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Combination of Anti-Optimization and Fuzzy-Set-Based Analyses for Structural Optimization under Uncertainty

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An approach to the optimum design of structures, in which uncertainties with a fuzzy nature in the magnitude of the loads are considered, is proposed in this study. The optimization process under fuzzy loads is transformed into a fuzzy optimization problem based on the notion of Werners' maximizing set by defining membership functions of the objective function and constraints. In this paper, Werner's maximizing set is defined using the results obtained by first conducting an optimization through anti-optimization modeling of the uncertain loads. An example of a ten-bar truss is used to illustrate the present optimization process. The results are compared with those yielded by other optimization methods.

Keywords: convex modeling; fuzzy set; maximizing set; membership function; optimization

1. INTRODUCTION

The optimization of structural behavior is usually performed for specified loading conditions. However, in most practical situations loads are uncertain, and the designer must contend with the effects of this uncertainty. Most researchers favor the use of probabilistic models to account for this uncertainty, and employ probabilistic structural optimization for the design process. The structure is hereby optimally designed so that the probability of failure is below any pre-specified threshold. Hilton and Feigin [1] appear to have been the first to use probabilistic optimization, and their work has been followed by numerous studies.

It was recognized since then that uncertainty is not the same as randomness, and the notions of probability does not exhaust our notions of uncertainty. One alternative to randomness for describing uncertainty is fuzziness. Indeed, imprecision is a property of physical phenomena. It arises from the intrinsic behavior of human reasoning and natural language, which is less specific than a numerical characterization. A fuzzy set membership function describes the degree to which an object belongs to a set with imprecise boundaries, while randomness deals with the uncertainty regarding the probabilistic occurrence or nonoccurrence of some event.

It is well known that in the field of design designers are often forced in practice to state their design problems in precise mathematical terms rather than in terms of the real world which may often be imprecise in nature. In many cases a complex real world design problem can be divided into a sequence of simpler sub-problems, which can be best solved by experienced designers using information expressed by statements such as words or phrases which are said to be values of given linguistic variables. The theory of fuzzy sets is a useful tool with which these statements can be interpreted with the use of membership functions, which express numerically the meaning of the linguistic variables. The construction of a membership function can be accomplished with the cooperation and assistance of a panel of experienced engineers in specific areas. The resulting design process can then be performed in a logical manner following the theory of fuzzy decision-making to obtain a meaningful answer to the originally complex problem.

The general framework for fuzzy optimum design follows the fuzzy decision-making proposed by Bellman and Zadeh [2], Zimmermann [3], Verdagay [4], Rao [5], etc. Their approach was to compute the confluence as a fuzzy set on decision space of the fuzzy goals and fuzzy constraints. Then the optimum design is a design which maximizes the membership function of the resultant fuzzy set.

The objective of the present paper is to formulate the problem of the structural optimization for the case of fuzzy load uncertainty. By defining the membership functions of the objective function and constraints, based on the notion of maximizing set by Werners [6], the optimization process is transformed into a fuzzy optimization problem, which, in turn, can be solved following the framework of conventional fuzzy decision making theory. The first step in the process is to perform the "crisp" optimization to obtain the results for the use of defining maximum set. A solution procedure based on sequential linear programming is utilized, and a well-known ten bar truss example (Haftka and Gürdal, [7]) is used to demonstrate the procedure. The results are compared with those obtained by the convex optimization and anti-optimization in our previous paper (Elishakoff, Haftka and Fang, [8]).

2. FORMULATION

Original Problem with Fuzzy Parameters

Consider the problem of designing a structure so as to minimize an objective function **φ(X),** where X is a vector of design variables. The problem is formulated as follows

$$\begin{aligned} &\underset{\mathrm{X}}{\text{minimize}}\ \varphi(\boldsymbol{X}) \\ &\text{such that } g_j(\boldsymbol{X},\boldsymbol{p}) \geq 0 \quad j = 1,\ldots,n_g \end{aligned} \tag{1}$$

where g_j are constraint functions, $\boldsymbol{p}$ is the uncertain parameter vector and n_g is the total number of constraints.

