutions, Convergence and Results

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In Ref. [6] the authors constructed analytical solutions including one arbitrary function for the problem of nonlinear, unsteady, supersonic flow analysis concerning slender bodies of revolution due to small amplitude oscillations. An application describing a flow past a right circular cone was presented and the constructed solutions were given in the form of infinite series through a set of convenient boundary and initial conditions in accordance with the physical problem. In the present paper we develop an appropriate convergence analysis concerning the before mentioned series solutions for the specific geometry of a rigid right circular cone. We succeed in estimating the limiting values of the series producing velocity and acceleration resultants of the problem under consideration. Several graphics for the velocity and acceleration flow fields are presented. We must underline here that the proposed convergence technique is unique and can be applied to any other geometry of the considered body of revolution.

Keywords: Unsteady supersonic flow; Convergence analysis; Right circular cone

Classification Categories: 76J99, 65M99, 65N20, 65N25

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MAIN NOTATION

(x, r, θ) = nondimensional cylindrical co-ordinates, normalized by the true body length;
U = freestream velocity;
T = true time;
$t = TU/L$, nondimensional time;
ω = true angular frequency;
$k = \omega L/U$, reduced frequency;
Ω, Φ = total and perturbed potentials, both normalized by $(UL)^{-1}$;
ε = maximum body radius/L, the body thickness ratio;
M = freestream Mach number;
γ = ratio of specific heats ($= 1.40$);
a = distance of the nonlifting position, normalized by L^{-1};
δ = oscillation amplitude;
u, v, w = nondimensional velocities, normalized by the freestream velocity U;
$\mathrm{d}u/\mathrm{d}t$; $\mathrm{d}v/\mathrm{d}t$; $\mathrm{d}w/\mathrm{d}t$ = nondimensional accelerations.

1 INTRODUCTION

In Ref. [6] (Part I) the authors constructed analytical solutions for the velocity potential of the problem of nonlinear supersonic flow past slender bodies of revolution. The bodies undergo small amplitude pitching oscillations around their nonlifting positions. The analysis was based on the small perturbation theory related with the unsteady transonic nonlinear PDE, as well as on the possibility of splitting this equation in a pure steady and a pure unsteady part. Several approximate methods and techniques were developed in constructing analytical solutions of the resulting new equations. We mention the well-known "parabolic method" [4,5], as well as the "local linearization method" [10], which succeeded in giving analytical solutions for convenient approximate forms of the above differential equations. Also, the "integral method" was often combined with the before mentioned techniques leading to satisfactory solutions of the problem under consideration [2,4,5,8,9]. Especially in Ref. [6], by means of the Monge method in combination with a convenient "ad hoc" assumption related to the steady nonlinear PDE, as well as by the separation of variables technique concerning the unsteady PDE, we succeeded in obtaining analytical solutions of the prescribed problem for Mach number $M_\infty > 1$ (supersonic flow). These solutions were expressed including one arbitrary function and several infinite series. The velocity potential was analytically defined for a specific geometry of the body (right circular cone).

In the present paper we continue the research of Ref. [6] firstly by presenting the analytical solutions of the velocity and acceleration unsteady supersonic flow fields related to a right circular cone performing small amplitude oscillations around its nonlifting position. Since the above solutions are expressed in the form of infinite series of various arguments, an appropriate convergence analysis is developed in order to evaluate the limiting values of the series. Thus, we focus on the "*destabilization*" points and the corresponding "*destabilization*" terms of the series expressions defining simultaneously the "*convergence*" terms included in these formulae. In the sequel, through a numerical analysis of some "*ideal*" sequences being constructed by the before mentioned terms, we finally prove the convergence of the original series. As an application we present several graphics concerning the velocity and acceleration flow fields for various values of the parameters being introduced in the solution of the problem under consideration. We must underline here that the developed convergence technique is unique and can be applied to any other geometry of the considered body of revolution.

2 MATHEMATICAL FORMULATION

Consider a rigid pointed body of revolution exposed to a steady uniform transonic flow U. The body performs harmonic, small amplitude pitching oscillations around its nonlifting position M, while it is assumed to be smooth and sufficiently slender so that the small disturbance concept can be applied. For the description of the problem we consider a space-fixed cartesian co-ordinate system

$M(x_s, y_s, z)$, the axis x_s of which coincides with the central axis of the body in its steady position (Fig. 1). The small amplitude oscillations occur in the x_s, y_s-plane and so a body-fixed cartesian co-ordinate system $M(x, y, z)$ is necessary to be introduced for our analysis. Finally, we consider a body-fixed cylindrical co-ordinate system (x, r, θ), where r is parallel to the yz-plane as Fig. 1 shows. The total velocity potential $\Omega(x, r, \theta, t)$ can be related to a perturbed velocity potential $\Phi(x, r, \theta, t)$ by the equation [7]

$$\Omega(x, r, \theta, t) = x\cos\delta + r\sin\delta\cos\theta + \Phi(x, r, \theta, t), \tag{2.1}$$

in which $\delta = \delta_0 \exp(ikt)$; δ_0 is the oscillation amplitude and k denotes the reduced frequency of the pitching motion. Using Eq. (2.1) one extracts the cylindrical velocity components [6] by the relations

$$V\,|_{\text{cylindrical}} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \Omega_x \\ \Omega_r \\ (1/r)\Omega_\theta \end{pmatrix} \cong \begin{pmatrix} 1 + \Phi_x \\ \delta_0 \cos kt \cos\theta + \Phi_r \\ -\delta_0 \cos kt \sin\theta + (1/r)\Phi_\theta \end{pmatrix}, \tag{2.2}$$

in which we have already introduced the expression for δ, while considering small amplitude oscillations ($\delta_0 \ll 1$) we have already

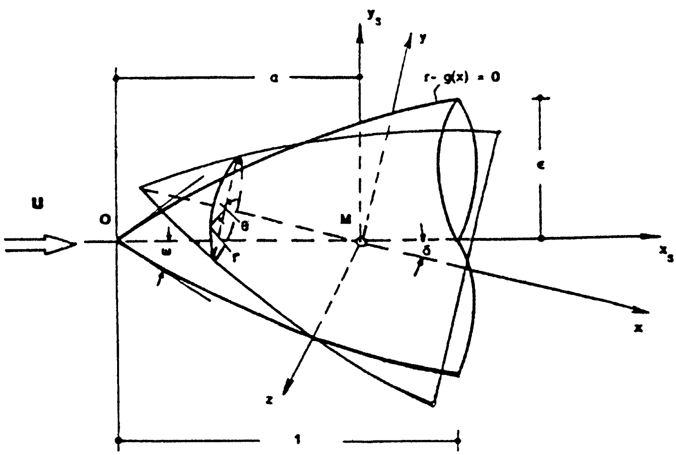

FIGURE 1 Geometry and sign convention of a rigid body of revolution under uniform flow.

