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Asymptotic Analysis of the Structure of a Steady Planar Detonation: Review and Extension

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The structure of a steady planar Chapman–Jouguet detonation, which is supported by a direct first-order one-step irreversible exothermic unimolecular reaction, subject to Arrhenius kinetics, is examined. Solutions are studied, by means of a limit-process-expansion analysis, valid for Λ, proportional to the ratio of the reaction rate to the flow rate, going to zero, and for β, proportional to the ratio of the activation temperature to the maximum flow temperature, going to infinity, with the product $\Lambda\beta^{1/2}$ going to zero. The results, essentially in agreement with the Zeldovich–von Neumann–Döring model, show that the detonation consists of (1) a three-region upstream shock-like zone, wherein convection and diffusion dominate; (2) an exponentially thicker five-region downstream deflagration-like zone, wherein convection and reaction dominate; and (3) a transition zone, intermediate to the upstream and downstream zones, wherein convection, diffusion, and reaction are of the same order of magnitude. It is in this transition zone that the ideal Neumann state is most closely approached.

Keywords: Asymptotic analysis; Planar detonation; Uniformly valid solution

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1 INTRODUCTION

The purpose of this paper is to review and extend the previous investigation [1] of the structure of a steady planar Chapman–Jouguet

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detonation wave, supported by a direct first-order one-step irreversible exothermic unimolecular reaction, subject to Arrhenius kinetics. For the physically realistic case of (1) Λ, proportional to the ratio of the reaction rate to the flow rate, going to zero, (2) β, proportional to the ratio of the activation temperature to the maximum flow temperature, going to infinity, with (3) $\Lambda\beta^{1/2}$ going to zero, solutions for this structure are obtained by means of limit-process-expansion techniques.

For $\Lambda \to 0$, $\beta \geq \boldsymbol{O}(1)$ (in the notation of [1]), Zeldovich, von Neumann, and Döring (ZND) proposed that the structure of the laminar Chapman–Jouguet detonation is separable into a thin nonreactive shock-like zone followed by a thicker nondissipative deflagration-like zone. For this ZND model, however, the exact method for the coupling of these two zones is not formulated.

For $\Lambda \to 0$, $\beta \to \infty$, it is shown in the following sections that, to describe the structure, it is necessary to introduce, as in [1], a three-zone model. In addition to an upstream shock-like and a downstream deflagration-like zones, a transition zone, intermediate to these two zones, in which there is a reaction–diffusion–convection balance, must be introduced. It is demonstrated that the solutions for the (three-region, instead of the two-region [1]) upstream zone, (five-region) downstream zone, and this transition zone span the domain of the detonation.

In Section 2, the governing equations for the model for the detonation structure are derived. A discussion of the results obtained from the analysis for the structure is given in Section 3. The essential mathematical features of the three-zone model are presented in Sections 4–6.

2 THE MODEL DETONATION PROBLEM

Consider the steady planar one-dimensional Chapman–Jouguet detonation wave produced by a binary mixture of ideal gases, undergoing a direct first-order one-step irreversible exothermic unimolecular chemical reaction, $R \to P$, with no inert species present. Let x be the spatial variable in the direction of the flow in a coordinate system at rest with respect to the wave; v, the velocity component of the gas in the x-direction; p, ρ, and T, the hydrostatic pressure, density, and temperature;

W_i, Y_i, ε_i, h_i, and w_i, the molecular weight, mass fraction, mass-flux fraction, specific enthalpy, and rate of production of species i by chemical reaction, with $i=1$ for reactant R and $i=2$ for product P; and μ'', λ, and $\bar{D}=\rho D$, the longitudinal-viscosity, thermal-conductivity, and mass-diffusion coefficients. The equations of conservation of mass, momentum, species, and energy, and the equation of state for the flow of such a gas are:

$$\rho v = \text{const.} = \rho_0 v_0 = \rho_\infty v_\infty; \tag{2.1a}$$

$$p + \rho v^2 - \mu'' \frac{\mathrm{d}v}{\mathrm{d}x} = \text{const.} = p_0 + \rho_0 v_0^2 = p_\infty + \rho_\infty v_\infty^2; \tag{2.1b}$$

$$\frac{\mathrm{d}\varepsilon_i}{\mathrm{d}x} = \frac{w_i}{\rho v}, \quad \varepsilon_i = Y_i - \frac{\bar{D}}{\rho v}\frac{\mathrm{d}Y_i}{\mathrm{d}x}, \text{ with } \sum_{i=1}^{2} Y_i, \sum_{i=1}^{2} \varepsilon_i = 1, \quad \sum_{i=1}^{2} w_i = 0; \tag{2.1c}$$

$$\rho v\left(\sum_{i=1}^{2} \varepsilon_i h_i + \frac{1}{2}v^2\right) - \lambda\frac{\mathrm{d}T}{\mathrm{d}x} - \mu'' v\frac{\mathrm{d}v}{\mathrm{d}x} = \text{const.}$$
$$= \rho_0 v_0\left(\sum_{i=1}^{2} \varepsilon_{i0} h_{i0} + \frac{1}{2}v_0^2\right) = \rho_\infty v_\infty\left(\sum_{i=1}^{2} \varepsilon_{i\infty} h_{i\infty} + \frac{1}{2}v_\infty^2\right); \tag{2.1d}$$

$$p = \rho R^\circ T \sum_{i=1}^{2}(Y_i/W_i) = \rho(R^\circ/W)T,$$
$$\text{with } W_i = W, \sum_{i=1}^{2} Y_i = 1, \ R^\circ = \text{universal gas const.} \tag{2.1e}$$

Here, the subscripts 0 and ∞ refer the cold- and hot-boundary states, respectively. It is presumed that $\varepsilon_{10} = Y_{10}$, so that $Y_{20}, \varepsilon_{20} = (1-Y_{10}), (1-\varepsilon_{10})$; and that $Y_{1\infty}, \varepsilon_{1\infty} = 0$, $Y_{2\infty}, \varepsilon_{2\infty} = 1$. With respect to the heat capacities of the species, it is taken that $c_{p1}, c_{p2} = c_p = \text{const.}$ In turn, this means that the specific enthalpies may be expressed as $h_1 = [h_1^\circ + c_p(T-T_0)]$, $h_2 = [h_2^\circ + c_p(T-T_0)]$, where h_1°, h_2° are the specific enthalpies of formation at the cold-boundary state. For the model, the longitudinal-viscosity, thermal-conductivity, and mass-diffusion coefficients are taken to be $(\mu''/\mu_0''), (\lambda/\lambda_0), (\bar{D}/\bar{D}_0) = (T/T_0)^\omega$, with $\omega = \text{const.} \sim \boldsymbol{O}(1)$.

