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Emergency Control of Unstable Behavior of Nonlinear Systems Induced by Fault

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This paper presents the emergency control approach intended to urgently return to the stability basin the system states affected by abrupt changes in certain system coefficients on a short time interval. Because of its short duration, the modeling of both the fault and controller involves δ-functions significantly simplifying analysis and control of fault phenomena. The design of an emergency controller is based on the technique for computing fault-induced jumps of the system states, which is described in the paper. An emergency controller instantaneously returning states of a sample nonlinear system to its stability basin is designed.

Keywords: Nonlinear systems; Emergency control; Fault modeling

AMS Subject Classification: 34K; 93D

1 INTRODUCTION

The impulse control technique based on δ-functions as controls was applied to the optimal control problems in the field of spacecraft navigation [1] and heat conduction [2], the filtering problems over discontinuous observations [3], and others. This paper makes an attempt to extend the application domain of impulse control to the

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problems of fault description and compensation as well as stability control. The impulse control designed for urgent fault compensation is called an emergency control, which underlines instantaneous action of the developed control methods.

This paper presents the concept of emergency control, which is applied to a dynamic system in the case of urgent necessity to change a system state affected by fault. It is assumed that system fault significantly affects the operation of a relatively small subsystem of the initial system on a short time interval and clears up at the end of this interval. Such fault results in pseudoimpulsive behavior of a relatively small number of the system coefficients, which abruptly increase to the peak values and abruptly return to the nominal values, as it occurs in transient stability problems for power systems. The pseudoimpulsive coefficients are modeled by δ-functions, which significantly simplify subsequent analysis and control of the faulted system. Thus, the initial system-governing equation becomes an equation in distributions, which describes the system fault. The solution of an equation in distributions is defined as a vibrosolution [4], whose jumps occur at points where δ-functions are activated. A number of examples, where jumps of the system state can be computed analytically, are given in the paper. Otherwise, system state jumps are computed through numerical integration of a subsystem, which is significantly reduced in comparison with the initial one.

The emergency control is designed to urgently return to the stability basin the system states affected by fault. For this purpose, the emergency controller introduces δ-functions into the system equation. This method is applied to a modification of the Van-der-Pol system, where emergency controller generates a jump of the system state into the stability basin, thus preventing the system state from transition to infinity.

The paper is organized as follows. A fault model is described in Section 2. The basic technique for computing jumps in system states affected by fault is given in Section 3. Examples of analytic computation of state jumps are presented in Section 4. The concept of emergency control is introduced in Section 5. The emergency control method is applied to a modified Van-der-Pol system in Section 6. Properties of the proposed method are discussed in Section 7. Section 8 concludes this study.

2 FAULT MODELING

Let us first describe the application of δ-functions to modeling of a system fault on a short time interval.

Consider an equation governing a dynamic system

$$\dot{x}(t) = f(x,t) + k(t)g(x,t), \quad x(t_0) = x_0, \quad x \in R^N, \quad f(x,t), g(x,t) \in R^N, \quad k(t) \in R^{N\times N}. \tag{1}$$

Assume that, due to a short system fault, the coefficients $k(t)$ change in a pseudoimpulsive manner on a short time interval $[t_0, t_0 + \Delta t]$. Namely, the coefficients $k(t)$ abruptly increase to their peak values and return to the pre-fault values. This induces abrupt changes in the system state on the interval $[t_0, t_0 + \Delta t]$. The problem is to find the post-fault system state $x(t_0 + \Delta t)$, or the system state jump $x(t_0 + \Delta t) - x(t_0)$, provided that a pre-fault state $x(t_0)$ is given. The determination of the post-fault system state or system state jump is necessary for forming the emergency control.

Suppose that a fault affects coefficients only from a small "fault" sub-system of the initial system. Because of lack of accurate knowledge and observation of the faulted coefficients $k(t)$, these coefficients are represented as δ-functions with the corresponding intensities, which are assumed the peak values of the faulted coefficients or estimated using a data record of the fault behavior. Such modeling of pseudoimpulsive behavior of the faulted coefficients is physically motivated and simplifies computation of state jumps.