retained only the real part of this expression. Also, subscript index denotes partial differentiation with respect to this index. Taking into account that the perturbed potential Φ satisfies the dimensionless equation [3]

$$\beta\Phi_{xx} + \Phi_{rr} + \frac{1}{r^2}\Phi_{\theta\theta} - M^2\Phi_{tt} - 2M^2\Phi_{xt} + \frac{1}{r}\Phi_r - (\gamma+1)M^2\Phi_x\Phi_{xx} = 0, \tag{2.3}$$

one writes Φ as the sum of the following two terms [5]

$$\Phi(x, r, \theta, t) = \varphi(x, r) + \psi(x, r, \theta, t). \tag{2.4}$$

In Eqs. (2.3) and (2.4) M denotes the freestream Mach number, while $\beta = 1 - M^2$; γ represents the ratio of the specific heats which is often taken equal to 1.40, while φ is the steady axisymmetric potential and ψ the oscillatory potential. Conveniently differentiating Eq. (2.4) and introducing the results into (2.3), we obtain

$$\begin{aligned}&\beta(\varphi+\psi)_{xx} + (\varphi+\psi)_{rr} + \frac{1}{r^2}\psi_{\theta\theta} - M^2\psi_{tt} - 2M^2\psi_{xt} + \frac{1}{r}(\varphi+\psi)_r \\ &\quad - (\gamma+1)M^2\varphi_x\varphi_{xx} - (\gamma+1)M^2(\varphi_x\psi_{xx} + \varphi_{xx}\psi_x) \\ &\quad - (\gamma+1)M^2\psi_x\psi_{xx} = 0.\end{aligned} \tag{2.5}$$

Because of the small amplitude oscillations one may neglect the term $(\gamma+1)M^2\psi_x\psi_{xx}$ in (2.5) and thus, one may split the resulting new equation into a nonlinear PDE corresponding to the steady flow and into one linear PDE corresponding to the unsteady flow, namely

$$\beta\varphi_{xx} + \varphi_{rr} + \frac{1}{r}\varphi_r = (\gamma+1)M^2\varphi_x\varphi_{xx}; \tag{2.6a}$$

$$\begin{aligned}&\beta\psi_{xx} + \psi_{rr} + \frac{1}{r^2}\psi_{\theta\theta} - M^2\psi_{tt} - 2M^2\psi_{xt} + \frac{1}{r}\psi_r \\ &\quad = (\gamma+1)M^2(\varphi_{xx}\psi_x + \varphi_x\psi_{xx}).\end{aligned} \tag{2.6b}$$

In Ref. [6] (Part I – Theory) a solution technique was developed in constructing analytical solutions for both previous PDEs. The obtained expressions for the potentials φ and ψ also include a one time–distance dependent arbitrary function concerning supersonic flow. The use of convenient boundary and initial conditions resulted in giving the following expression for the total perturbed potential $\Phi(x, r, \theta, t)$

$$\Phi(x, r, \theta, t) = \frac{1}{(\gamma+1)M^2}x + A(\varepsilon)\ell nr + B + \delta_0\varepsilon(x - t + a)$$
$$\times \sum_{i=1}^{\infty}\left[\frac{1}{(1 + (\mu_i/\varepsilon))^2}\frac{1}{H(\xi_i)}J_1\left(\frac{\mu_i}{\varepsilon}r\right)\cos\left(\frac{\mu_i}{\varepsilon M}t\right)\right]\cos\theta, \tag{2.7}$$

in which

$$\xi_i = \mu_i(x - t + a); \tag{2.8a}$$

$$H(\xi_i) = J_1(\xi_i) - \xi_i J_0(\xi_i) \tag{2.8b}$$

and

$$A(\varepsilon) = \varepsilon\left(1 + \frac{1}{(\gamma+1)M^2}\right)\exp\left[-\frac{1}{\varepsilon^2[1 + (\gamma+1)M^2]}\right]. \tag{2.8c}$$

In these equations μ_i $(i = 1, 2, \ldots)$ represent an infinite number of positive eigenvalues of the transcendental equation

$$\left(\frac{3}{\mu^2} - 1\right)\left(1 + \frac{J_1(\mu)}{H(\mu)}\right)J_1(\mu) + 2\Lambda(\mu)\left(1 - \frac{\mu^2 J_1(\mu)\Lambda(\mu)}{H^2(\mu)}\right) = 0, \tag{2.9a}$$

in which

$$H(\mu) = J_1(\mu) - \mu J_0(\mu); \quad \Lambda(\mu) = \left(\frac{2}{\mu^2} - 1\right)J_1(\mu) - \frac{1}{\mu}J_0(\mu). \tag{2.9b}$$

The already introduced functions J_1 and J_0 are the Bessel functions of the first kind of integer order. Finally, ε is the maximum thickness of the slender body of revolution, while a is the distance of the nonlifting position M from the origin O (Fig. 1). By now, making use of the well-known recurrence formulae of Bessel's functions [1], we obtain the relations

$$\dot{J}_1(z) = -\frac{H(z)}{z}; \quad \dot{H}(z) = -\frac{(1-z^2)J_1(z)}{z}, \tag{2.10}$$

where dot means differentiation with respect to z. Thus, partially differentiating relation (2.7) with respect to $x; r; \theta$, using Eqs. (2.10), and introducing the results into the matrix equation (2.2), we deduce the following analytical expressions for the dimensionless velocity components u, v and w:

$$\begin{aligned}
u &= 1 + \frac{1}{(\gamma+1)M^2} \\
&\quad + \delta_0 \varepsilon \sum_{i=1}^{\infty} \left[\frac{1}{(1+(\mu_i/\varepsilon))^2} \frac{1}{H(\xi_i)} \left(1 + \frac{(1-\xi_i^2)J_1(\xi_i)}{H(\xi_i)}\right) J_1\left(\frac{\mu_i}{\varepsilon} r\right) \cos\left(\frac{\mu_i}{\varepsilon M} t\right) \right] \cos\theta; \\
v &= \delta_0 \cos kt \cos\theta + \frac{A(\varepsilon)}{r} \\
&\quad - \frac{\delta_0 \varepsilon}{r}(x - t + a) \sum_{i=1}^{\infty} \left[\frac{1}{(1+(\mu_i/\varepsilon))^2} \frac{1}{H(\xi_i)} H\left(\frac{\mu_i}{\varepsilon} r\right) \cos\left(\frac{\mu_i}{\varepsilon M} t\right) \right] \cos\theta; \\
w &= -\delta_0 \cos kt \sin\theta \\
&\quad - \frac{\delta_0 \varepsilon}{r}(x - t + a) \sum_{i=1}^{\infty} \left[\frac{1}{(1+(\mu_i/\varepsilon))^2} \frac{1}{H(\xi_i)} J_1\left(\frac{\mu_i}{\varepsilon} r\right) \cos\left(\frac{\mu_i}{\varepsilon M} t\right) \right] \sin\theta.
\end{aligned} \tag{2.11}$$

Furthermore, differentiating (2.11) with respect to time t and using Eqs. (2.10), one derives analytically the dimensionless acceleration components $\mathrm{d}u/\mathrm{d}t$; $\mathrm{d}v/\mathrm{d}t$; and $\mathrm{d}w/\mathrm{d}t$, that is

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{\delta_0 \varepsilon}{x - t + a} \sum_{i=1}^{\infty} \frac{1}{(1+(\mu_i/\varepsilon))^2} J_1\left(\frac{\mu_i}{\varepsilon} r\right) \\
&\quad \times \left[\left[\frac{(\xi_i^2 - 1)J_1(\xi_i)}{H^2(\xi_i)} \left(1 + \frac{(1-\xi_i^2)J_1(\xi_i)}{H(\xi_i)}\right) \right.\right. \\
&\quad \left. + \frac{1-\xi_i^2}{H(\xi_i)} + \frac{2\xi_i^2 J_1(\xi_i)}{H^2(\xi_i)} - \frac{(\xi_i^2-1)^2 J_1^2(\xi_i)}{H^3(\xi_i)} \right] \\
&\quad \left. \times \cos\left(\frac{\mu_i}{\varepsilon M} t\right) - \frac{\xi_i}{\varepsilon M} \frac{1}{H(\xi_i)} \left(1 + \frac{(1-\xi_i^2)J_1(\xi_i)}{H(\xi_i)}\right) \sin\left(\frac{\mu_i}{\varepsilon M} t\right) \right];
\end{aligned}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\delta_0 k \sin kt \cos\theta + \frac{\delta_0\varepsilon}{r}\sum_{i=1}^{\infty}\frac{1}{(1+(\mu_i/\varepsilon))^2} H\left(\frac{\mu_i}{\varepsilon}r\right)$$
$$\times\left[\frac{1}{H(\xi_i)}\left(1+\frac{(1-\xi_i^2)J_1(\xi_i)}{H(\xi_i)}\right)\cos\left(\frac{\mu_i}{\varepsilon M}t\right)\right.$$
$$\left.+\frac{\xi_i}{\varepsilon M}\frac{1}{H(\xi_i)}\sin\left(\frac{\mu_i}{\varepsilon M}t\right)\right]\cos\theta;$$
$$\frac{\mathrm{d}w}{\mathrm{d}t} = \delta_0 k \sin kt \sin\theta + \frac{\delta_0\varepsilon}{r}\sum_{i=1}^{\infty}\frac{1}{(1+(\mu_i/\varepsilon))^2} J_1\left(\frac{\mu_i}{\varepsilon}r\right)$$
$$\times\left[\frac{1}{H(\xi_i)}\left(1+\frac{(1-\xi_i^2)J_1(\xi_i)}{H(\xi_i)}\right)\cos\left(\frac{\mu_i}{\varepsilon M}t\right)\right.$$
$$\left.+\frac{\xi_i}{\varepsilon M}\frac{1}{H(\xi_i)}\sin\left(\frac{\mu_i}{\varepsilon M}t\right)\right]\sin\theta. \qquad (2.12)$$

3 CONVERGENCE ANALYSIS

3.1 Relative Order between Two Sequences – "Destabilization" Points

If ξ_i becomes equal to a root of the equation

$$H(x) = J_1(x) - xJ_0(x) = 0, \qquad (3.1)$$

then the series included in the expressions (2.11) and (2.12) diverge. Furthermore, in case ξ_i lies inside a small area of a root of the above equation, then $H(\xi_i)$ becomes very small and the series in (2.11) and (2.12) are destabilized. Thus, one should study the relative order between the terms of the sequences ξ_i and s_r as $i; r \to \infty$. Here s_r $(r=1,2,\ldots)$ represent the negative eigenvalues of the equation $H(s)=0$. We note that in Eq. (2.8a), which expresses the quantity ξ_i, μ_i is a positive root of the Eq. (2.9a). Since $(x-t+a)\in(-t,-t+1]$, ξ_i can take negative values. On the other hand, the terms of the sequences μ_i and $|s_i|$ have the same period, which is approximately constant as $i\to\infty$. More specifically, this period takes values in the interval [3.141, 3.142], varying slightly and periodically but at the same rate for the two sequences. Furthermore, comparing Eqs. (2.9a) and (3.1) one concludes that, $\mu_i \neq |s_i|$, $i=1,2,\ldots$ Thus, we deduce that, for $i \to \infty$ the distance between μ_i and $|s_i|$ remains constant. Note that $|s_r|$ are also roots of equation $H(s)=0$, because for $s>0$ it is valid

$$J_1(-s) = -J_1(s), \quad J_0(-s) = J_0(s) \;\Rightarrow\; H(-s) = -H(s).$$

As an example, we write here a number of the roots of the above two equations:

μ_i: $\ldots$, 522.2878, 525.4294, 528.571, 531.7126, 534.8542, $\ldots$, 1452.2, 1455.342, 1458.484, 1461.625, 1464.767, 1467.908, $\ldots$,

$|s_i|$: $\ldots$, 523.8589, 527.0005, 530.1421, 533.2837, $\ldots$, 1453.771, 1456.913, 1460.055, 1463.196, 1466.338, 1469.479, $\ldots$

Consequently, as ξ_i increases absolutely with step $(x-t+a)\times$ period, its values periodically approach those of the eigenvalues s_r. These values, which will be called "*destabilization*" points or d-points, constitute a subsequence ξ_{i_n}. One writes that $\xi_{i_n} \in (s_r - d, s_r + d)$ where $d = \mathrm{O}(10^{-3})$.

