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A Collocation Method for Optimal Control of Linear Systems with Inequality Constraints

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A numerical method for solving linear quadratic optimal control problems with control inequality constraints is presented in this paper. The method is based upon hybrid function approximations. The properties of hybrid functions which are the combinations of block-pulse functions and Legendre polynomials are first presented. The operational matrix of integration is then utilized to reduce the optimal control problem to a set of simultaneous nonlinear equations. The inequality constraints are first converted to a system of algebraic equalities, these equalities are then collocated at Legendre–Gauss–Lobatto nodes. An illustrative example is included to demonstrate the validity and applicability of the technique.

Keywords: Optimal control; Inequality constraints; Legendre polynomials; Legendre–Gauss–Lobatto nodes

1. INTRODUCTION

Orthogonal functions, often used to represent an arbitrary time function, have recently been used to solve various problems of dynamic systems. Typical examples are the Walsh function [1], block-pulse functions [2], Laguerre polynomials [3], Legendre polynomials [4], Chebyshev series [5] and Fourier series [6].

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The design of optimal feedback was obtained by Kleinman [7]. Similar problems for linear systems have been studied either by means of the Walsh functions [1] or by means of the block-pulse functions [2]. Due to the nature of these functions, the solutions obtained were piecewise constant. Most of these methods successfully solve the unconstrained problem, but the presence of inequality constraints often resulted in both analytical and computational difficulties. Theoretical aspects of trajectories inequality constraint have been studied in [8,9]. Early contribution to the numerical computation were due to [10,11]. Mehra and Davis [11] described that difficulties arising from handling trajectories inequality constraints are due to the exclusive use of control variables as independent variables and presented the so-called generalized gradient technique.

In the present paper we introduce a new direct computational method for solving linear quadratic optimal control with control inequality constraints. The method consists of reducing the optimal control problem to a set of nonlinear algebraic equations by first expanding the candidate function as a hybrid function with unknown coefficients. The hybrid functions which are the combinations of block-pulse functions and Legendre polynomials are introduced. The operational matrix of integration is then utilized to evaluate the hybrid function coefficients. The control inequality constraints are first transferred into a system of algebraic equalities. The given system are collocated at Legendre–Gauss–Lobatto (LGL) nodes [12]. When the optimal control inequalities constraints are not satisfied in the whole desired interval, in each subintervals for which the constraints is violated, we change the given LGL nodes with the extremum of the solution in that subinterval. An illustrative example is given to demonstrate the application of the proposed method.

2. PROPERTIES OF HYBRID FUNCTIONS

2.1. Hybrid Functions

Hybrid functions $b(n, m, t)$, $n = 1, \ldots, N$, $m = 0, \ldots, M$ defined on $[0, t_f)$, have three arguments, m and n are the order for Legendre polynomials and block-pulse functions respectively and t is the normalized time and is defined as

$$b(n,m,t) = \begin{cases} P_m\left(\dfrac{2N}{t_f}t - 2n + 1\right), & t \in \left[\dfrac{n-1}{N}t_f, \dfrac{n}{N}t_f\right), \quad \begin{matrix} n = 1,2,3,\ldots,N, \\ m = 0,1,2,\ldots,M, \end{matrix} \\ 0 & \text{elsewhere.} \end{cases} \tag{1}$$

Here, $P_m(t)$ are the well-known Legendre polynomials of order m which are orthogonal in the interval $[-1,1]$ and satisfy the following recursive formula:

$$P_0(t) = 1, \tag{2}$$

$$P_1(t) = t \tag{3}$$

and

$$P_{m+1}(t) = \frac{2m+1}{m+1} t P_m(t) - \frac{m}{m+1} P_{m-1}(t). \tag{4}$$

Since $b(n,m,t)$ is the combination of Legendre polynomials and block-pulse functions which are both complete and orthogonal, thus the set of hybrid functions are complete orthogonal set. Figure 1 shows the hybrid functions for $M=2$ and $N=4$.