3 COMPUTATION OF SYSTEM STATE JUMPS

The coefficients $k(t)$ in (1) are replaced by δ-functions with intensities $\mu \in R^{N\times N}$. The intensities are measured or computed as $\mu = M\Delta t$, where $M = \sup_t k(t)$, $t \in [t_0,\ t_0 + \Delta t]$. Then, Eq. (1) takes the form

$$\dot{x}(t) = f(x,t) + \mu g(x,t)\delta(t - t_0), \quad x(t_0) = x_0. \tag{2}$$

Let us denote $b(x,\ t) = \mu g(x,\ t)$ and $u(t) = \chi(t - t_0)$, where $\chi(t - t_0)$ is a Heaviside function and $\mathrm{d}\chi(t - t_0)/\mathrm{d}t = \delta(t - t_0)$. Thus, Eq. (2) can be rewritten as

$$\dot{x}(t) = f(x,t) + b(x,t)\dot{u}(t), \quad x(t_0) = x_0. \tag{3}$$

Equation (3) with a bounded variation function $u(t)$ is an equation in distributions, whose solution (that is referred to as a vibrosolution in [4]) is a discontinuous function of bounded variation. The definition of a vibrosolution is presented below and is followed by the theorems describing computation of vibrosolution jumps. The theorem proofs are given in Appendix.

DEFINITION The left-continuous function $x(t)$ is said to be a vibrosolution of Eq. (3) if the $*$-weak convergence of an arbitrary sequence of absolutely continuous nondecreasing functions $u^k(t)$ to a nondecreasing function $u(t)$ in the bounded variation functions space

$$*\text{-}\lim u^k(t) = u(t), \quad k \to \infty, \ \ t \geq t_0,$$

implies the analogous convergence

$$*\text{-}\lim x^k(t) = x(t), \quad k \to \infty, \ \ t \geq t_0,$$

of the corresponding solutions $x^k(t)$ of the equation

$$\dot{x}^k(t) = f(x^k,t) + b(x^k,t)\dot{u}^k(t), \quad x^k(t_0) = x_0,$$

and $x(t)$ is obtained regardless of a choice of an approximating sequence $\{u^k(t)\}$, $k = 1,2\ldots$

Remark The $*$-weak convergence in the bounded variation functions space

$$*\text{-}\lim x^k(t) = x(t), \quad k \to \infty, \ \ t \geq t_0,$$

takes place if and only if the following conditions hold

(1) $\lim \|x^k(t_0) - x(t_0)\| = 0, \ k \to \infty, \ t \geq t_0$,
(2) $\lim \|x^k(t) - x(t)\| = 0, \ k \to \infty, \ t \geq t_0$, in all continuity points of the function $x(t)$,
(3) $\sup_k Var[t_0,T]x^k(t) < \infty$ for any $T \geq t_0$, where $Var[a,b]f(t)$ denotes variation of a function $f(t)$ on an interval $[a,b]$.

THEOREM 1 *Let* (1) *the functions* $f(x,t)$ *and* $b(x,t)$ *be piecewise continuous in* x, t *and satisfy the one-side Lipschitz condition in* x (*see* [5]), (2) *the function* $b(x,t)$ *have piecewise continuous derivatives in* x *and* t: $\partial b(x,t)/\partial x$, $\partial b(x,t)/\partial t$, *and* (3) *the* $N \times N$*-dimensional system in differentials*

$$\frac{\mathrm{d}\xi(z,\omega,u,s)}{\mathrm{d}u} = b(\xi,s), \quad \xi(\omega) = z, \tag{4}$$

be solvable for arbitrary initial values $\omega \in R^N$, $z \in R^N$ *inside a cone of positive directions* $K = \{u \geq \omega \mid u_i \geq \omega_i,\ i = 1,\dots,N\}$ *and* $s \geq t_0$. *Then*: *there exists the only vibrosolution* $x(t)$ *of Eq.* (3).