3.2 "Destabilization" Terms – "Convergence" Terms

In order to investigate the convergence of the formulae (2.11) and (2.12) expressing the flow fields, we must focus on the "*destabilization*" terms or d-terms of the series being introduced. In other words, we must investigate the terms which destabilize the series at the d-points ξ_{i_n}. For this purpose we shall try to obtain simpler approximate expressions of the before mentioned formulae.

One easily observes that the following inequalities hold true:

$$1 \ll \frac{\mu_i}{\varepsilon}; \tag{3.2a}$$

$$1 \ll \xi_i^2, \quad 2 \ll \xi_i^2; \tag{3.2b}$$

$$|\xi_{i_n} J_1(\xi_{i_n})| \gg |J_0(\xi_{i_n})|. \tag{3.2c}$$

The inequality (3.2c) results from the different phase between the Bessel functions J_1 and J_0 shown in Fig. 2, in combination with the

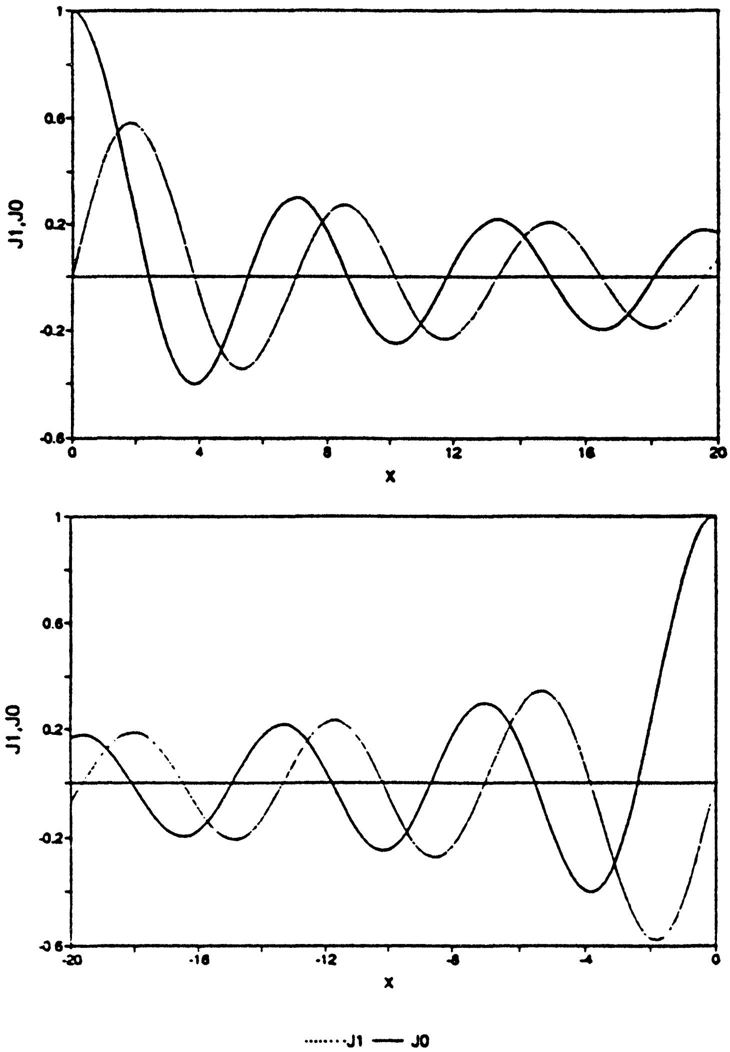

FIGURE 2 Different phase of the Bessel functions J_1 and J_0.

form of the function H given in (2.8b). Note that the value ξ_{i_n} neither coincides with a root of the function J_1 or J_0, nor belongs to a very small area of a root of the function J_1.

Taking into account the inequalities (3.2a)–(3.2c), the expressions (2.11) and (2.12) concerning the velocity and acceleration components become:

$$\Sigma_u \cong -\delta_0\varepsilon^3(x-t+a)^2\sum_{i=1}^{\infty}\left[\frac{J_1(\xi_i)}{H^2(\xi_i)}J_1\left(\frac{\mu_i}{\varepsilon}r\right)\cos\left(\frac{\mu_i}{\varepsilon M}t\right)\right]\cos\theta; \tag{3.3a}$$

$$\Sigma_{v,w} \cong -\frac{\delta_0\varepsilon^3(x-t+a)^3}{r}\sum_{i=1}^{\infty}\left[\frac{1}{\xi_i^2 H(\xi_i)}G\left(\frac{\mu_i}{\varepsilon}r\right)\cos\left(\frac{\mu_i}{\varepsilon M}t\right)\right]g(\theta); \tag{3.3b}$$

$$\Sigma_{\mathrm{d}u} = -\delta_0\varepsilon^3(x-t+a)\sum_{i=1}^{\infty}\left\{J_1\left(\frac{\mu_i}{\varepsilon}r\right)\left[\frac{\xi_1^2\left[2J_1^2(\xi_i)+J_0^2(\xi_i)\right]}{H^3(\xi_i)}\cos\left(\frac{\mu_i}{\varepsilon M}t\right) - \frac{1}{\varepsilon M}\frac{\xi_i J_1(\xi_i)}{H^2(\xi_i)}\sin\left(\frac{\mu_i}{\varepsilon M}t\right)\right]\right\}\cos\theta; \tag{3.4a}$$

$$\Sigma_{\mathrm{d}v,\mathrm{d}w} \cong \frac{\delta_0\varepsilon^3(x-t+a)^2}{r}\sum_{i=1}^{\infty}\left\{G\left(\frac{\mu_i}{\varepsilon}r\right)\left[\frac{J_1(\xi_i)}{H^2(\xi_i)}\cos\left(\frac{\mu_i}{\varepsilon M}t\right) + \frac{1}{\varepsilon M}\frac{1}{\xi_i H(\xi_i)}\sin\left(\frac{\mu_i}{\varepsilon M}t\right)\right]\right\}g(\theta), \tag{3.4b}$$