The objective function and constraints may depend on parameters, such as external forces, material density and allowable stresses, that are best described as fuzzy variables, with specified membership functions. That is, we assume that the set of parameters $\boldsymbol{p}$ describing the problem is known to belong to a fuzzy set $\boldsymbol{C_P}$

$$\boldsymbol{p} \in C_p \tag{2}$$

Transformed System

While solving the present optimization problem with fuzzy parameters involved is usually difficult and numerically cumbersome, it is suitable to seek a somewhat "equivalent" problem where only a fuzzy objective function and fuzzy constraints instead of several fuzzy parameters are involved. From the notion of maximizing set by Werners [6], it is possible to define the membership functions of the objective function and the constraints, and hence transform the original problem into an ordinary fuzzy optimization problem.

The "equivalent" problem in fuzzy environment, can be stated as where the fuzzy subset $\boldsymbol{G}_j$ denotes the allowable region for the constraint function g_j, the bold face symbols of $\boldsymbol{\varphi}$ and $\boldsymbol{G}_j$ indicate that the operators or variables contain fuzzy information. The

$$\begin{aligned}&\underset{\mathbf{X}}{\text{minimize}}\ \boldsymbol{\varphi}(\boldsymbol{X})\\&\text{such that } g_j(\boldsymbol{X}) \in \boldsymbol{G}_j \quad j = 1, \dots, n_g\end{aligned} \tag{3}$$

constraint $g_j \in \boldsymbol{G}_j$ means that g_j is a member of a fuzzy subset $\boldsymbol{G}_j$ in the sense of $\mu_{\boldsymbol{G}j}(g_j) \geq 0$, where $\mu(.)$ is the membership function. The fuzzy feasible region concerning all the constraints is defined as

$$S = \bigcap_{j=1}^{m} \boldsymbol{G}_j \tag{4}$$

The membership degree of any design vector $\boldsymbol{X}$ to fuzzy feasible region $\boldsymbol{S}$ is given by

$$\mu_s(\boldsymbol{X}) = \min_{j=1,2,\cdot\ ,m} \{\mu_{\boldsymbol{G}_j}[g_j(\boldsymbol{X})]\} \tag{5}$$

i.e. the minimum degree of satisfaction of the design vector $\boldsymbol{X}$ to all of the constraints. A design vector $\boldsymbol{X}$ is considered feasible provided $\mu_s(\boldsymbol{X}) \geq 0$, and the differences in the membership degrees of two design vectors $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ imply nothing but variations in the minimum degrees of satisfaction of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ to the constraints.

The fuzzy feasible region concerning the objective function is denoted as $\boldsymbol{T}$, determined by $\mu_\varphi(\boldsymbol{X}) \geq 0$, after the membership function of the objective function $\mu_\varphi(\boldsymbol{X})$ is defined (which will be discussed in the next section). Thus the optimum solution should be found within a fuzzy domain $\boldsymbol{D}$,

$$\boldsymbol{D} = \boldsymbol{S} \cap \boldsymbol{T} \tag{6}$$

Its membership function is

$$\mu_D(\boldsymbol{X}) = \min\{\mu_\varphi(\boldsymbol{X}), \min_{j=1,2,\cdot\ ,m} \mu_{\boldsymbol{G}_j}[g_j(\boldsymbol{X})]\} \tag{7}$$

If the membership function of $\boldsymbol{D}$ is unimodal and has a unique maximum, then the optimum solution $\boldsymbol{X}^*$ is one for which the membership function is maximum:

$$\mu_D(\boldsymbol{X}) = \max \mu_{\boldsymbol{D}}(\boldsymbol{X}), \quad \boldsymbol{X} \in \boldsymbol{D} \tag{8}$$

The fuzzy optimization problem can then be solved using ordinary nonlinear programming techniques as follows:

Find $\boldsymbol{X}$ which maximizes λ

$$\underset{\mathbf{X}}{\text{maximize}}\,\lambda \tag{9}$$

subject to

$$\begin{array}{lll} \lambda \leq \mu_{\varphi}(X) & & \\ \lambda \leq \mu_{g_j}(X), & j=1,2,\cdot,m & \\ g_i(X,p^0) \geq 0, & i=n_1,n_2,\cdot,\ n_m & \text{“crisp” constraints} \end{array} \tag{10}$$

where $\mathbf{p^0}$ is the mid-point value of the fuzzy set C_p in (2), i.e. with its membership function equals 0.5,