THEOREM 2 *A vibrosolution is also the only solution of the following equation with a measure*

$$\mathrm{d}x(t) = f(x,t)\,\mathrm{d}t + \sum_{t_j} G(x_{0_j}, u_{0_j}, \Delta u(t_j), t_j)\,\mathrm{d}\chi(t - t_j), \tag{5}$$

$$x(t_0) = x_0,$$

where $G(z,\omega,u,s) = \xi(z,\omega,\omega+u,s) - z$, *and* $\xi(z,\omega,u,s)$ *is a solution of the system in differentials* (4); t_j *are points where* δ*-function is activated*, $\chi(t - t_j)$ *is a Heaviside function*, x_{0_j} *is a value of* $x(t)$ *before a jump*, u_{0_j} *is a value of* $u(t)$ *before a jump*, *and* $\Delta u(t_j)$ *is a jump of* $u(t)$ *at a point* t_j.

Thus, Eqs. (4) and (5) enable us to compute the jumps of the Eq. (1) state $x(t)$, which are induced by pseudoimpulsive behavior of the coefficients $k(t)$. Explicit analytic formulas for a jump $\Delta x(t_0) = G(x_0, 0, 1, t_0)$ can be obtained in special cases, and numerical simulation of the significantly reduced fault subsystem yields the jumps values in other cases. The stability of a vibrosolution (in particular, a value of its jump) with respect to $*$-weak approximations of a Heaviside function enables us to use any approximation for numerical computation of a jump. For example, a pseudoimpulsive matrix $k(t)$ with intensity $\mu\Delta t$ can be represented as a constant μ on an interval of length Δt, 2μ on an interval of length $\Delta t/2$, or another $*$-weak approximation of a Heaviside function. All possible approximations yield the same limit, which is equal to a vibrosolution jump as $\Delta t \to 0$. Moreover, it can be proved that if

$g(x)$ is time-invariant, $k(t)$ remains constant during a time interval with length ΔT, and $k\Delta T = \mu\Delta t$, then the integral expression $x(t) = \int_{t_0}^{t_0+\Delta T} k(t)g(x)\,\mathrm{d}t$, which should be used for numerical computation, is the same as for computing a jump by virtue of Eq. (5).

The numbers of both coupled equations and terms in each of these equations in the fault subsystem, which is used for computation of a vibrosolution jump, are significantly reduced in comparison with the initial system. This allows fast and, possibly, on-line numerical computation of the state jumps.

4 EXAMPLES OF EXPLICIT COMPUTATION OF STATE JUMPS

In Examples 1 and 2, only the fault subsystems are given. The nonimpulsive terms are insignificant for computation of jumps (because they do not affect the function $G(z, \omega, u, s)$ in Eq. (5)) and are omitted for simplicity.

1. Let us consider a system

$$\dot{x} = k(t)x^n, \quad x(t_0) = x_0, \ x \in R,$$

which will be used later for design of an emergency controller for a modified Van-der-Pol system. Assuming that the intensity of the coefficient $k(t)$ is equal to μ, we obtain

$$\Delta x = \begin{cases} ((1-n)\mu + x_0^{1-n})^{1/(1-n)} - x_0, & \text{if } n \neq 1, \\ x_0(\exp(\mu) - 1), & \text{if } n = 1. \end{cases}$$

This result readily follows from the fact that $\xi = ((1-n)\mu u + x_0^{1-n})^{1/(1-n)}$ and $\xi = x_0\exp(\mu u)$ are the solutions of the systems (4) in these cases, respectively. For $n > 1$, Δx is equal to ∞, if x_0 and μ satisfy the condition

$$x_0^{1-n} = (n-1)\mu.$$

2. Consider another system equation related to the theory of transient stability of power networks,

$$\dot{x} = k_1(t)\sin(ax) + k_2(t)\cos(ax), \quad x(t_0) = x_0, \ x \in R^N,$$

where μ_1 and μ_2 are intensities of $k_1(t)$ and $k_2(t)$ at a point t_0. Since $\xi = a^{-1}\{2\arctan[\exp(a(\mu_1^2+\mu_2^2)^{1/2}u) + \tan((ax_0+\theta)/2)] - \theta\}$ is the solution of the system (4) in this case, we obtain

$$\Delta x = a^{-1}\{2\arctan[\exp(a(\mu_1^2+\mu_2^2)^{1/2}) + \tan((ax_0+\theta)/2)] - \theta\} - x_0,$$

where $\theta = \mu_1^{-1}\mu_2 e$, and $e = (1,\ldots,1)$ is the unit N-dimensional vector.