For the Arrhenius kinetics of the model, the reaction rates are given by

$$w_1 = -w_2 = -\rho B(T - T_0)(1 - Y_2)\exp(-T_A/T), \tag{2.2}$$

with B, the (const.) amplitude of the frequency factor, and T_A, the (const.) activation temperature. In this formulation, the reaction rates go to zero algebraically as the cold-boundary temperature is approached.†

It is convenient to introduce nondimensional variables. The spatial variable is defined by $\eta = (\rho_0 v_0/\mu_0'')x$. The velocity, temperature, and pressure are given by $v = v_0\phi$, $\rho = \rho_0 s = \rho_0(1/\phi)$, $T = T_0\theta$, $p = p_0\pi = \rho_0(R^\circ/W)T_0(s\theta) = \rho_0 R_W T_0(\theta/\phi)$, respectively. The mass fractions and the mass-flux fractions are written as $Y_1 = Y_{10}(1 - Y)$, $Y_2 = [1 - Y_{10}(1 - Y)]$, and $\varepsilon_1 = Y_{10}(1 - \varepsilon)$, $\varepsilon_2 = [1 - Y_{10}(1 - \varepsilon)]$. The specific total enthalpy is expressed as $H = (c_p T + (1/2)v^2) = [H_0 + (H_\infty - H_0)\tau] = [H_0 + q\tau] = [H_0 + Y_{10}(h_1^\circ - h_2^\circ)\tau]$.

For the postulated model, the governing equations can now be written as

$$\theta^\omega \frac{\mathrm{d}\phi}{\mathrm{d}\eta} = \frac{1}{\gamma M_0^2}\left(\frac{\theta}{\phi} - 1\right) - (1 - \phi), \tag{2.3a}$$

$$\frac{1}{Le_0 Pr_0''}\theta^\omega \frac{\mathrm{d}Y}{\mathrm{d}\eta} = (Y - \varepsilon), \tag{2.3b}$$

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\eta} = \Lambda_0 \frac{(\theta - 1)}{\phi}(1 - Y)\exp[-(\theta_A/\theta)], \tag{2.3c}$$

$$\frac{1}{Pr_0''}\theta^\omega\left[\frac{\mathrm{d}\tau}{\mathrm{d}\eta} - (1 - Pr_0'')\frac{(1/2)(\gamma - 1)M_0^2}{\alpha'}\frac{\mathrm{d}(\phi^2)}{\mathrm{d}\eta}\right] = (\tau - \varepsilon), \tag{2.3d}$$

with

$$\theta = [1 + \alpha'\tau + \tfrac{1}{2}(\gamma - 1)M_0^2(1 - \phi^2)]. \tag{2.3e}$$

†In the previous analysis [1], the reaction rates are given by $w_1 = -w_2 = -\rho BT^a(1 - Y_2)\exp[-T_A/(T - T_0)]$, with $a = \text{const.} \sim O(1)$, such that the reaction rates go to zero exponentially as the cold-boundary temperature is approached. Thus, the present "algebraic" reaction-rate model attacks the "cold-boundary difficulty", rather than circumvents it, as does the (previous) "exponential" model.

The boundary conditions for these equations are

$$\phi \to 1,\ Y \to 0,\ \varepsilon \to 0,\ \tau \to 0,\ \theta \to 1 \quad \text{as } \eta \to -\infty; \tag{2.4a}$$

$$\phi \to \phi_\infty,\ Y \to 1,\ \varepsilon \to 1,\ \tau \to 1,\ \theta \to \theta_\infty \quad \text{as } \eta \to \infty. \tag{2.4b}$$

In (2.3), the parameters introduced are: the ratio of the specific heats, $\gamma = c_p/(c_p - R_W)$; the initial Mach number, $M_0 = v_0/(\gamma R_W T_0)^{1/2}$; the normalized heat release, $\alpha' = q/(c_p T_0)$; the initial Lewis number, $Le_0 = \lambda_0/(c_p \bar{D}_0)$; the initial (longitudinal) Prandtl number, $Pr_0'' = (c_p \mu_0'')/\lambda_0$; the initial first Damköhler number, $\Lambda_0 = \mu_0''(BT_0)/(\rho_0 v_0^2)$; and the normalized activation temperature, $\theta_A = T_A/T_0$.

For a Chapman–Jouguet detonation, with $M_\infty = v_\infty/(\gamma R_W T_\infty)^{1/2} \equiv 1$, it follows from the thermodynamics relating the bounding states that the normalized heat release and the downstream value of the velocity function are

$$\alpha' = \left[\frac{(M_0^2 - 1)^2}{2(\gamma + 1)M_0^2}\right] > 1, \tag{2.5a}$$

$$\phi_\infty = \left[\frac{(\gamma M_0^2 + 1)}{(\gamma + 1)M_0^2}\right], \quad \text{with } (1 - \phi_\infty) = \left[\frac{(M_0^2 - 1)}{(\gamma + 1)M_0^2}\right] > 0 \ \text{ for } M_0^2 > 1, \tag{2.5b}$$

such that the two are related by

$$\frac{\alpha'}{(1/2)(\gamma - 1)M_0^2} = \frac{(\gamma + 1)}{(\gamma - 1)}(1 - \phi_\infty)^2. \tag{2.5c}$$