The hybrid representation of a function $f(t)$ defined over $[0, t_f)$ is given by a finite series as

$$f(t) \cong \sum_{m=0}^{M} \sum_{n=1}^{N} c(n,m) b(n,m,t) = C^{\mathrm{T}} B(t), \tag{5}$$

where

$$c(n,m) = \frac{(2m+1)}{t_f} N \int_{(n-1)/t_f}^{n/t_f} f(t) b(n,m,t)\,\mathrm{d}t, \qquad n = 1,\ldots,N, \quad m = 0,\ldots,M \tag{6}$$

and

$$C = [c(1,0),\ldots,c(N,0),c(1,1),\ldots,c(N,1),\ldots,c(1,M),\ldots,c(N,M)]^{\mathrm{T}}, \tag{7}$$

$$B(t) = [b(1,0,t),\ldots,b(N,0,t),b(1,1,t),\ldots,b(N,1,t),\ldots, b(1,M,t),\ldots,b(N,M,t)]^{\mathrm{T}}. \tag{8}$$

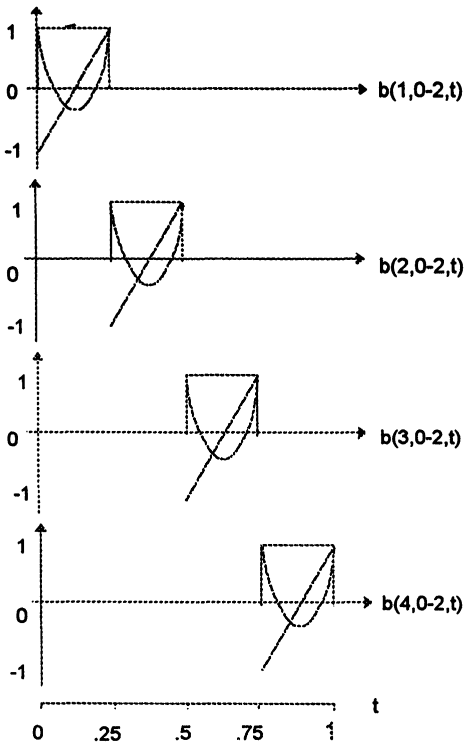

FIGURE 1 Hybrid functions for $N=4$ and $M=2$.

The orthogonality property is given by

$$\int_0^{t_f} B(t)B^{\mathrm{T}}(t)\,\mathrm{d}t = \frac{t_f}{N}\,\mathrm{diag}\left[I, \frac{I}{3}, \ldots, \frac{I}{2M+1}\right]. \tag{9}$$

2.2. Operational Matrix of Integration

The integration of the vector $B(t)$ defined in Eq. (8) can be approximated by

$$\int_0^t B(t')\,\mathrm{d}t' \cong PB(t), \tag{10}$$

where P is the $NM \times NM$ operational matrix for integration and is given by

$$P = \frac{t_f}{N}\begin{bmatrix} P_0 & \frac{I}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ -\frac{I}{6} & 0 & \frac{I}{6} & 0 & \dots & 0 & 0 & 0 \\ 0 & -\frac{I}{10} & 0 & \frac{I}{10} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -\frac{I}{2(2M+1)} & 0 & \frac{I}{2(2M+1)} \\ 0 & 0 & 0 & 0 & \dots & 0 & -\frac{I}{2(2M+1)} & 0 \end{bmatrix}. \tag{11}$$

In Eq. (11), I is an $N \times N$ identity matrix and

$$P_0 = \begin{bmatrix} \frac{1}{2} & 1 & \dots & 1 & 1 \\ 0 & \frac{1}{2} & \dots & 1 & 1 \\ \dots & \dots & \dots & 1 & 1 \\ 0 & 0 & \dots & \frac{1}{2} & 1 \\ 0 & 0 & \dots & 0 & \frac{1}{2} \end{bmatrix}, \tag{12}$$

3. OPTIMAL CONTROL PROBLEM

Consider the following class of linear systems with control inequality constraints:

$$\dot{x}(t) = Ax(t) + Eu(t), \quad x(0) = x_0, \tag{13}$$

$$|u_j(t)| \leq u_j^*, \quad j = 1, \dots, s \tag{14}$$

with the cost functional

$$J = \frac{1}{2}\int_0^{t_f} \left(x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\right) \mathrm{d}t. \tag{15}$$

Here, $x(t)$ and $u(t)$ are $r \times l$ and $s \times l$ state and control vectors respectively, A, E, Q and R are matrices of appropriate dimensions, t_f is the final time which is given, Q and R are symmetric semidefinite and positive-definite matrices respectivity. The problem is to find the optimal control $u(t)$ which minimizes Eq. (15), subject to the constraints of Eq. (13) and the control inequality constraints of Eq. (14).