3. Finally, consider a Riccati equation for the estimate variance in the Kalman–Bucy filter

$$\dot{P} = AP + PA^* + GG^* - PC^*HCP, \quad P(t_0) = P_0, \;\; P \in R^{N\times N},$$

where P is the estimate variance, G and H^{-1} are variances of Gaussian noises, and C is a transition matrix in an observation equation. If H changes pseudoimpulsively on an interval $[t_0, t_0+\Delta t]$, then the corresponding jump of the variance matrix P is equal to

$$\Delta P = P_0[I + C^*hCP_0]^{-1} - P_0,$$

where h is the intensity matrix for the matrix H, and I is the $N\times N$-dimensional identity matrix. The function $\xi = P_0[I + C^*hCP_0u]^{-1}$ is the solution of the system (4) for this example.

Thus, the application of δ-functions to computation of the fault-induced jumps of system states enables us either to obtain explicit analytic formulas or to significantly simplify their numerical computation.

5 CONCEPT OF EMERGENCY CONTROL

The emergency control is applied to a dynamic system in the case of urgent necessity to change back the system states affected by fault. The nominal equilibrium position of the system is considered stable with a compact stability basin, whose boundary can be estimated. Let us assume that the system state leaves the stability basin due to short fault and its further motion produces severe problems in the system operation. The emergency controller is designed to urgently return the system state to the stability basin. The fault modeling via δ-functions, which was described in Section 2, motivates the application of δ-functions to the design of emergency control. Such emergency control adequately responds to the pseudoimpulsive behavior of the faulted system.

An emergency controlled dynamic system can be written in the form

$$\dot{x}(t) = f(x,t) + \mu g(x,t)\dot{u}(t), \quad x(t_1) = x^*, \ \ x \in R^N, \tag{6}$$

where $\mu\dot{u}(t)$ is an emergency control, $\dot{u}(t) = \delta(t - t_1)$, t_1 is the point where the emergency control is active, and μ is the intensity matrix of emergency control. If $f(x,t) = 0$ and $g(x,t) = 1$, the emergency control is additive.

Equation (6), as well as (3), is an equation in distributions. The solution of (6) is defined as a vibrosolution, and its jumps are computed in accordance with Theorems 1 and 2. As noted, jumps of the emergency controlled system state (6) can be analytically computed in special cases. A number of examples are given below, where the emergency control method is applied to a modified Van-der-Pol system. Even if jumps of the system state (6) cannot be analytically computed, the number of terms necessary for numerical jump computation is reduced in comparison with the total number of terms in (6).

Let us note that the faulted system (3) and the emergency controlled system (6) are governed by equations in distributions in the same form. Thus, one can readily design the emergency control $\mu\dot{u}(t)$ compensating for the fault action. For example, if the pseudoimpulsive coefficients $k(t)$ affected by fault are represented as δ-functions with intensities ν, then the emergency control returning the system to the pre-fault state can be designed by changing sign of intensities, i.e., is equal to $-\nu\delta(t - t_1)$. However, in practice, this simple approach may fail due to control limitations.

Consider a general method for design of an emergency control $\mu\delta(t - t_1)$ moving a state of the system (6) into its stability basin. The second addition in (6) is equal to 0 everywhere, except for the point t_1 where emergency control is active. Let the initial state x^* be disposed outside the stability basin. Assume that there exists a