The temperature function can now be expressed as

$$\theta = \left[1 + \frac{1}{2}(\gamma - 1)M_0^2\left\{(1 - \phi^2) + \frac{(\gamma + 1)}{(\gamma - 1)}(1 - \phi_\infty)^2\tau\right\}\right], \tag{2.6a}$$

so that its downstream value is

$$\theta_\infty = \left[\left\{\frac{(\gamma M_0^2 + 1)}{(\gamma + 1)M_0^2}\right\}^2\right]. \tag{2.6b}$$

With $Le_0, Pr_0'' = 1$, such that, consistent with the equations and boundary conditions, $Y = \tau$; and with the introduction of ξ, the modified spatial variable, defined by

$$\xi = \xi^* + \int_0^{\eta} \theta^{-\omega}\,\mathrm{d}z, \tag{2.7}$$

where the arbitrary constant ξ^* denotes explicitly the translational invariance of the equations, the model reduces to

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = -Z(\phi,\tau); \tag{2.8a}$$

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\xi} = \Lambda X(\phi,\tau)\exp[-\beta Y(\phi,\tau)]; \tag{2.8b}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}\xi} = (\tau-\varepsilon). \tag{2.8c}$$

Here,

$$\begin{aligned} Z(\phi,\tau) &= \frac{(\gamma+1)}{2\gamma}\left[\frac{1}{\phi}\{(1-\phi_\infty)^2(1-\tau)-(\phi-\phi_\infty)^2\}\right] \\ &= \frac{(\gamma+1)}{2\gamma}\left[\frac{1}{\phi}\{(1-\phi)(\phi-\phi_S)-(1-\phi_\infty)^2\tau\}\right], \\ \text{with } \phi_S &= (2\phi_\infty-1) = \left[\frac{(\gamma-1)M_0^2+2}{(\gamma+1)M_0^2}\right]; \end{aligned} \tag{2.9a}$$

$$\begin{aligned} X(\phi,\tau) &= \frac{(1-\tau)}{\phi}[\theta(\phi,\tau)]^{\omega}[\theta(\phi,\tau)-1] \\ &= \frac{(1-\tau)}{\phi}\left[1+\frac{1}{2}(\gamma-1)M_0^2\left\{(1-\phi^2)+\frac{\gamma+1}{\gamma-1}(1-\phi_\infty)^2\tau\right\}\right]^{\omega} \\ &\quad\times\left[\frac{1}{2}(\gamma-1)M_0^2\left\{(1-\phi^2)+\frac{\gamma+1}{\gamma-1}(1-\phi_\infty)^2\tau\right\}\right]; \end{aligned} \tag{2.9b}$$

$$\begin{aligned} Y(\phi,\tau) &= \frac{[\theta(\phi_M,\tau_M)-\theta(\phi,\tau)]}{\theta(\phi,\tau)} \\ &= \frac{[(1/2)(\gamma-1)M_0^2\{(\phi^2-\phi_M^2)+[(\gamma+1)/(\gamma-1)](1-\phi_\infty)^2(\tau_M-\tau)\}]}{[1+(1/2)(\gamma-1)M_0^2\{(1-\phi^2)+[(\gamma+1)/(\gamma-1)](1-\phi_\infty)^2\tau\}]}. \end{aligned} \tag{2.9c}$$

The upstream and downstream boundary conditions are

$$\phi \to 1,\ \varepsilon \to 0,\ \tau \to 0 \quad \text{as } \xi \to -\infty; \tag{2.10a}$$

$$\phi \to \phi_\infty = \left[\frac{(\gamma M_0^2 + 1)}{(\gamma + 1)M_0^2}\right] < 1,\ \varepsilon \to 1,\ \tau \to 1 \quad \text{as } \xi \to \infty. \tag{2.10b}$$

The parameters β and Λ are introduced in (2.8b). These parameters, which depend upon the maximum flow temperature,

$$\begin{aligned}\theta_M &= (T_M/T_0) = \theta(\phi_M, \tau_M)\\ &= \left[1 + \frac{1}{2}(\gamma - 1)M_0^2\left\{(1 - \phi_M^2) + \frac{\gamma + 1}{\gamma - 1}(1 - \phi_\infty)^2 \tau_M\right\}\right],\end{aligned} \tag{2.11}$$

are defined by

$$\beta = (\theta_A/\theta_M) = (T_A/T_M); \quad \Lambda = \Lambda_0 \exp(-\beta). \tag{2.12a, b}$$

The asymptotic solutions to the boundary-value problem of (2.8)–(2.10), in the limit of $\beta \to \infty$, $\Lambda \to 0$, such that

$$\Lambda\beta^{1/2} = \Lambda_0\beta^{1/2}\exp(-\beta) \to 0, \tag{2.13}$$

for all of the other parameters of $\boldsymbol{O}(1)$, are presented in succeeding sections. It is noted that ϕ_M, τ_M, and θ_M are quantities to be determined from these solutions.

3 DISCUSSION OF RESULTS

Here, a discussion is presented of the detonation-structure results that are obtained from an asymptotic analysis, valid for $\beta \to \infty$, $\Lambda \to 0$, such that $\Lambda\beta^{1/2} \to 0$, of the boundary-value problem of (2.8)–(2.10). The salient mathematical features of this analysis are presented in Sections 4–6.

Four "distinguished" states, two bounding states and two interior states, of the gas characterize the detonation structure. The two bounding states are (1) the upstream ($\xi \to -\infty$) cold-boundary state (denoted by the subscript 0), where the combustible mixture enters the wave at a supersonic speed, i.e., $M_0 > 1$; and (2) the downstream ($\xi \to \infty$) hot-boundary state (denoted by the subscript ∞), where pure product leaves the wave at the local equilibrium speed of sound, i.e., $M_\infty = 1$. The two interior states are (1) the shock state (denoted by the subscript S), with $\xi = \xi_S = 0$; and (2) the maximum-temperature state (denoted by the subscript M), with $0 < \xi = \xi_M < \infty$.