4. HYBRID FUNCTION APPROXIMATIONS

4.1. The Approximation of the System Dynamics

By expanding each of the s control vectors and the derivative of each of the r state vectors by hybrid functions, we get

$$\dot{x}_i(t) = \sum_{m=0}^{M}\sum_{n=1}^{N} c_i(n,m)b(n,m,t) = C_i^{\mathrm{T}}B(t), \quad i = 1,2,\ldots,r, \tag{16}$$

$$u_j(t) = \sum_{m=0}^{M}\sum_{n=1}^{N} y_j(n,m)b(n,m,t) = Y_j^{\mathrm{T}}B(t), \quad j = 1,2,\ldots,s, \tag{17}$$

where

$$C_i = [c_i(1,0),\ldots,c_i(N,0),\ldots,c_i(1,M),\ldots,c_i(N,M)]^{\mathrm{T}}, \quad i = 1,\ldots,r, \tag{18}$$

$$Y_i = [y_i(1,0),\ldots,y_i(N,0),\ldots,y_i(1,M),\ldots,y_i(N,M)]^{\mathrm{T}}, \quad i = 1,\ldots,s. \tag{19}$$

Using Eqs. (16)–(19), the state rate variable $\dot{x}(t)$ and the control vector $u(t)$ can be represented as

$$\dot{x}(t) = [C_1^{\mathrm{T}}, C_2^{\mathrm{T}},\ldots,C_r^{\mathrm{T}}]\hat{B}(t) = \hat{C}^{\mathrm{T}}\hat{B}(t), \tag{20}$$

$$u(t) = [Y_1^{\mathrm{T}}, Y_2^{\mathrm{T}},\ldots,Y_s^{\mathrm{T}}]\hat{B}(t) = \hat{Y}^{\mathrm{T}}\hat{B}(t), \tag{21}$$

where

$$\hat{B}(t) = [B^{\mathrm{T}}(t), B^{\mathrm{T}}(t),\ldots,B^{\mathrm{T}}(t)]^{\mathrm{T}}. \tag{22}$$

Using Eq. (20), $x(t)$ can be represented as

$$x(t) = \int_0^t \hat{C}^{\mathrm{T}}\hat{B}(t')\,\mathrm{d}t' + x(0) = \hat{C}^{\mathrm{T}}P\hat{B}(t) + \hat{C}_0^{\mathrm{T}}\hat{B}(t) = (\hat{C}^{\mathrm{T}}P + \hat{C}_0^{\mathrm{T}})\hat{B}(t), \tag{23}$$

where

$$\hat{C}_0 = [x_1(0),0,\ldots,0,0,x_2(0),0,\ldots,0,0,\ldots,x_r(0),0,\ldots,0]^{\mathrm{T}}. \tag{24}$$

Substituting Eqs. (20), (21) and (23) in Eq. (13) we get the following algebraic equations:

$$\hat{C} = \hat{Y}E^{\mathrm{T}} + (P^{\mathrm{T}}\hat{C} + \hat{C}_0)A^{\mathrm{T}}. \tag{25}$$

Now, the inequality constraints (14) are incorporated into the above scheme. Equation (14) can be replaced by the finite number of algebraic equalities as

$$u_j^2(t) + z_j^2(t) = u_j^{*2}, \quad j = 1, 2, \ldots, s, \tag{26}$$

where $z_j(t)$ is an auxiliary function.