Lyapunov function $L(x, t)$ such that

$$S(x,t) = \mathrm{d}L(x,t)/\mathrm{d}t\,|_{\dot{x}(t)=f(x,t)} < 0$$

for $t \geq t_1$, $x \in \omega \subset \omega_0$, and ω_0 is the stability basin of (6). To move a state of (6) into the stability basin, an emergency control should generate a jump of the state in such a way that the Lyapunov function derivative is negative at the post-jump state x_1

$$S(x,t)|_{x=x_1,t=t_1} < 0, \quad x_1 = x^* + \Delta x(t_1). \tag{7}$$

In accordance with Theorem 2, the jump corresponding to an initial point x^* and an intensity vector μ is equal to

$$\Delta x(t_1) = G(x^*, 0, 1, t_1), \tag{8}$$

where $G(z, \omega, u, s) = \xi(z, \omega, \omega + u, s) - z$, and $\xi(z, \omega, u, s)$ is a solution of (4). Thus, the expressions (7) and (8) compose a closed system for determination of an intensity vector μ and, therefore, an emergency control $\mu\dot{u}(t)$. The optimal emergency control minimizing the Lyapunov function derivative after the jump can be determined as follows

$$S(x,t)|_{x=x_1,t=t_1} \to \min_{\mu}, \quad \|\mu\| = \|\mu_{\max}\|,$$
$$x_1 = x^* + \Delta x(t_1), \quad \Delta x(t_1) = G(x^*, 0, \mu, 1, t_1),$$

where the function $S(x, t)$ should be minimized over all possible intensities with the maximum available intensity norm.

6 EMERGENCY CONTROL OF A MODIFIED VAN-DER-POL SYSTEM

Let us consider the application of the emergency control method to a modified Van-der-Pol system, where the control objective is to return the system state to the stability basin, preventing it from transition to infinity.

Consider a system described by the equation

$$\mathrm{d}^2x/\mathrm{d}t^2 + \omega x + \alpha\,\mathrm{d}x/\mathrm{d}t - \beta(\mathrm{d}x/\mathrm{d}t)^3 = 0, \quad x(t_0) = x_0,\ \alpha, \beta > 0. \tag{9}$$

This system has the stable equilibrium at the origin with the stability basin bounded by the solution $x^2+(\mathrm{d}x/\mathrm{d}t)^2=r^2$ where $r=\sqrt{\alpha/\beta}$. Upon introducing the variable $v=\mathrm{d}x/\mathrm{d}t$, the Eq. (9) can be written as the system of first-order equations

$$\begin{aligned} &\mathrm{d}x/\mathrm{d}t = v, \qquad &&\mathrm{d}v/\mathrm{d}t + \omega x + \alpha v - \beta v^3 = 0, \\ &x(t_0) = x_0, \qquad &&v(t_0) = \mathrm{d}x(t_0)/\mathrm{d}t. \end{aligned} \tag{10}$$

Each trajectory outgoing from the interior of the stability basin approaches zero, i.e., $\|x(t)\| \to 0$, as $t \to \infty$, if $x_0^2 + v_0^2 < r^2$, and each trajectory starting from a point outside the stability basin tends to infinity, i.e., $\|x(t)\| \to \infty$, as $t \to \infty$, if $x_0^2 + v_0^2 > r^2$. Assume that the initial point (x_0, v_0) jumps out of the stability basin due to fault. The control objective is to urgently return the system state to the interior of the stability basin. A number of emergency controllers solving this problem are considered below.

1. Assume that additive emergency control is available. If a fault moves the system state to a position $(0, v_0)$, where $v_0 > r$, then an additive control $\mu\delta(t-t_0)$ solving the problem is included in the second equation of (10)

$$\mathrm{d}v/\mathrm{d}t + \omega x + \alpha v - \beta v^3 + \mu\dot{u}(t) = 0.$$

The intensity μ should belong to the range $v_0 - r < \mu < v_0 + r$.

If a fault moves the system to a position $(x_0, 0)$, where $x_0 > r$, then an additive control is included in the first equation of (10)

$$\mathrm{d}x/\mathrm{d}t = v - \mu\dot{u}(t).$$

The intensity μ should belong to the range $x_0 - r < \mu < x_0 + r$.

Both equations of (10) should be controlled, if a fault moves the system state to a position (x_0, v_0), $x_0 \neq 0$, $v_0 \neq 0$, $x_0^2 + v_0^2 > r^2$, which is located beyond the phase plane axes and stability basin.