According to the ZND detonation model, a "chemically frozen" shock-like compression zone precedes a "diffusion-free" (high-speed) deflagration-like expansion zone. In the notation of this paper, the normalized velocity, ϕ, decreases monotonically from its initial value, $\phi_0 = 1$, at the upstream boundary ($\xi \to -\infty$), toward a minimum value, ϕ_S, at the shock state ($\xi = 0$), and, then, increases monotonically to its final value, ϕ_∞, at the downstream boundary ($\xi \to \infty$). That is, ϕ_S is related to the conditions at the bounding states by $\phi_S = (2\phi_\infty - 1) = [(\gamma - 1)M_0^2 + 2]/[(\gamma + 1)M_0^2]$, where the second equality is the Rankine–Hugoniot relation between the upstream and downstream velocities in a normal shock. From the above relations, it is seen that $0 < \phi_S < \phi_\infty < 1$, i.e., the flow decelerates through the shock-like upstream zone, and, then, it accelerates through the deflagration-like downstream zone. Further, the normalized stagnation enthalpy, τ, increases monotonically from its cold-boundary value, $\tau_0 = 0$, to its hot-boundary value, $\tau_\infty = 1$, remaining exponentially small at the shock state, i.e., $\tau_S \approx 0$. The normalized temperature, θ, increases from its cold-boundary value, $\theta_0 = 1$, toward a local stationary value, θ_S, at the shock state, continues to rise through the initial part of the deflagration, reaches an absolute maximum value, θ_M, at the maximum-temperature state, and, then, in the final part of the deflagration, decreases to its hot-boundary value, θ_∞. If, as is consistent with the ZND model, it is taken that diffusion is negligible in the deflagration, it is found that the maximum-temperature state in velocity space is defined by $\phi_M = [(\gamma + 1)/(2\gamma)]\phi_\infty = [(\gamma + 1)/(4\gamma)](1 + \phi_S)$, such that $\phi_S < \phi_M < \phi_\infty < 1$ for $\gamma > 1$, $M_0^2 > [(3\gamma - 1)/(\gamma(3 - \gamma))] > 1$, which reinforces the concept that the flow accelerates through the deflagration-like downstream zone. In turn, it is determined that, for the downstream zone, $\theta_S < \theta_\infty < \theta_M$ for $\gamma > 1$, $M_0^2 > [(2\gamma - 1)/(\gamma(2 - \gamma))] > [(3\gamma - 1)/(\gamma(3 - \gamma))] > 1$.

The aforementioned four distinguished states are introduced within the framework of an asymptotic analysis, in which approximations are developed that produce (simplified) systems of equations that define the flow process in regions (of reduced size) within the detonation. The criterion of consistency, with respect to the approximations that are developed and the resulting sequences of quadratures that are performed, is the ability of the solutions obtained for a given region to "match" to the solutions for the adjacent regions.

The results of the analysis for the present model are in general agreement with those for the ZND model, in that both models produce a detonation structure consisting of an upstream "chemically frozen" shock-like zone and a downstream "diffusion-free" deflagration-like zone. However, in the present analysis, a zone, intermediate to these upstream and downstream zones, is introduced to describe the transition of the detonation from its shock-like behavior to its deflagration-like behavior.

The regions into which the flow is categorized by this analysis are schematically sketched in Figure 1.

In Section 4, the downstream zone is analyzed. For this zone, the flow variables, ϕ, τ, and ε, are of (at most) order unity, and the spatial variable, ξ, is much greater than order unity as $\beta \to \infty$, $\Lambda \to 0$: $\Lambda\beta^{1/2} \to 0$. Thermal-conduction and viscous-diffusion effects are rendered to be of exponentially higher order. Five regions are required to describe the flow in this downstream zone.

The first of these five regions is (denoted as) the chemical-reaction "induction region". In this region, wherein the "initial state" is the shock state, the flow begins its inviscid acceleration (due to heat addition). For $L_r \equiv [\Lambda\beta\exp(-\beta G_S)]^{-1} \to \infty (G_S = \text{const.} \sim \boldsymbol{O}(1))$, this region is characterized by $\xi = \boldsymbol{O}(L_i) = \boldsymbol{O}(\log L_r) \to \infty$; and by $(\phi - \phi_S), \tau, \varepsilon \sim \boldsymbol{O}((\beta L_r)^{-1} \log L_r) \to 0$.

The second downstream region is (designated as) the "first-reaction region". In this region, the chemical reaction is of order unity, rather than exponentially small, and the velocity and temperature increase from their shock-state values to their maximum-temperature-state values. Thus, this region is defined by $\xi \sim \boldsymbol{O}(L_r)$; and $\phi, \tau, \varepsilon \sim \boldsymbol{O}(1)$.

The third is the "maximum-temperature region". Here, the velocity continues to increase, passing through its maximum-temperature-state value; the temperature increases to its maximum value at the region's