By expanding each $z_j(t), j = 1, \ldots, s$ in hybrid functions we get

$$z_j(t) = \sum_{m=0}^{M}\sum_{n=1}^{N} Z_j(n, m)b(n, m, t) = \hat{Z}_j^{\mathrm{T}} B(t), \quad j = 1, 2, \ldots, s. \tag{27}$$

Using Eqs. (17), (26) and (27) we obtain

$$Y_j^{\mathrm{T}} B(t)B^{\mathrm{T}}(t)Y_j + \hat{Z}_j^{\mathrm{T}} B(t)B^{\mathrm{T}}\hat{Z}_j(t) = u_j^{*2}, \quad j = 1, 2, \ldots, s. \tag{28}$$

4.2. The Performance Index Approximation

The performance index J can also be expressed as a function of the unknown $\hat{C}$ and $\hat{Y}$. Using Eqs. (15), (21) and (23) we obtain

$$\begin{aligned} J = {} & \frac{1}{2}[\hat{C}_0^{\mathrm{T}} + \hat{C}^{\mathrm{T}}P]\left(\int_0^{t_f} \hat{B}(t)Q\hat{B}^{\mathrm{T}}(t)\,\mathrm{d}t\right)[\hat{C}_0 + P^{\mathrm{T}}\hat{C}] \\ & + \frac{1}{2}\hat{Y}^{\mathrm{T}}\left(\int_0^{t_f} \hat{B}(t)R\hat{B}^{\mathrm{T}}(t)\,\mathrm{d}t\right)\hat{Y}. \end{aligned} \tag{29}$$

By applying Kronecker product [13] we have

$$\hat{B}(t)Q\hat{B}^{\mathrm{T}}(t) = (\hat{B}(t) \otimes \hat{B}(t))Q_{\mathrm{v}}, \tag{30}$$

$$\hat{B}(t)R\hat{B}^{\mathrm{T}}(t) = (\hat{B}(t) \otimes \hat{B}(t))R_{\mathrm{v}}, \tag{31}$$

where Q_v and R_v are the vector form for the matrices Q and R and $\otimes$ denotes Kronecker product. Equation (29) can be written as

$$J = \tfrac{1}{2}[\hat{C}_0^T + \hat{C}^T P] D_v Q_v [\hat{C}_0 + P^T \hat{C}] + \tfrac{1}{2}[\hat{Y}^T D_v R_v \hat{Y}], \tag{32}$$

where D_v is the vector form for the matrix D in Eq. (9).

4.3. Replacing the Inequality Control Constraints Using Collocation Method

To satisfy the inequality constraints in the desired interval we first collocate Eq. (28) at $M \times N$ points; thus we get

$$\begin{gathered} Y_j^T B(t_k) B^T(t_k) Y_j + \hat{Z}_j^T B(t_k) B^T(t_k) \hat{Z}_j = u_j^{*2}, \\ k = 1, 2, \ldots, M \times N,\ j = 1, 2, \ldots, s. \end{gathered} \tag{33}$$

For a suitable collocation points t_k, $k = 1, \ldots, M \times N$ we use the LGL nodes in $[0, t_f/N]$ which are defined in [12] as follows.

Let $P_M(t)$, $-1 \le t \le 1$ denote the Legendre polynomials of order M; then the LGL nodes are defined by

$$t_0 = -1,\ t_M = 1,\quad t_k \text{ are the zeros of } \dot{P}_M(t),\ 1 \le k \le M-1, \tag{34}$$

where $\dot{P}_M(t)$ denotes the 1st derivative of $P_M(t)$. No explicit formula of the nodes in (34) is known. However, they can be computed numerically.

In order to use LGL nodes for Eq. (1) we transfer $[-1, 1]$ into

$$\left[\frac{n-1}{N} t_f, \frac{n}{N} t_f\right].$$

It is noted that Eq. (1) has an extremum point at LGL nodes. Once the optimal control for Eqs. (13)–(16) is obtained the intervals for which the inequality constraints (14) are violated would be considered again. The new collocation points in these intervals are obtained from the extremum of the pervious solution. This process is continued until the inequality constraints (14) are satisfied in the whole interval.