2. Assume that multiplicative emergency control $\mu\dot{u}(t) = \alpha v - \beta v^3$, where α and β are impulsive coefficients, is available. Let $\alpha = \mu\delta(t-t_0)$ and $\beta = 0$. The value of μ returning the system state to the interior of the stability basin is determined as follows. Due to Theorem 2, the jump $\Delta v(t_0)$ inspired by the control $\mu\delta(t-t_0)v$ is equal to $\Delta v(t_0) = -v_0(\exp(\mu) - 1)$. Thus, the desired intensity is $\mu = \ln(1 - \Delta v(t_0)/v_0)$, where $|\Delta v| > |v_0 - \sqrt{r^2 - x_0^2}|$. This result readily follows from Example 1 of Section 4 for $n = 1$.

Analogously, if $\alpha = 0$ and $\beta = -\mu\delta(t - t_0)$, then the jump $\Delta v(t_0)$ inspired by the control $\mu\delta(t - t_0)v^3$ is equal to $\Delta v(t_0) = v_0 - (-2\mu + v_0^{-2})^{-1/2}$. Thus, the desired intensity is $\mu = (1/2) \times (v_0^{-2} - (v_0 - \Delta v(t_0))^{-2})$, where $|\Delta v| > |v_0 - \sqrt{r^2 - x_0^2}|$. This result readily follows from Example 1 of Section 4 for $n = 3$.

7 DISCUSSION

Using an additive emergency control $\mu\dot{u}(t)$ with an appropriate intensity, it is possible to return a state of the faulted system (1) to the stability basin from any post-fault position. However, the intensity resource of additive emergency control can be insufficient to return a state of the system (1) to the stability basin from any post-fault position (for example, if $Var[t_0, T](\mu u(t)) < C = const$, where $\dot{u}(t) = \delta(t - t_0)$, i.e., emergency control intensity $\mu < C$). In this case, several additive controllers $\mu_0 u(t - t_0)$, $\mu_1 u(t - t_1), \ldots, \mu_m u(t - t_m)$ operating subsequently at $t = t_0, t_1, \ldots, t_m$ solve the emergency control problem. If additive emergency control is unavailable, the question whether it is possible to return a state of the system (1) to the stability basin from a post-fault position should be resolved in each specific case. This study is simplified due to a significant system reduction associated with impulse control, which can be designed in a closed form in the case of analytic computation of state jumps.

Modeling of short impulsive behavior of system or controller coefficients by δ-functions is physically motivated and highly simplifies subsequent mathematical analysis. Intensities of an emergency control should be chosen to obey the controller objectives. Note that design of an emergency controller requires only observation of the state jumps in the fault subsystem. The jumps can be measured directly or, as shown in Section 3, can be computed if intensities of the faulted coefficients are estimated.

8 CONCLUSION

This paper presents the emergency control approach intended to urgently return the state of a dynamic system affected by fault to the stability basin. A method for design of an emergency controller is addressed and applied to a modified Van-der-Pol system, thus preventing its state from transition to infinity. The fault-modeling approach based on modeling a fault via δ-functions and computing the fault-induced jumps of the system state is described.

9 APPENDIX

Proof of Theorem 1 By virtue of the theorem conditions, the system (4) has the solution $\xi(z, \omega, u, t)$ on the cone K for $t > t_0$. Let us seek the solution of (3) corresponding to a nondecreasing function $u(t)$ in the form

$$x(t) = \xi(z(t), u_0, u(t), t), \tag{11}$$

where $u_0 = u(t_0)$ and $u(t) \geq u_0$.

In accordance with the definition of a solution of the system in differentials (4), the expression (11) implies the representation

$$x(t) = z(t) + \int_{u_0}^{u(t)} b(\xi(v), v, t)\,\mathrm{d}v,$$

or

$$x(t) = z(t) + \int_0^T b(z(t) + y(r), u_0 + w(r), t)\dot{w}(r)\mathrm{d}r, \tag{12}$$

where $T = t - t_0$ is the time, for which the trajectory of (4) reaches the point $x(t)$, and $y(r)$ is the solution of (4) corresponding to the nondecreasing function $w(r) = u(r) - u_0$.