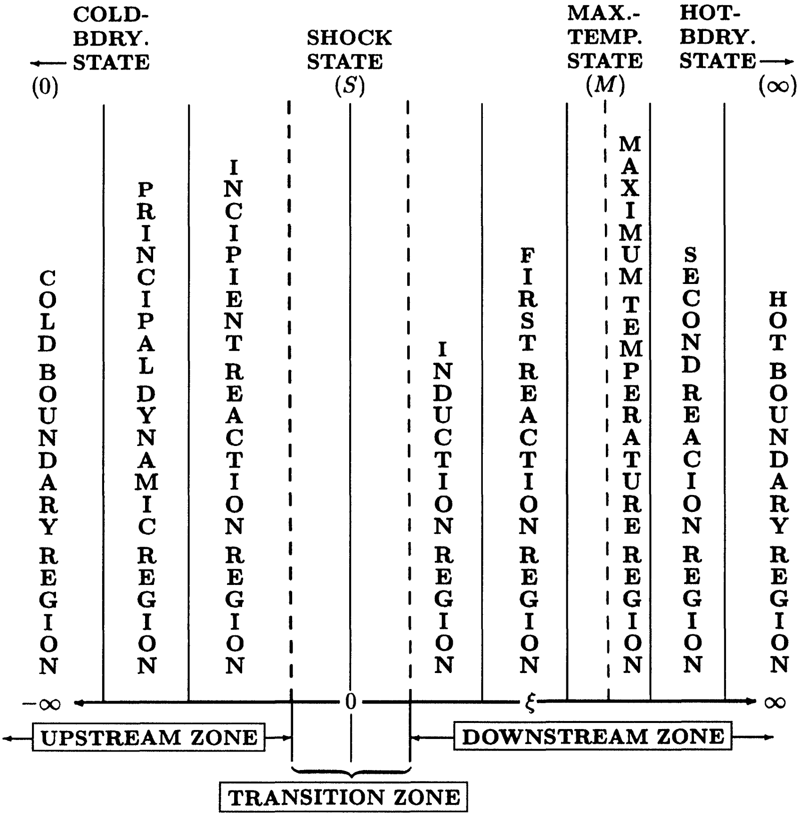

FIGURE 1 Schematic diagram of detonation structure.

"origin", and, then, begins to decrease. For $L_m \equiv [\Lambda\beta^{1/2}]^{-1} \to \infty$ (i.e., $\Lambda\beta^{1/2} \to 0$), this region is characterized by $\{\xi - (L_r\xi_r^\circ)\} \sim \boldsymbol{O}(L_m) \to \infty(\xi_r^\circ = \text{const.} \sim \boldsymbol{O}(1))$; and by $(\phi - \phi_M), (\tau - \tau_M), (\varepsilon - \varepsilon_M) \sim \boldsymbol{O}(\beta^{-1/2}) \to 0$.

The fourth region is (designated as) the "second-reaction region", in which the velocity and temperature proceed to increase and decrease, respectively, from their maximum-temperature-state values toward their hot-boundary-state values. For $L_q \equiv [\Lambda\beta\exp(-\beta G_\infty)]^{-1} \to \infty$ $(G_\infty = \text{const.} \sim \boldsymbol{O}(1))$, with $(L_q/L_r) = \exp\{-\beta(G_S - G_\infty)\} \to 0$, since $(G_S - G_\infty) = \text{const.} > 0$, this region is defined by $\{\xi - (L_r\xi_r^\circ)\} \sim \boldsymbol{O}(L_q) \to \infty$; and $\phi, \tau, \varepsilon \sim \boldsymbol{O}(1)$.

The "hot-boundary region" is the fifth (and last) region of the downstream zone. It is in this region that the chemical reaction is completed, and the flow accelerates and cools to achieve its hot-boundary-state values. For $L_h \equiv [\Lambda \exp(-\beta G_\infty)]^{-1} \to \infty$, such that $(L_h/L_q) = \beta \to \infty$, yet $(L_h/L_r) = \beta \exp\{-\beta(G_S - G_\infty)\} \to 0$, this region has $\{\xi - (L_r\xi_r^\circ)\} \sim \boldsymbol{O}(L_h) \to \infty$; and $(\phi_\infty - \phi) \sim \boldsymbol{O}(\beta^{-1}) \to 0$, $(1-\tau)$, $(1-\varepsilon) \sim \boldsymbol{O}(\beta^{-2}) \to 0$.

It follows that the effective thickness of the downstream zone (and, hence, of the detonation, itself) in ξ-space is $\bar{L}_d \approx (L_r\xi_r^\circ) \sim \boldsymbol{O}([\Lambda\beta \times \exp(-\beta G_S)]^{-1}) \to \infty$. It is the requirement that the maximum-temperature-region length-scale, L_m, be much greater than order unity that produces the third limit of this analysis, namely, $\Lambda\beta^{1/2} \to 0$.

In Section 5, the upstream zone is analyzed. For this zone, ϕ is of order unity, τ, ε are exponentially small, and $\{\xi + (L_u^*\xi_u^*)\}$ is of order unity as $\beta \to \infty, \Lambda \to 0: \Lambda\beta^{1/2} \to 0$. The chemical reaction is rendered an exponentially higher-order effect. Three regions are required to describe the flow in this upstream zone.

The first upstream region considered is (designated as) the "principal dynamic region". In this region, the chemical reaction is exponentially small, and, to the approximation considered, the velocity and temperature satisfy the Becker solutions for a steady planar "frozen shock", as they decrease and increase, respectively, from their cold-boundary-state values toward their shock-state values. This region has $\xi_p = \{\xi + (L_p^*\xi_p^*)\} \sim \boldsymbol{O}(1)$, with $(L_p^*\xi_p^*) \sim \boldsymbol{O}(B\log(\beta L_r)) \to \infty$; and $\phi \sim \boldsymbol{O}(1)$, $\tau \sim \boldsymbol{O}(\beta^{1+B}L_r)^{-1} \to 0$, $\varepsilon \sim \boldsymbol{O}(\beta^2 L_r)^{-1} \to 0$ for $0 < B =$ const. < 1. As the cold-boundary state is approached, i.e., $\xi_p \to -\infty$, $\phi \to 1$ (exponentially), $\tau \to 0$ (exponentially), and the upstream boundary conditions for these functions are achieved; however, in this limit, due to the reaction-rate model employed, $\varepsilon \not\to 0$ (exponentially), and the upstream boundary condition is *not* achieved for this function.

The upstream nonuniformity of the principal-dynamic-region ε-solution is studied in the "cold-boundary region". This region is characterized by $\xi_c = \{\xi + (L_c^*\xi_c^*)\} \sim \boldsymbol{O}(1)$, with $(L_c^*\xi_c^*) \sim \boldsymbol{O}(B\log L_r + (A+B)\log\beta) \to \infty$ ($A, B =$ consts. $\sim \boldsymbol{O}(1)$); and $(1-\phi) \sim \boldsymbol{O}(\beta^{-1}) \to 0$, $\tau \sim \boldsymbol{O}(\beta^{1+A+B}L_r)^{-1} \to 0$, $\varepsilon \sim \boldsymbol{O}((\beta^2 L_r)^{-1}\exp(-\beta R_0)) \to 0$ ($R_0 =$ const. $\sim \boldsymbol{O}(1)$). As $\xi_c \to -\infty$, the solutions for the cold-boundary region satisfy all three of the upstream boundary conditions of the detonation-structure problem. [As $\xi_c \to \infty$, the cold-boundary-region solutions match to those of the principal dynamic region (as $\xi_p \to -\infty$).]