Definition of the membership functions

In the present case of fuzzy parameters $\boldsymbol{p}$, the definition by Werners (1984) is extended to construct the membership functions of φ and g_j. Let φ: $\boldsymbol{X} \rightarrow \boldsymbol{R^1}$ be the objective function, R is the fuzzy feasible region of $\boldsymbol{p}$, $S(R)$ is the support of R, and R_1 is α-level cut of R for $\alpha = 1$. The membership function of the goal (objective function) given solution space R is then defined as

$$\mu_{\varphi}(\boldsymbol{X}) = \begin{cases} 1 & if\ \varphi(\boldsymbol{X}) \leq \inf\limits_{R_1} \varphi \\ \dfrac{\sup\limits_{S(R)} - \varphi(\boldsymbol{X})}{\sup\limits_{S(R)} - \inf\limits_{R_1} \varphi} & if\ \inf\limits_{R_1} \varphi \leq \varphi(\boldsymbol{X}) \leq \sup\limits_{S(R)} \varphi \\ 0 & if\ \sup\limits_{S(R)} \varphi \leq \varphi(\boldsymbol{X}) \end{cases} \tag{11}$$

where $\varphi(\boldsymbol{X})$ is the objective function, $\inf\limits_{R_1} \varphi$ is the optimized objective function from the "crisp" optimization for the loading condition corresponding to $\alpha = 1$; $\sup\limits_{S(R)} \varphi$ is the optimized objective function from the "crisp" optimization through convex modeling of loads whose multi-dimensional bounded load space is produced by setting $\alpha = 0$. For the details of the convex modeling of uncertainty, the reader may consult with the monograph by Ben-Haim and Elishakoff [9] and with review by Elishakoff [10].

The membership functions of the constraints is defined as
where $\inf \boldsymbol{G}_j$ denotes minimum value of the fuzzy set $\boldsymbol{G}_j$ and $\sup \boldsymbol{G}_j$ denotes maximum value of $\mathbf{G}_j$ with $\boldsymbol{p} = \boldsymbol{p}^0$. In this way, the optimization problem with fuzzy parameters can be transformed into an ordinary fuzzy optimization problem, which contains only fuzzy objective function and fuzzy constraints, from the nominal values and bounded ranges of the fuzzy parameters.

3. TEN-BAR TRUSS EXAMPLE

3.1 Analysis

$$\mu_{g_j}(\boldsymbol{X}) = \begin{cases} 1 & if\ g_j(\boldsymbol{X}) \leq\ inf\ \boldsymbol{G}_j \\ \dfrac{g_j(\boldsymbol{X}) - inf\,\boldsymbol{G}_j}{sup\,\boldsymbol{G}_j - inf\,\boldsymbol{G}_j} & if\ inf\,\boldsymbol{G}_j \leq g_j(\boldsymbol{X}) \leq sup\,\boldsymbol{G}_j \\ 0 & if\ g_j(\boldsymbol{X}) \geq sup\,\boldsymbol{G}_j \\ with\ \boldsymbol{p} = \boldsymbol{p}^0 & and\ j = 1, 2, \cdot, m \end{cases} \tag{12}$$

Consider the simple ten-bar truss (Fig. 1), which has been investigated in various optimization contexts (Haftka and Gürdal [7]; Elishakoff, Haftka and Fang [8]; Vanderplaats and Salajeghah [11]; Zhou and Rozvany [12]). The minimum weight design obtained by varying the cross-sectional areas of the truss members is sought subject to the vertical displacement constraint at joint 2, stress constraints and minimum gage constraints of 0.1 in^2. The maximum allowable stress in each member is same in tension and compression. The allowable vertical displacement of joint 2 is 5 inches and the allowable stress is 25 ksi for all bars except bar 9, whose allowable stress is 75 ksi. The truss is made of aluminum with weight density of 0.1 lb/in^3 and elasticity modulus of 10^4 ksi. The bar length L is 360 in. Joint 4 is subjected to vertical load P_1, joint 2 is subjected to vertical load P_2 and horizontal load P_3. Haftka and Gürdal [7], Vanderplaats and Salajeghah [11], Zhou and Rozvany [12] treated the problem of optimization of the truss for a specific combination of the loads, namely for $P_1 = P_2 = 100$ kips, $P_3 = 0$. For the purpose of illustration we consider the optimization subject to stress and displacement constraints with fuzzy parameters of loads. Without loss of generality, the displacement constraints are treated as fuzzy constraints while stress constraints will be taken as crisp constraints. The support of $\boldsymbol{p}$, namely when $\alpha = 0$, P_k, (k = 1, 2, 3) are varying in the following three-dimensional box