The solvability of the system in differentials (4) on the cone K implies (see [6] for further details) that the integral form

$$\int_0^{T'} b(z(t) + y(r), u_0 + w(r), t)\dot{w}(r)\,\mathrm{d}r = 0 \tag{13}$$

is equal to zero for any nonnegative function $w(r)$ in R^m: $w(r) \in R^m$, $w_i(r) \geq 0$, $i = 1, \ldots, m$, which is piecewise smooth on the interval $[0, T']$ and equal to zero at its terminal points 0 and T'. In other words, the integral form (13) is equal to zero for any piecewise smooth loop $w(r) \in R^m$ inside the nonnegative orthant of R^m, which starts and ends at zero. Here, T' is the time of passing the loop.

Let $w(r)$ be such a piecewise smooth loop in R^m that the corresponding solution of (4) with the initial value $z(t)$ reaches the point $x(t)$ for the time $T = t - t_0$, where $w(r) \geq 0$. Then, the equality (13) takes the form

$$\begin{aligned}
&\int_0^T b(z(t) + y(r), u_0 + w(r), t)\dot{w}(r)\,\mathrm{d}r \\
&\quad = \int_T^{T'} b(z(t) + y(r), u_0 + w(r), t)\dot{w}(r)\,\mathrm{d}r \\
&\quad = \int_0^T b(z(t) + y(r), u_0 + w(r), t)\dot{w}(r)\,\mathrm{d}r \\
&\qquad + \int_0^{T'-T} b(x(t) + y(r), u(t) + w(r), t)\dot{w}(r)\,\mathrm{d}r \\
&\quad = 0. \qquad (14)
\end{aligned}$$

Upon substituting the representation (12) into (14), we obtain

$$x(t) - z(t) + \int_0^{T'-T} b(x(t) + y(r), u(t) + w(r), t)\dot{w}(r)\,\mathrm{d}r = 0,$$

i.e.,

$$z(t) = x(t) + \int_0^{T'-T} b(x(t) + y(r), u(t) + w(r), t)\dot{w}(r)\,\mathrm{d}r,$$

or

$$z(t) = x(t) + \int_{u(t)}^{u_0} b(\xi(v), v, t)\,\mathrm{d}v. \qquad (15)$$

The representation (15) implies that the inversion formula

$$z(t) = \xi(x(t), u(t), u_0, t) \qquad (16)$$

is valid for $u(t) \geq u_0$. In particular, $z(t_0) = x(t_0) = x_0$.

By the definition, a solution of the system in differentials (4) can be represented as

$$\xi(z,\omega,u,s) = z + \int_{\omega}^{u} b_1(\xi(z,\omega,v,s),v,s)\,\mathrm{d}v, \quad u \geq \omega,$$

where $b_1(\xi, v, s)$ either coincides with $b(\xi, v, s)$ or belongs to the right-hand side of the corresponding differential inclusion. The derivatives $\partial\xi/\partial z$, $\partial\xi/\partial\omega$, and $\partial\xi/\partial s$ satisfy almost everywhere, similarly to the case of a continuous right-hand side (see [7]), certain systems of linear differential equations in variations, which can be derived using the technique of differential inclusions from [5]. In the examined case, these systems are composed of linear differential equations with discontinuous right-hand sides, and their solvability on the cone K follows from the solvability of the system in differentials (4) and the existence theorem for solutions of differential equations with discontinuous right-hand sides (see [5]). The solvability of the systems of equations in variations yields the almost everywhere existence and piecewise continuity of the derivatives $\partial\xi/\partial z$, $\partial\xi/\partial\omega$, and $\partial\xi/\partial s$. Since the solution $\xi(z,\omega,u,s)$ of the system (4) is absolutely continuous, the derivative $\partial\xi/\partial u$ also exists.