In the third region, (designated as) the "incipient-reaction region", of the upstream zone, the chemical reaction, although more intense than in the principal dynamic region, is still exponentially small, and the velocity and temperature behaviors, to leading order, remain independent of the chemical reaction as the shock state is more closely approached. This region is characterized by $\xi_v = \{\xi + (L_v^* \xi_v^*)\} \sim \boldsymbol{O}(1)$, with $(L_v^* \xi_v^*) \sim \boldsymbol{O}(B \log L_r) \to \infty$; and $(\phi - \phi_S) \sim \boldsymbol{O}(\beta^{-1}) \to 0, \tau, \varepsilon \sim \boldsymbol{O}(\beta L_r)^{-1} \to 0$.

The transition zone, intermediate to the upstream and downstream zones, is analyzed in Section 6. In this zone, there is a convection–diffusion–chemical-reaction balance, with (1) the chemical-reaction contribution's going to zero as the upstream zone is approached, and (2) the diffusion contribution's going to zero as the downstream zone is approached. For this zone, $\xi \sim \boldsymbol{O}(1)$; and $\{(\phi - \phi_S) - (\lambda_t^* \phi_t^*)\}$, $\{\tau - (\lambda_t^* \tau_t^*)\}, \{\varepsilon - (\lambda_t^* \varepsilon_t^*)\} \sim \boldsymbol{O}(\beta L_r)^{-1} \to 0$, with $\lambda_t^* \sim \boldsymbol{O}((\beta L_r)^{-1} \times \log L_r) \to 0$ ($\phi_t^*, \tau_t^*, \varepsilon_t^* =$ consts. $\sim \boldsymbol{O}(1)$). The solutions of this transition zone match directly to the incipient-reaction-region solutions of the shock-like zone and to the induction-region solutions of the deflagration-like zone. With respect to the (ideal) Neumann state, $(\phi - \phi_S), \tau, \varepsilon \equiv 0$, when $\xi \equiv 0$, it is in this transition zone that the detonation approaches this state to within a factor of $\boldsymbol{O}((\beta L_r)^{-1} \times \log L_r) \to 0$.

4 THE DOWNSTREAM ZONE

For the downstream zone, which is the inviscid subsonic deflagration-like zone of the detonation, the appropriate representations for the spatial coordinate and the flow variables are

$$\xi_d = \xi / L_d, \quad \text{with } L_d \to \infty; \tag{4.0.1}$$

$$\phi(\xi) \approx \phi_d(\xi_d), \quad \tau(\xi) \approx \tau_d(\xi_d), \quad \varepsilon(\xi) \approx \varepsilon_d(\xi_d). \tag{4.0.2}$$

In the limit of $L_d \to \infty$, for $\xi_d, \phi_d, \tau_d, \varepsilon_d$ fixed, from (2.8) and (2.9), the governing equations in velocity space are

$$\tau_d(\phi_d), \varepsilon_d(\phi_d) \approx J_d(\phi_d); \tag{4.0.3a, b}$$

$$\frac{\mathrm{d}\xi_d}{\mathrm{d}\phi_d} \approx (L_d\Lambda\beta)^{-1}\beta\frac{\exp\{\beta G_d(\phi_d)\}}{E_d(\phi_d)}:$$

$$\xi_d(\phi_d) \approx \xi_d^{\circ} + (L_d\Lambda\beta)^{-1}\frac{\exp\{\beta G_d(\phi_d)\}}{E_d(\phi_d)F_d(\phi_d)}. \tag{4.0.3c}$$

Here,

$$J_d(\phi_d) = \left[1 - \frac{(\phi_\infty - \phi_d)^2}{(1-\phi_\infty)^2}\right] = \left[\frac{(1-\phi_d)(\phi_d - \phi_S)}{(1-\phi_\infty)^2}\right],$$

$$\frac{\mathrm{d}J_d}{\mathrm{d}\phi_d} = H_d(\phi_d) = \left[\frac{2(\phi_\infty - \phi_d)}{(1-\phi_\infty)^2}\right]; \tag{4.0.4a}$$

$$E_d(\phi_d) = \frac{X(\phi_d, J_d(\phi_d))}{H_d(\phi_d)}$$

$$= \left[\frac{(\phi_\infty - \phi_d)}{2\phi_d}\right][1 + \gamma M_0^2\{(1-\phi_M)^2 - (\phi_d - \phi_M)^2\}]^{\omega}$$

$$\times [\gamma M_0^2\{(1-\phi_M)^2 - (\phi_d - \phi_M)^2\}]; \tag{4.0.4b}$$

$$G_d(\phi_d) = Y(\phi_d, J_d(\phi_d)) = \frac{[\gamma M_0^2(\phi_d - \phi_M)^2]}{[1 + \gamma M_0^2\{(1-\phi_M)^2 - (\phi_d - \phi_M)^2\}]},$$

$$\frac{\mathrm{d}G_d}{\mathrm{d}\phi_d} = F_d(\phi_d) = \frac{[1 + \gamma M_0^2(1-\phi_M)^2][2\gamma M_0^2(\phi_d - \phi_M)]}{[1 + \gamma M_0^2\{(1-\phi_M)^2 - (\phi_d - \phi_M)^2\}]}. \tag{4.0.4c}$$

In the foregoing expressions, the following temperature function has been employed:

$$\theta(\phi_d, J_d(\phi_d)) = \theta_d(\phi_d) = [1 + \gamma M_0^2\{(1-\phi_M)^2 - (\phi_d - \phi_M)^2\}]. \tag{4.0.5}$$