where with $\Sigma_u; \Sigma_{v,w}; \Sigma_{\mathrm{d}u}; \Sigma_{\mathrm{d}v,\mathrm{d}w}$ we symbolize the series terms of the before mentioned components. Also, G denotes one of the functions H; J_1, while g one of the functions (cos); (sin), if we refer to the series Σ_v; Σ_w and $\Sigma_{\mathrm{d}v}$; $\Sigma_{\mathrm{d}w}$, respectively. In relations (3.3a,b) and (3.4a,b) the d-terms are the quantities $H^{-2}(\xi_{i_n}); H^{-1}(\xi_{i_n})$ and $\xi_{i_n}^2 H^{-3}(\xi_{i_n}); H^{-2}(\xi_{i_n})$, respectively. These terms are reduced significantly because they are multiplied by the Bessel functions with arguments ξ_{i_n} and $(\mu_{i_n}r/\varepsilon)(\xi_{i_n} = \mu_{i_n}(x-t+a))$. The convergence or divergence of the above products will be investigated in our further analysis. Thus, referring to the Bessel functions with argument ξ_{i_n}, we may consider as "*convergence*" terms, or c-terms, the following expressions:

$$CVA(\xi_{i_n}) = \frac{J_1(\xi_{i_n})}{H^2(\xi_{i_n})}; \quad n = 1, 2, \ldots \tag{3.5}$$

corresponding to the resultants u; $\mathrm{d}v/\mathrm{d}t$; $\mathrm{d}w/\mathrm{d}t$,

$$CVW(\xi_{i_n}) = \frac{1}{\xi_{i_n}^2 H(\xi_{i_n})}; \quad n = 1, 2, \ldots \tag{3.6}$$

corresponding to the resultants v; w, and

$$CAX(\xi_{i_n}) = \frac{\xi_{i_n}^2[2J_1^2(\xi_{i_n}) + J_0^2(\xi_{i_n})]}{H^3(\xi_{i_n})}; \quad n = 1, 2, \ldots, \tag{3.7}$$

corresponding to the resultant $\mathrm{d}u/\mathrm{d}t$.

3.3 "Ideal" Sequences – Convergence of the Series

Since the relations (3.5)–(3.7) represent a combination of d-terms with their "dumper" (Bessel functions), one concludes that the limiting values of the sequences constructed by c-terms, namely the limiting values of the sequences

$$CVA(\xi_{i_n}), \quad CVW(\xi_{i_n}), \quad CAX(\xi_{i_n}); \qquad n = 1, 2, \ldots,$$

are decisive for the convergence of the series (3.3a)–(3.4b). More especially, the main variation of the values of these series occur at the d-points ξ_{i_n}, as these points have been already determined in Section 3.1. The above variations are insignificant for any other value of the sequence ξ_i. Furthermore, the "*convergence*" terms (3.5)–(3.7) affect the values of the before mentioned series at the particular points ξ_{i_n}. We are now able to formulate the following propositions.

PROPOSITION 1 "*A necessary and sufficient condition for the convergence of the series* (3.3a)–(3.4b) *is that the sequences* (3.5)–(3.7) *must have limiting value equal to zero.*"

Since the series (3.3a)–(3.4b) are approximate expressions of the flow field components (2.11) and (2.12), it is well understood that their convergence secure the correct evaluation of the above components. The study of the limiting values of the sequences (3.5)–(3.7) can be achieved if the argument ξ_{i_n} is replaced by the corresponding p_r such that

$$p_r \in [s_r - \varepsilon_0, s_r); \quad r = 1, 2, \ldots; \quad \varepsilon_0 \in (0, 10^{-6}]. \tag{3.8}$$

Here s_r denote the roots of the equation

$$H(s) = J_1(s) - sJ_0(s) = 0.$$

The introduction of p_r as a new argument in the "convergence" terms (3.5)–(3.7) is necessary because the distance of the d-points ξ_{i_n} from the real roots s_r varies inside an interval $(0, d]$, where $d = O(10^{-3})$. Thus, since the above distance is not constant, the study of the ξ_{i_n}-sequences does not provide an "absolute" criterion for the convergence of the c-terms (3.5)–(3.7). On the contrary, the distance between the points p_r and the roots s_r is approximately constant and always very small, lying inside an interval of our choice, for example $(0, 10^{-6}]$. We call the sequences $CVA(p_r)$, $CVW(p_r)$, $CAX(p_r)$, $r = 1, 2, \ldots$ being constructed from the expressions (3.5)–(3.7) if ξ_{i_n} is replaced by p_r, "*ideal*" sequences or i-sequences. They play the role of an "ideal" criterion as the following proposition reads.

PROPOSITION 2 "*The convergence of the "ideal" sequences towards zero, is a sufficient condition for the convergence of the c-terms* (3.5)–(3.7) *to zero.*"

Considering now that ξ_{i_n}, $n = 1, 2, \ldots$, increases absolutely, that is for any real function $f(x)$ the limit $\lim_{n\to\infty} f(\xi_{i_n}) = \beta$ reads

$$\forall\varepsilon > 0 \;\; \exists\xi_{i_0} \in \boldsymbol{R}^+\colon \; \forall\xi_{i_n} \text{ with } |\xi_{i_n}| > \xi_{i_0} \Rightarrow |f(\xi_{i_n}) - \beta| < \varepsilon,$$

then, the sequence p_r, $r = 1, 2, \ldots$, increases absolutely and we are able to state the following proposition.