Based on the almost everywhere existence of the derivatives, we obtain, using the transforming technique from [8], that $z(t)$ satisfies the equation with a discontinuous right-hand side

$$\dot{z}(t) = \varphi(z(t), u_0, u(t), t), \quad z(t_0) = x_0, \tag{17}$$

where

$$\begin{aligned}\varphi(z,u_0,u,t) = & \frac{\partial\xi(\xi(z,u_0,u,t),u,u_0,t)}{\partial z} \\ & \times f(\xi(z,u_0,u,t),u,t) \\ & + \frac{\partial\xi(\xi(z,u_0,u,t),u,u_0,t)}{\partial s}.\end{aligned}$$

The function $\varphi(z,\ u_0,\ u,\ t)$ is piecewise continuous as a combination of the piecewise continuous functions $\partial\xi/\partial z, f$, and $\partial\xi/\partial s$. Thus, using results from [5], we conclude that a solution of Eq. (17) exists.

Let $u^k(t)$, $k = 1, 2, \ldots$, where $*$-lim $u^k(t) = u(t)$, $k \to \infty$, $t \geq t_0$, be a sequence of absolutely continuous nondecreasing functions converging to $u(t)$ in the $*$-weak topology of the bounded variation functions space. Equation (3) with functions $u^k(t)$ in the right-hand side becomes an ordinary differential equation with a discontinuous right-hand side

$$\dot{x}^k(t) = f(x^k, u^k, t) + b(x^k, u^k, t)\dot{u}^k(t), \quad x^k(t_0) = x_0. \tag{18}$$

The existence and uniqueness theorem (see [5]) for a solution of an ordinary differential equation with a discontinuous right-hand side implies, in view of the theorem conditions (1) and (2), that a solution of Eq. (18) exists and is unique. The inversion formula (16) implies, in turn, that $\xi(x^k(t), u^k(t), u_0, t) = z^k(t)$ is the unique solution of the equation

$$\dot{z}^k(t) = \varphi(z^k(t), u_0, u^k(t), t), \quad z^k(t_0) = x_0. \tag{19}$$

Let $*$-lim $u^k(t) = u(t)$, $k \to \infty$, $t \geq t_0$. Based on the continuous dependence of a solution of a differential equation on its right-hand side [5], we obtain that $*$-lim $z^k(t) = z^*(t)$, $k \to \infty$, $t \geq t_0$, where $z^*(t)$ is a solution of Eq. (17). This solution is unique due to uniqueness of solutions $z^k(t)$ of (19) for pre-limiting functions $u^k(t)$. Thus, $z^*(t)$ is the vibrosolution of (17). Based on the continuity and one-to-one correspondence of the relation (11), we conclude that $x^*(t) = \xi(z^*(t), u_0, u(t), t)$ is the desired vibrosolution of Eq. (3). Moreover, $\sup_k Var[t_0, t]x^k(t) < \infty$ for $t \geq t_0$, in view of uniform boundedness of variations of the functions $z^k(t)$ and $u^k(t)$, $k = 1, 2, \ldots$ The uniform boundedness of variations of functions $z^k(t)$ and $u^k(t)$ follows from the convergence

$$*\text{-lim}\, u^k(t) = u(t), \quad *\text{-lim}\, z^k(t) = z^*(t), \quad k \to \infty, \; t \geq t_0,$$

in the $*$-weak topology of the bounded variation functions space.

Proof of Theorem 2 A proposition similar to Theorem 2 is proved in [8], assuming continuity of the functions $f(x, u, t)$, $b(x, u, t)$, $\partial b(x, u, t)/\partial x$, $\partial b(x, u, t)/\partial t$. The continuity condition is used in [8] only for proving the existence and uniqueness of a solution of Eq. (5). However, the existence and uniqueness of this solution can be proved under the conditions of Theorem 2, using the existence and uniqueness theorem [5] for a solution of a differential equation with a discontinuous right-hand side, as it was done in Theorem 1.

Substantiation of other statements of Theorem 2 does not require the continuity of the functions $f(x, u, t)$, $b(x, u, t)$, $\partial b(x, u, t)/\partial x$, $\partial b(x, u, t)/\partial t$ and can be carried over from [8].

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