$$C_p = \left\{ \begin{array}{c} P_1^l \leq P_1 \leq P_1^u \\ P_2^l \leq P_2 \leq P_2^u \\ P_3^l \leq P_3 \leq P_3^u \end{array} \right\} \tag{13}$$

where P_k^l and P_k^u are lower and upper bounds of the load P_k, respectively. Note that the structure is statically indeterminate, with the degree of indeterminacy equal to two. To determine the stresses in the bars we employ the standard flexibility method. The unknown axial forces in the members are denoted N_i, where i is the sequence number of the bar, $i = 1, 2, ..., 10$. They satisfy the set of following equilibrium and compatibility equations

$$
\begin{aligned}
&N_1 = P_2 - \tfrac{\sqrt{2}}{2}N_8\\
&N_2 = -\tfrac{\sqrt{2}}{2}N_{10}\\
&N_3 = -P_1 - 2P_2 + P_3 - \tfrac{\sqrt{2}}{2}N_8\\
&N_4 = -P_2 + P_3 - \tfrac{\sqrt{2}}{2}N_{10}\\
&N_5 = -P_2 - \tfrac{\sqrt{2}}{2}N_8 - \tfrac{\sqrt{2}}{2}N_{10}\\
&N_6 = -\tfrac{\sqrt{2}}{2}N_{10}\\
&N_7 = \sqrt{2}(P_1 + P_2) + N_8\\
&N_9 = \sqrt{2}P_2 + N_{10}\\
&a_{11}N_8 + a_{12}N_{10} = b_1\\
&a_{21}N_8 + a_{22}N_{10} = b_2
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
a_{11} &= \left(\frac{1}{A_1} + \frac{1}{A_3} + \frac{1}{A_5} + \frac{2\sqrt{2}}{A_7} + \frac{2\sqrt{2}}{A_8}\right)\frac{L}{2E}\\
a_{12} &= a_{21} + \frac{L}{2A_5E}\\
a_{22} &= \left(\frac{1}{A_2} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} + \frac{2\sqrt{2}}{A_9} + \frac{2\sqrt{2}}{A_{10}}\right)\frac{L}{2E}\\
b_1 &= \left[\frac{P_2}{A_1} - \frac{P_1 + 2P_2 - P_3}{A_3} - \frac{P_2}{A_5} - \frac{2\sqrt{2}(P_1 + P_2)}{A_7}\right]\frac{\sqrt{2}L}{2E}\\
b_2 &= \left(\frac{\sqrt{2}(P_3 - P_2)}{A_4} - \frac{\sqrt{2}P_2}{A_5} - \frac{4P_2}{A_9}\right)\frac{L}{2E}
\end{aligned}
\tag{15}
$$

where A_i is the cross-sectional area of bar i, and E is the modulus of elasticity. Once the axial loads N_i in the members are calculated, the vertical displacement δ_2 of joint 2 can be found from expression

$$
\delta_2 = \left[\sum_{i=1}^{6}\frac{N_i^0 N_i}{A_i} + \sqrt{2}\sum_{i=7}^{10}\frac{N_i^0 N_i}{A_i}\right]\frac{L}{E},
\tag{16}
$$

where N_i^0 are found from equations, similar to Eqs. (14) and (15) with a substitution $P_1 = P_3 = 0$ and $P_2 = 1$.