PROPOSITION 3 "*The i-sequences $CVA(p_r)$, $CVW(p_r)$ and $CAX(p_r)$, $r = 1, 2, \ldots$, converge towards zero as $r \to \infty$, namely*

$$\forall\varepsilon > 0 \;\; \exists\xi_{va}, \xi_{vw}, \xi_{ax} \in \boldsymbol{R}_+$$

such that:

$$\begin{aligned}
\forall p_r\colon \quad & |p_r| > \xi_{va} \;\Rightarrow\; |CVA(p_r)| < \varepsilon,\\
\forall p_r\colon \quad & |p_r| > \xi_{vw} \;\Rightarrow\; |CVW(p_r)| < \varepsilon,\\
\forall p_r\colon \quad & |p_r| > \xi_{ax} \;\Rightarrow\; |CAX(p_r)| < \varepsilon.
\end{aligned}$$

Computational study can prove the validity of the above proposition. The kind of this convergence is an oscillation of the absolute values of the sequences $CVA(p_r)$, $CVW(p_r)$ and $CAX(p_r)$ about a center, the distance of which decreases absolutely towards zero. Taking the positive number $\varepsilon = 10^{-5}$ one evaluates

$$\xi_{ax} > \xi_{va} > \xi_{vw} > 0;$$
$$\xi_{va} \cong 1.2 \times 10^6;\ \xi_{vw} \cong \mathrm{O}(10^4);\ \xi_{ax} > 10^7.$$

Combining the before mentioned propositions 1–3 one asserts that the points ξ_{va}, ξ_{vw}, ξ_{ax} are the "stabilization" points of the series (3.3a)–(3.4b). In other words, the c-terms (3.5)–(3.7) with $|\xi_{i_n}| > \xi_{va}$ or $|\xi_{i_n}| > \xi_{vw}$ or, finally, $|\xi_{i_n}| > \xi_{ax}$, do not vary the values of the series obtained so far; namely, for the value $\varepsilon = 10^{-5}$ the variations of the series (3.3a)–(3.4b) tend to zero as $n \to \infty$.

4 RESULTS

By means of the main notations we consider the true time $t = TU/L$ and the reduced frequency $k = \omega L/U$ and we derive

$$\bar{p} = pL/U, \quad p = 2\pi/k,$$

where $\bar{p}$ represents the true period of the body oscillation and p the nondimensional one. For the right circular cone shown in Fig. 3 we evaluate the results presented in Table I. Making use of the well-known formula resulting from the small perturbation theory

$$1 - M_{\mathrm{loc}}^2 \approx 1 - M^2 - (1+\gamma)M_\infty^2(\varphi_x + \psi_x),$$

in which M_{loc} is the local Mach number on the body, and introducing the expression

$$\varphi_x = 1/(1+\gamma)M^2$$

for the supersonic case [6], we deduce

$$M_{\mathrm{loc}}^2 \cong 1 + M^2[1 + (1+\gamma)\psi_x].$$

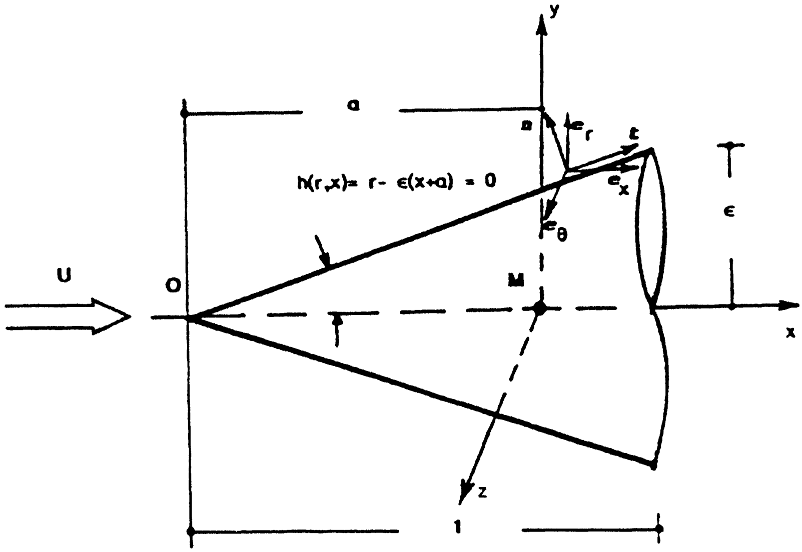

FIGURE 3 Geometry and sign convention of a right circular one.

TABLE I Basic parameters for a rigid body of revolution

M	U (km/h)	U/L (1/s)	$\bar{p}(s)\begin{pmatrix} k=0.03 \\ p=66.67\pi \end{pmatrix}$	$\bar{p}(s)\begin{pmatrix} k=0.1 \\ p=20\pi \end{pmatrix}$	$\bar{p}(s)\begin{pmatrix} k=0.2 \\ p=10\pi \end{pmatrix}$
1.00	1224	68	3.08	0.92	0.46
1.60	1958	136	1.54	0.46	0.23

Here, the quantity ψ_x can be evaluated by the series term of the expression (2.7), so that one can estimate the local Mach number M_{loc}. Furthermore, combining the asymptotic expansions of the Bessel functions for large values of their arguments given in the APPENDIX with the values of ξ_{AX} ($> 10^7$) according to Proposition 3, and taking into account the demanded accuracy in the numerical evaluation of the J_1 and J_0 functions, we derive the following statement: "It is not possible to obtain numerical results for the axial acceleration du/dt, except in case when the convergence occurs at values $\xi_i < 4 \times 10^6$ (Figs. 6 and 7)".

Figure 4(a) depicts the variation of the dimensionless velocity u and the local Mach number M_{loc} versus the dimensionless distance $(x + a)$. Figures 4(b)–(e) show the variation of the dimensionless