The minimization problem of Eq. (15) subject to Eqs. (13) and (14) is reduced to a parameter optimization problem which can be stated as follows. Find $\hat{C}$, $\hat{Y}$ and $\hat{Z}$ which minimizes the following equations:

$$L(\hat{C}, \hat{Y}, \hat{Z}, \lambda) = J(\hat{C}, \hat{Y}) + \lambda^{\mathrm{T}}[\hat{Y}E^{\mathrm{T}} + (P^{\mathrm{T}}\hat{C} + \hat{C}_0)A^{\mathrm{T}} - \hat{C}], \qquad (35)$$

where λ is the Lagrange multiplier. The determining equations for the unknowns $\hat{C}$, $\hat{Y}$, $\hat{Z}$ and λ are

$$\frac{\partial}{\partial \hat{C}} L(\hat{C}, \hat{Y}, \hat{Z}, \lambda) = 0, \qquad (36)$$

$$\frac{\partial}{\partial \hat{Y}} L(\hat{C}, \hat{Y}, \hat{Z}, \lambda) = 0, \qquad (37)$$

$$\frac{\partial}{\partial \hat{Z}} L(\hat{C}, \hat{Y}, \hat{Z}, \lambda) = 0, \qquad (38)$$

$$\frac{\partial}{\partial \lambda} L(\hat{C}, \hat{Y}, \hat{Z}, \lambda) = 0. \qquad (39)$$

Equations (36)–(39) are nonlinear equations which can be solved by Newton's iterative method.

5. ILLUSTRATIVE EXAMPLE

Consider the linear system with inequality control constraint [14],

$$x(t) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(0) = [0 \quad 10]^{\mathrm{T}}, \qquad (40)$$

$$|u(t)| \leq 1, \qquad (41)$$

with the cost functional

$$J = \frac{1}{2}\int_0^1 (x_1^2(t) + u^2(t))\,\mathrm{d}t. \qquad (42)$$

The problem is to find the optimal control $u(t)$ which minimizes Eq. (42) subject to Eqs. (40) and (41). The exact solutions for this example are

$$u(t) = \begin{cases} -1 & 0 \le t \le 0.29115 \\ 6.26\sinh\left(\frac{\sqrt{3}}{2}t\right)\sin\left(\frac{t}{2}+1.14\right)^{-1} - 0.31\cosh\left(\frac{\sqrt{3}}{2}t\right)\sin\left(\frac{t}{2}+0.19\right) \\ & 0.29115 \le t \le 1 \end{cases} \tag{43}$$

and

$$J = 7.99455. \tag{44}$$

We determine the hybrid functions approximation for $N=3$ and $M=2$. Let

$$\dot{x}_i(t) = \sum_{m=0}^{2}\sum_{n=1}^{3} c_i(n,m)b(n,m,t) = C_i^{\mathrm{T}}B(t), \quad i=1,2, \tag{45}$$

$$u(t) = \sum_{m=0}^{2}\sum_{n=1}^{3} y(n,m)b(n,m,t) = Y^{\mathrm{T}}B(t). \tag{46}$$

Using Eq. (25) we get

$$C_1 = P^{\mathrm{T}}C_2 + C_{02}, \qquad C_2 = -P^{\mathrm{T}}C_2 - C_{02} + Y. \tag{47}$$

By applying Eq. (32) we obtain

$$\begin{aligned} J = \frac{1}{81}\{&c_1(1,0)[7c_1(1,0) + 0.5c_1(1,1) - 0.1c_1(1,2) + 9c_1(2,0) \\ &- c_1(2,1) + 3c_1(3,0) - c_1(3,1)] + c_1(2,0)[4c_1(2,0) \\ &- 0.5c_1(2,1) - 0.1c_1(2,2) + 3c_1(3,0) - c_1(3,1)] \\ &+ 0.01c_1(1,2)^2 + 0.01c_1(2,2)^2 + 0.01c_1(3,2)^2 \\ &+ 0.1c_1(1,1)^2 + 0.1c_1(2,1)^2 + 0.1c_1(3,1)^2 + c_1(3,0) \\ &\times [c_1(3,0) - 0.5c_1(3,1) - 0.1c_1(3,2)]\} \\ + \sum_{m=0}^{2}\sum_{n=1}^{3}&\frac{y(n,m)^2}{2m+1}. \end{aligned} \tag{48}$$