Since

$$\theta_d(\phi_\infty) = \theta_\infty = \left[\left\{\frac{(\gamma M_0^2 + 1)}{(\gamma + 1)M_0^2}\right\}^2\right],$$

from (4.0.5), it is determined that

$$\phi_M = \frac{\gamma+1}{2\gamma}\phi_\infty = \frac{\gamma+1}{4\gamma}(1+\phi_S) = \left[\frac{(\gamma M_0^2 + 1)}{2\gamma M_0^2}\right]. \tag{4.0.6}$$

Thus, the velocity functions at the shock, maximum-temperature, and hot-boundary states are related by

$$\phi_S < \phi_M < \phi_\infty < 1 \quad \text{for } \gamma > 1,\ M_0^2 > \frac{(3\gamma - 1)}{\gamma(3-\gamma)} > 1. \tag{4.0.7}$$

In turn,

$$\tau_M, \varepsilon_M = J_d(\phi_M) = J_M = \left[1 - \frac{(\phi_\infty - \phi_M)^2}{(1-\phi_\infty)^2}\right] = \left[\frac{(1-\phi_M)(\phi_M - \phi_S)}{(1-\phi_\infty)^2}\right]. \tag{4.0.8}$$

Employing the notation $\theta_d(\phi_M) = \theta_M$, $\theta_d(\phi_\infty) = \theta_\infty$, and $\theta_d(\phi_S) = \theta_S$, the following holds:

$$\begin{aligned}
\theta_M &= [1 + \gamma M_0^2 (1-\phi_M)^2],\\
\theta_\infty &= [1 + \gamma M_0^2\{(1-\phi_M)^2 - (\phi_\infty - \phi_M)^2\}],\\
\theta_S &= [1 + \gamma M_0^2\{(1-\phi_M)^2 - (\phi_M - \phi_S)^2\}],\\
&\text{with } \theta_S < \theta_\infty < \theta_M \quad \text{for } \gamma > 1,\ M_0^2 > \frac{(2\gamma-1)}{\gamma(2-\gamma)} > \frac{(3\gamma-1)}{\gamma(3-\gamma)} > 1.
\end{aligned} \tag{4.0.9}$$

Thus, (4.0.9) provides the requirement that the upstream Mach number, M_0, must satisfy in order to be compatible with the Chapman–Jouguet condition.

It is now possible to evaluate $J_d(\phi_d)$, $E_d(\phi_d)$, and $G_d(\phi_d)$ at the maximum-temperature, shock, and hot-boundary states. At the maximum-temperature state, $(\phi_d - \phi_M) \to 0$,

$$\begin{aligned}
J_d(\phi_d) &\sim J_M + H_M(\phi_d - \phi_M)(1 + \cdots) \to J_M,\\
&\text{with } J_M = \left[1 - \frac{(\phi_\infty - \phi_M)^2}{(1-\phi_\infty)^2}\right] = \left[\frac{(1-\phi_M)(\phi_M - \phi_S)}{(1-\phi_\infty)^2}\right] > 0,\\
H_M &= \left[\frac{2(\phi_\infty - \phi_M)}{(1-\phi_\infty)^2}\right] > 0;
\end{aligned} \tag{4.0.10a}$$

$$\begin{aligned}
E_d(\phi_d) &\sim E_M(1 + \cdots) \to E_M,\\
&\text{with } E_M = \left[\frac{(\phi_\infty - \phi_M)}{2\phi_M}\right][1 + \gamma M_0^2(1-\phi_M)^2]^\omega[\gamma M_0^2(1-\phi_M)^2] > 0;
\end{aligned} \tag{4.0.10b}$$

$$G_d(\phi_d) \sim K_M^2(\phi_d - \phi_M)^2(1 + \cdots) \to 0,$$
$$F_d(\phi_d) \sim 2K_M^2(\phi_d - \phi_M)(1 + \cdots) \to 0,$$
$$\text{with } K_M^2 = \frac{\gamma M_0^2}{[1 + \gamma M_0^2(1 - \phi_M)^2]} > 0. \tag{4.0.10c}$$

At the shock state, $(\phi_d - \phi_S) \to 0$,

$$J_d(\phi_d) \sim H_S(\phi_d - \phi_S)(1 + \cdots) \to 0, \quad \text{with } H_S = \left[\frac{4}{(1 - \phi_S)}\right] > 0; \tag{4.0.11a}$$

$$E_d(\phi_d) \sim E_S(1 + \cdots) \to E_S,$$

$$\text{with } E_S = \left[\frac{(1 - \phi_S)}{4\phi_S}\right][1 + \gamma M_0^2\{(1 - \phi_M)^2 - (\phi_M - \phi_S)^2\}]^\omega$$
$$\times [\gamma M_0^2\{(1 - \phi_M)^2 - (\phi_M - \phi_S)^2\}] > 0; \tag{4.0.11b}$$

$$G_d(\phi_d) \sim G_S + F_S(\phi_d - \phi_S)(1 + \cdots) \to G_S,$$

$$\text{with } G_S = \frac{[\gamma M_0^2(\phi_M - \phi_S)^2]}{[1 + \gamma M_0^2\{(1 - \phi_M)^2 - (\phi_M - \phi_S)^2\}]} > 0,$$

$$F_S = -\frac{[1 + \gamma M_0^2(1 - \phi_M)^2][2\gamma M_0^2(\phi_M - \phi_S)]}{[1 + \gamma M_0^2\{(1 - \phi_M)^2 - (\phi_M - \phi_S)^2\}]^2} < 0. \tag{4.0.11c}$$