The minimum weight design formulation corresponding to (3) is

$$\begin{aligned}
&\underset{A_i}{\text{minimize}}\ W = \sum_{i=1}^{10}(\rho L_i A_i) = \rho L\left(\sum_{i=1}^{6} A_i + \sqrt{2}\sum_{i=7}^{10} A_i\right)\\
&\text{such that } A_i \geq A_0 = 0.1\ in^2\\
&g_j(A) = |\sigma_j(A)| = \max_{P_k^\ell \leq P_k \leq P_k^u} \frac{|N_j(p,A)|}{A_j} \leq \sigma_{j,allow}, \quad (k=1,2,3)\\
&\max_{P_k^\ell \leq P_k \leq P_k^u} \delta_2(p,A) \leq \delta_{2,allow} = 5\ in
\end{aligned} \tag{17}$$

where σ_j and $\sigma_{j,allow}$ are the stress and maximum allowable stress, for bar j, and δ_i and $\delta_{i,\ allow}$ are the displacement and maximum allowable displacement for bar i.

For some specific values *of* $\boldsymbol{p}^0$, the problem can be solved through sequential linear programming (SLP) (Haftka and Gürdal [7]). SLP starts with a trial design $A_i^{(0)}$, and replaces the constraints with linear approximation obtained from a Taylor series expansion about $A_i^{(0)}$. Thus the problem (17) is replaced by

$$\begin{aligned}
&\underset{A_j}{\text{minimize}}\, W = \rho L\left(\sum_{i=1}^{6} A_i + \sqrt{2}\sum_{i=7}^{10} A_i\right)\\
&\text{such that } A_i \geq 0.1\\
&\left.\begin{aligned}
&g_j(P^0, A_1^{(0)}, A_2^{(0)}, \ldots, A_{10}^{(0)}) + \sum_{i=1}^{10}(A_i - A_i^{(0)})\left(\frac{\partial g_j}{\partial A_i}\right)_{A_i^{(0)}} - \sigma_{j,allow} \leq 0\\
&\delta_2(P^0, A_1^{(0)}, A_2^{(0)}, \ldots, A_{10}^{(0)}) + \sum_{i=1}^{10} + \frac{\partial \delta_2}{\partial A_i}(A_i - A_i^{(0)}) \leq 5
\end{aligned}\right\} P_k \in (P_k^l, P_k^u)
\end{aligned} \tag{18}$$

For the fuzzy parameters, according to the formulation addressed in section 2, the present optimization problem with fuzzy parameters can be transformed into an ordinary fuzzy optimization process by seeking an "equivalent" problem. $\underset{S(R)}{sup}$ W, $\underset{R_1}{inf}$ *W*, $sup\boldsymbol{G}_j$, $inf\boldsymbol{G}_j$ will be needed in the definition of membership functions of objective function and constraints.

Because N_j and δ_2 are linear functions of $\boldsymbol{p}$, the extreme cases of the loads can be found at the vertices of the box. Thus $\underset{S(R)}{sup}$ W may be obtained through the following optimization process:

$$\begin{aligned}
&\underset{A_i}{\text{minimize}}\ W = \rho L\left(\sum_{i=1}^{6} A_i + \sqrt{2}\sum_{i=7}^{10} A_i\right)\\
&\text{such that } A_i \geq 0.1\\
&\left.\begin{aligned}&\frac{|N_j|}{A_j} \leq \sigma_{j,allow}\\ &\delta_2 \leq \delta_{2,allow}\end{aligned}\right\} P_k \in (P_k^l, P_k^u)
\end{aligned} \tag{19}$$

That is, the stress and displacement constraints are enforced at all 8 vertices of the load-domain box.

On the other hand, $\underset{R_1}{inf}\, W$ can be obtained by choosing minimum weight for all optimization processes at the vertices of the loads.

$$\underset{R_1}{inf}\, W = \min(W_{v_1}, W_{v_2}, \ldots, W_{v_8}) \tag{20}$$

where W_{vl} and $g_{i,\ vi}$ can be obtained as follows:

Then the membership function of the objective function, i.e. weight W, given solution space R is then defined as follows (Fig. 2) where ***W(X)*** is the objective function, $\underset{S(R)}{sup}\, W$ is the optimized objective function from the "crisp" optimization under non-fuzzy uncertainties of loads, whose magnitude belongs to the bounded

$$\begin{aligned}
&\underset{A_i}{\text{minimize}}\ W_{v_i} = \rho L\left(\sum_{i=1}^{6} A_i + \sqrt{2}\sum_{i=7}^{10} A_i\right)\\
&\text{such that } A_i \geq 0.1\\
&\left.\begin{aligned}&\frac{|N_j|}{A_j} \leq \sigma_{j,allow}\\ &\delta_2 \leq \delta_{2,allow}\end{aligned}\right\} P_k\ is\ at\ vertex\ v_i
\end{aligned} \tag{21}$$

$$\mu_W(\boldsymbol{X}) = \begin{cases} 1 & if\ W(\boldsymbol{X}) \leq \underset{R_1}{inf}\, W \\ \dfrac{\underset{S(R)}{sup}\, W - W(\boldsymbol{X})}{\underset{S(R)}{sup}\, W - \underset{R_1}{inf}\, W} & if\ \underset{R_1}{inf}\, W \leq W(\boldsymbol{X}) \leq \underset{S(R)}{sup}\, W \\ 0 & if\ \underset{S(R)}{sup}\, W \leq W(\boldsymbol{X}) \end{cases} \tag{22}$$

set, and $\underset{R_1}{inf}\, w$ is the optimized objective function from the "crisp" optimization for one specific point in the multi-dimensional load space under which the corresponding *W(X)* is the minimum among the *W(X)*'s of all other points of loads in the bounded set.

According to Eq. (12), the membership functions of the constraint of displacement 62 can be represented as follows (Fig. 3)

$$\mu_{\delta_2}(\boldsymbol{X}) = \begin{cases} 1 & if\ \delta_2(\boldsymbol{X}) \leq inf\ \delta_2 \\ \dfrac{\delta_2(\boldsymbol{X}) - inf\ \delta_2}{sup\ \delta_2 - inf\ \delta_2} & if\ inf\ \delta_2 \leq \delta_2(\boldsymbol{X}) \leq sup\ \delta_2 \\ 0 & if\ \delta_2(\boldsymbol{X}) \geq sup\ \delta_2 \end{cases} \tag{23}$$

where $inf\delta_2$ and $sup\delta_2$ are chosen according to the problem itself. Usually $sup\delta_2$ is chosen as $\delta_{2,\text{allow}}$ and $inf\delta_2$ is chosen in the neighborhood of $\delta_{2,\text{allow}}$.

3.2 Numerical results

Minimum weight designs were obtained first with no uncertainty (that is $P_k^l = P_k^u$; $k = 1, 2, 3$). The results obtained for $P_1 = P_2 = 100$ kips, $P_3 = 0$ and no displacement constraint coincide with those reported ed by Haftka and Gürdal [7], who utilized the finite element method in conjunction with several optimization techniques. For the same loads but with the additional displacement constraint, the results coincide with those derived by Zhou and Rozvany [12]. The results of optimization for nominal values of the loads $P_1 = P_2 ==$ 100 kips and $P_3 = 400$ kips are listed in Table I which indicates a total weight of 1598.6 lb.

TABLE I The Cross-Sectional Areas of Optimal Truss under Nominal Loads

Bar's Serial Number	*Cross-Sectional Areas (in^2)*
1	4.0354
2	0.1000
3	4.0354
4	12.1000
5	3.8646
6	0.1000
7	11.2637
8	0.1000
9	2.7577
10	0.1414
Weight: 1598.62 lb	

For the case of uncertain loads, the stress and displacement constraints in (19) are enforced at the 8 vertices

$$\begin{aligned} v_1 &= (P_1^l, P_2^l, P_3^l) \\ v_2 &= (P_1^u, P_2^l, P_3^l) \\ v_3 &= (P_1^l, P_2^l, P_3^u) \\ v_4 &= (P_1^u, P_2^l, P_3^u) \\ v_5 &= (P_1^l, P_2^u, P_3^l) \\ v_6 &= (P_1^u, P_2^u, P_3^l) \\ v_7 &= (P_1^l, P_2^u, P_3^u) \\ v_8 &= (P_1^u, P_2^u, P_3^u) \end{aligned} \tag{24}$$

We consider as an example a case with 10 percent load uncertainty, that is $P_l^l = P_2^l = 90$ kips, $P_l^u = P_2^u = 110$ kips, $P_3^l = 360$ kips, $P_3^u = 440$ kips. The results of optimization are included in Table II. The objective function weight equals 1949.9 lb, indicating a 22 percent increase over the nominal case due to the uncertainty. This weight is taken as $\sup_{S(R)} W$. Table III shows the weight of optimal truss under the loads corresponding to eight different vertices, and vertical displacement at joint 2. $sup\delta_2$ and $inf\delta_2$ are 5 (in) and 0 (in), respectively.