By using LGL nodes in [0, 1] and Eq. (33) the inequality constraints in Eq. (41) is replaced by

$$\begin{aligned}
&-1+[y(n,1)+y(n,2)-y(n,0)]^2+[z(n,1)+z(n,2)-z(n,0)]^2=0,\\
&\quad n=1,2,3,\\
&-1+[y(n,0)-0.5y(n,2)]^2+[z(n,0)-0.5z(n,2)]^2=0,\quad n=1,2,3, \qquad (49)\\
&-1+[y(n,0)+y(n,1)+y(n,2)]^2+[z(n,0)+z(n,1)+z(n,2)]^2=0,\\
&\quad n=1,2,3.
\end{aligned}$$

Further, Eqs. (36)–(39) give five nonlinear algebraic equations from which $x_1(t)$, $x_2(t)$ and $u(t)$ can be calculated. The computational results for $x_1(t)$, $x_2(t)$ and $u(t)$ together with the exact solution for $u(t)$ are given in Fig. 2. The curves of exact and computational solutions for $x_1(t)$ and $x_2(t)$ are the same. In this case the minimum value for J is

$$J = 8.05035. \qquad (50)$$

It is noted that the inequality constraint is not satisfied in the interval $[0, 0.2]$. In this interval we replace the LGL nodes $t_2 = 0.16667$ with $t_2^1 = 0.0825$, where t_2^1 is the minimum $u(t)$ in $[0, 0.2]$. Figure 3 shows state vector and control input together with the exact solution for $u(t)$ for new collocation points. The curves of the exact and computational solutions for $x_1(t)$ and $x_2(t)$ are the same.

It is seen that Eq. (41) is satisfied in [0, 1]. In this case the minimum value for the J is

$$J = 8.03060. \qquad (51)$$

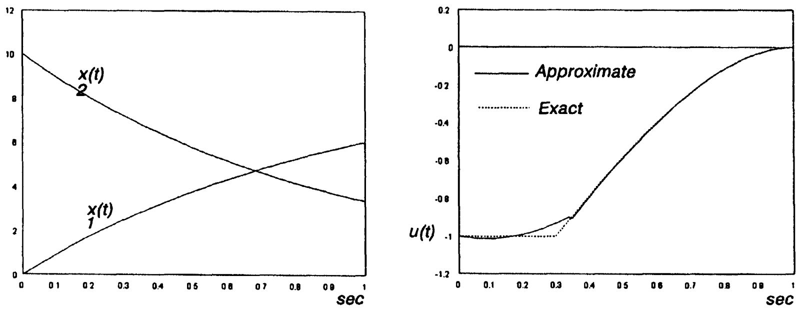

FIGURE 2 Approximate and exact solutions for $x_1(t)$, $x_2(t)$ and $u(t)$ for LGL nodes.

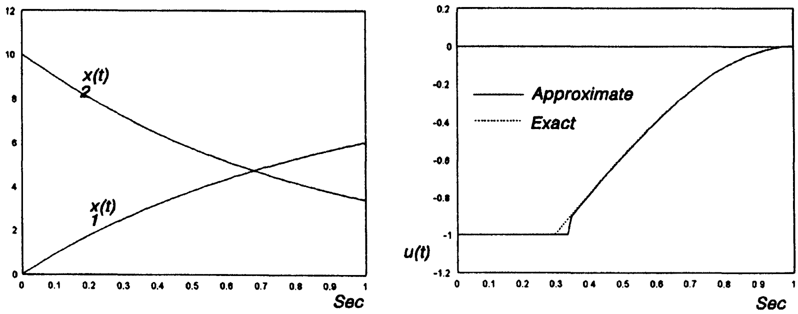

FIGURE 3 Approximate and exact solutions for $x_1(t)$, $x_2(t)$ and $u(t)$ for new collocation points.

6. CONCLUSION

In the present work the hybrid functions which are the combinations of block-pulse functions and Legendre polynomials are used to solve the optimal control of linear systems subject to a quadratic cost criteria with control inequality constraints. The problem has been reduced to a problem of solving a system of algebraic equations. The control inequality constraints are adjoined into the optimization problem by using LGL nodes. An example is given to demonstrate the application of the proposed method.

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