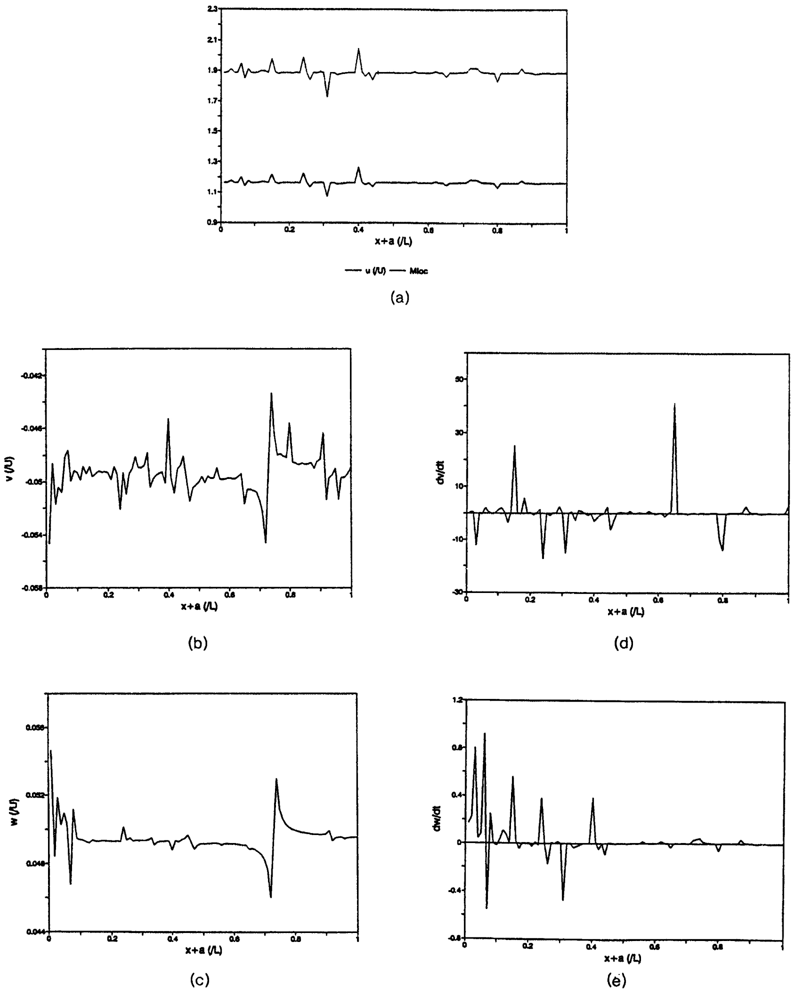

FIGURE 4(a)–(e) Variations of the dimensionless velocities, the local Mach number and the accelerations versus the dimensionless distance $(x + a)$.

velocities $v; w$ and accelerations $\mathrm{d}v/\mathrm{d}t$; $\mathrm{d}w/\mathrm{d}t$ versus $(x + a)$ from the top to the base of the right circular cone of Fig. 3. The parameters being introduced are $a = 0.60$; $\varepsilon = 0.067$; $\delta_0 = 0.07$; $\theta = \pi/4$; $k = 0.20$; $M = 1.6$; $t = 20.50p$ and $p = 10\pi$.

Figures 5(a)–(d) depict the variation of the dimensionless cylindrical velocity component w versus $(x + a)$, at a time step equal to $p/4$

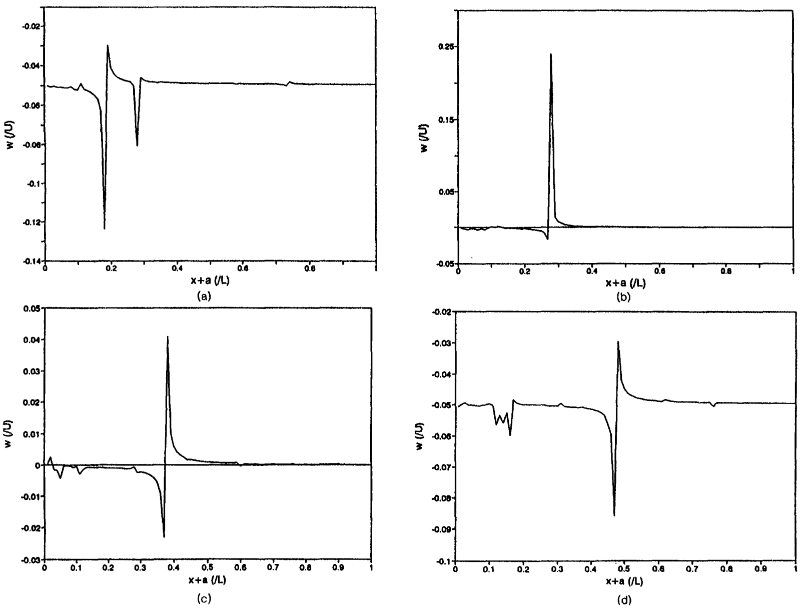

FIGURE 5(a)–(d) Variations of the dimensionless cylindrical velocity component w versus $(x+a)$ at a time step equal to $p/4$ ($\cong 0.23$ s).

($\cong 0.23$ s) for the cone shown in Fig. 3. The parameters being introduced are $a=0.50$; $\varepsilon=0.67$; $\delta=0.07$; $\theta=\pi/4$; $k=0.10$; $M=1.00$; $t=20p$; $20.25p$; $20.50p$; $20.75p$ and $p=20\pi$.

Figure 6(a) shows the variation of the dimensionless velocity u and the local Mach number M_{loc} versus the normalized cone thickness ratio ε with $a=0.60$; $k=0.03$; $M=1.60$; $\delta_0=0.07$; $x=-0.30$; $\theta=\pi/4$; $t=10p$ and $p=66.67\pi$. Figures 6(b)–(f) depict the variations of the dimensionless velocities v; w and the accelerations $\mathrm{d}u/\mathrm{d}t$; $\mathrm{d}v/\mathrm{d}t$ and $\mathrm{d}w/\mathrm{d}t$ versus the cone thickness ratio ε, where the introduced parameters are the same as in Fig. 6(a).

Figure 7(a) and Figs. 7(b)–(f) show the variations of u; M_{loc} and v; w; $\mathrm{d}u/\mathrm{d}t$; $\mathrm{d}v/\mathrm{d}t$ and $\mathrm{d}w/\mathrm{d}t$ versus the Mach number M, for the same parameters as in Fig. 6, if instead of the Mach number M, the cone thickness ratio $\varepsilon=0.067$ is introduced.

From the above graphics one observes the rapid variations of the flow characteristics, namely of the velocity and the acceleration components, denoting the unstable and dynamic character of the