TABLE II The Cross-Sectional Areas of Optimal Truss Under Uncertainty

Bar's Serial Number	*Cross-Sectional Areas (in^2)*
1	4.4638
2	0.1000
3	6.8230
4	14.0414
5	4.3064
6	0.1000
7	12.3776
8	0.1000
9	4.6256
10	0.1000
δ_2: 2.2038 in Weight: 1949.89 lb	

3.3 Comparison with α-cut method

In this section the result from introducing the membership function of the loads directly is obtained through the level-cut method, which is often used in optimization studies associated with fuzzy-subsets modeling of

TABLE III The Weight of Optimal Truss at Various Load Vertices

Load Vertices Number				*Weight (lb)*
100,	100,	0,	400	1598.62
90,	90,		360	1440.22
90,	90,		440	1670.62
90,	110,		360	1497.82
90,	110,		440	1728.22
110,	90,		360	1469.02
110,	90,		440	1699.42
110,	110,		360	1526.62
110,	110,		440	1757.02

TABLE IV The Cross-Sectional Areas of Optimal Truss Under Fuzzy Environment

Bar's Serial Number	*Cross-Sectional Areas (in^2)*
1	4.2750
2	0.1000
3	4.0353
4	12.0381
5	4.1661
6	0.1000
7	11.2638
8	0.1000
9	5.9143
10	0.1000
Weight: 1774.47 lb	

uncertainty. The reader is referred to several studies (Adali [13]; Dong [14]; Mohandas, et al, [15]; Yeh and Hsu [16]; Wang and Wang [17]) for examples of the application of the fuzzy set theory to optimal design problems. In this study two membership functions are used to compare the results with those yielded by the method proposed. First the following triangular membership function is utilized where $P_l^l = P_2^l = 90$ kips, $P_i^u = P_2^u = 110$ kips, $P_3^l = 360$ kips, $P_3^u = 440$ kips (Fig. 4). According to Adali (1991), the optimization procedure proceeds as follows: first the uncertainty level is chosen, i.e. is specified. Making α-cuts yields the values of $P_{i,\alpha}^{max}$ and $P_{i,a}^{min}$. Then loads are varied in the box $P_{i,\alpha}^{min} \le P_i \le P_{i,\alpha}^{max}$ with attendant evaluation of optimal weight. For ten-bar truss problem, it is found that when α=0.4628, the corresponding minimized weight equals

1774.48 lb. In the real-world design problems selecting a is subjective, although use of

$$\mu_{P_i}(P_i) = \begin{cases} 0 & if\ P_i \leq P_i^l\ or\ P_i \geq P_i^u \\ \dfrac{P_i - P_i^l}{P_i^0 - P_i^l} & if\ P_i^l \leq P_i \leq P_i^0 \\ \\ \dfrac{P_i - P_i^u}{\quad} & if\ P_i^0 < P_i < P_i^u \end{cases} \tag{25}$$

the α-cut method is computationally inextensive. The present method does not require the above subjective choice of parameters, and therefore appears to be superior to the α level-cut method.

4. CONCLUDING REMARKS

The idea of combining the fuzzy-sets-based and anti-optimization modeling of uncertainty is proposed in this paper. An example of minimum weight design of a ten-bar truss structures subjected to the set of uncertain bounded loads is employed to illustrate the applications of the proposed method. The uncertainty in variation of the loads is assumed to be confined to a multi-dimensional box, with vertices corresponding to lower and/or upper bounds of different loads. The previous results from the optimization through convex modeling of the loads are utilized to define the membership functions of both the objective function and constraints. Comparison with the α-cut method is performed.

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