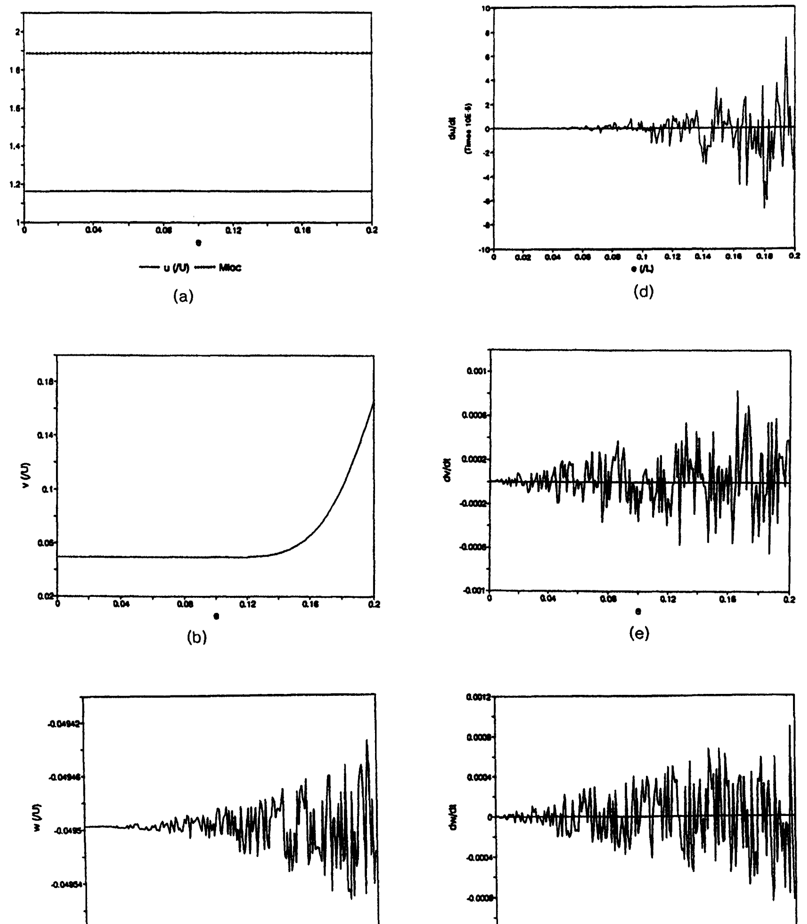

FIGURE 6(a)–(f) Variations of the dimensionless velocities, the local Mach number and the accelerations versus the cone thickness ratio ε.

problem under consideration. Especially, in Fig. 5 one can see a disturbance wave with a specific shape, which is removed with time towards the base of the cone, while in Figs. 6 and 7 a strong variation versus the cone thickness ratio ε and the Mach number M appear during the supersonic flow.

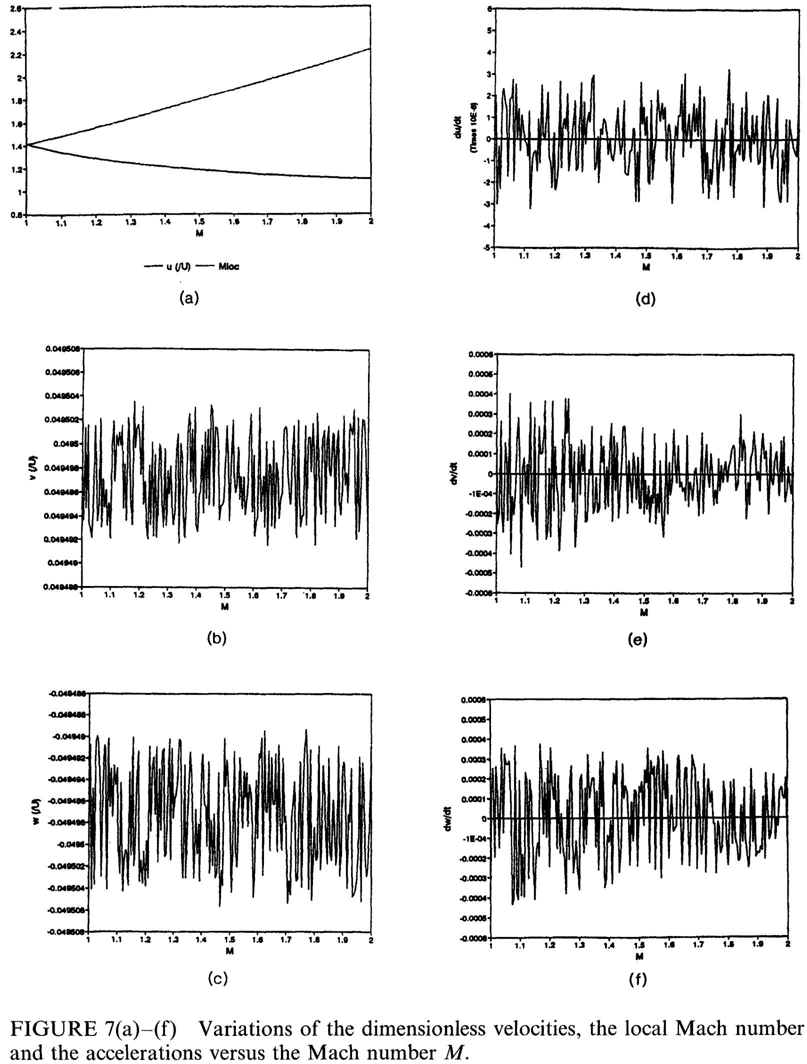

FIGURE 7(a)–(f) Variations of the dimensionless velocities, the local Mach number and the accelerations versus the Mach number M.

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APPENDIX

In order to estimate the Bessel functions for large arguments one can make use of several asymptotic expansions. Many tests of various formulae concerning the convergence of the above expansions and a comparison with the results presented in Ref. [1] (Tables 9.1 to 9.4) lead to the conclusion that the Hankel formulae are the more accurate for our problem under consideration.

Thus, for large values of the argument, one may write ([1], p. 364, types 9.2.5; 9.2.6; 9.2.9, and 9.2.10):

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}[P(\nu, x)\cos X - Q(\nu, x)\sin X]$$

and

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}}[P(\nu, x)\sin X + Q(\nu, x)\cos X],$$

where $X = x - (\nu\pi/2) - (\pi/4)$ and

$$P(\nu, x) = 1 + \sum_{k=1}^{\infty}\left[(-1)^k \frac{\prod_{\lambda=0}^{2k-1}\left[\mu - (2\lambda+1)^2\right]}{(2k)!(8x)^{2k}}\right];$$

$$Q(\nu, x) = \sum_{k=0}^{\infty}\left[(-1)^k \frac{\prod_{\lambda=0}^{2k}\left[\mu - (2\lambda+1)^2\right]}{(2k+1)!(8x)^{2k+1}}\right]$$

with $\mu = 4\nu^2$.

As x increases, the convergence of the above types occurs faster, but the accuracy of the results decreases. More specifically these asymptotic types do not give the variation of the Bessel functions J_ν, Y_ν for very large values of the variable x. On the contrary, the types furnish the same values for the functions J_ν, Y_ν for an interval of the variable x with amplitude d_0. This amplitude increases as $|x|$ increases. In particular we estimate the following values:

$$\text{for } x \geq 4.2 \times 10^6, \quad d_0 = 0.5$$

and

$$\text{for } x \geq 8.4 \times 10^6, \quad d_0 = 1.$$