At the hot-boundary state, $(\phi_\infty - \phi_d) \to 0$,

$$J_d(\phi_d) = 1 - I_\infty(\phi_\infty - \phi_d)^2 \to 1, \quad \text{with } I_\infty = \left[\frac{1}{(1 - \phi_\infty)^2}\right] > 0; \tag{4.0.12a}$$

$$E_d(\phi_d) \sim D_\infty(\phi_\infty - \phi_d)(1 + \cdots) \to 0,$$
$$\text{with } D_\infty = \frac{1}{2\phi_\infty}[1 + \gamma M_0^2\{(1 - \phi_M)^2 - (\phi_\infty - \phi_M)^2\}]^\omega$$
$$\times [\gamma M_0^2\{(1 - \phi_M)^2 - (\phi_\infty - \phi_M)^2\}] > 0; \tag{4.0.12b}$$

$$G_d(\phi_d) \sim G_\infty - F_\infty(\phi_\infty - \phi_d)(1 + \cdots) \to G_\infty,$$

$$\text{with } G_\infty = \frac{[\gamma M_0^2(\phi_\infty - \phi_M)^2]}{[1 + \gamma M_0^2\{(1 - \phi_M)^2 - (\phi_\infty - \phi_M)^2\}]} > 0,$$

$$F_\infty = \frac{[1 + \gamma M_0^2(1 - \phi_M)^2][2\gamma M_0^2(\phi_\infty - \phi_M)]}{[1 + \gamma M_0^2\{(1 - \phi_M)^2 - (\phi_\infty - \phi_M)^2\}]^2} > 0. \qquad (4.0.12c)$$

From (4.0.11c), (4.0.12c), and (4.0.9), it is determined that $G_S > G_\infty$.

4.1 The Induction Region

In the induction region, for which the "initial state" is the shock state, the variables are determined to be

$$\xi_i = (L_d\xi_d)/L_i = \xi/L_i, \quad \text{with } L_i = \log L_r \to \infty,$$

$$\text{for } L_r = [\Lambda\beta\exp(-\beta G_S)]^{-1} \to \infty; \qquad (4.1.1)$$

$$\phi_d(\xi_d) \approx \phi_S + \delta_i\phi_i(\xi_i), \quad \tau_d(\xi_d) \approx \sigma_i\tau_i(\xi_i), \quad \varepsilon_d(\xi_d) \approx \pi_i\varepsilon_i(\xi_i),$$

$$\text{with } \delta_i, \sigma_i, \pi_i := \lambda_i = (\beta L_r)^{-1}\log L_r \to 0. \qquad (4.1.2)$$

For these scalings, the introduction of (4.1.1) and (4.1.2) into (4.0.3) yields

$$\tau_i(\phi_i),\ \varepsilon_i(\phi_i) \approx H_S\phi_i; \qquad (4.1.3a, b)$$

$$\frac{d\xi_i}{d\phi_i} \approx \frac{1}{E_S}: \quad \xi_i(\phi_i) \approx \frac{1}{E_S}\phi_i - \xi_i^\circ, \qquad (4.1.3c)$$

where $\xi_i^\circ = \text{const.} = B$, defined in Section 5. In turn, in terms of ξ_i, these solutions can be written as

$$\phi_i(\xi_i) \approx E_S(\xi_i + \xi_i^\circ); \quad \tau_i(\xi_i),\ \varepsilon_i(\xi_i) \approx H_S E_S(\xi_i + \xi_i^\circ). \qquad (4.1.4a, b, c)$$

Upstream, as $\xi_i \to (L_{ti}\xi_{ti})/L_i \to 0$, these solutions can be expressed as

$$\phi_{(i)} \approx \phi_S + (\lambda_i/L_i)E_S\{(L_i\xi_i^\circ) + (L_{ti}\xi_{ti})\}$$

$$= \phi_S + (\beta L_r)^{-1}\frac{Q_S}{H_S}\{(B\log L_r) + (L_{ti}\xi_{ti})\}, \qquad (4.1.5a)$$

$$\tau_{(i)},\ \varepsilon_{(i)} \approx (\lambda_i/L_i)H_S E_S\{(L_i\xi_i^\circ) + (L_{ti}\xi_{ti})\}$$

$$= (\beta L_r)^{-1}Q_S\{(B\log L_r) + (L_{ti}\xi_{ti})\}, \qquad (4.1.5b, c)$$

where $H_S E_S = \text{const.} = Q_S$, defined in Section 5.

As $\xi_i \to (L_{ir}\xi_{ir})/L_i \to \infty$, the solutions of (4.1.4) yield

$$\phi_{(i)} \approx \phi_S + (\lambda_i/L_i)E_S(L_{ir}\xi_{ir}) = \phi_S + (\beta L_r)^{-1}E_S(L_{ir}\xi_{ir}), \qquad (4.1.6a)$$

$$\tau_{(i)},\ \varepsilon_{(i)} \approx (\lambda_i/L_i)H_S E_S(L_{ir}\xi_{ir}) = (\beta L_r)^{-1}H_S E_S(L_{ir}\xi_{ir}). \qquad (4.1.6b, c)$$

4.2 The First-Reaction Region

For the first-reaction region, in which the flow accelerates from the shock state to the maximum-temperature state,

$$\xi_r = (L_d\xi_d)/L_r = \xi/L_r, \quad \text{with } L_r = [\Lambda\beta\exp(-\beta G_S)]^{-1} \to \infty; \qquad (4.2.1)$$

$$\phi_d(\xi_d) \approx \phi_r(\xi_r), \quad \tau_d(\xi_d) \approx \tau_r(\xi_r), \quad \varepsilon_d(\xi_d) \approx \varepsilon_r(\xi_r). \qquad (4.2.2)$$

With these representations for the variables, the governing equations for this region are

$$\tau_r(\phi_r),\ \varepsilon_r(\phi_r) \approx J_d(\phi_r); \qquad (4.2.3a, b)$$

$$\frac{d\varepsilon_r}{d\phi_r} \approx \beta\frac{\exp\{-\beta[G_S - G_d(\phi_r)]\}}{E_d(\phi_r)}:$$

$$\xi_r(\phi_r) \approx \xi_r^\circ + \frac{\exp\{-\beta[G_S - G_d(\phi_r)]\}}{E_d(\phi_r)F_d(\phi_r)},$$

$$\text{with } \xi_r^\circ = \left[-\frac{1}{E_S F_S}\right] > 0. \qquad (4.2.3c)$$

As the upstream shock state is approached, i.e., $(\phi_r - \phi_S) \to 0$, the asymptotic behaviors of (4.2.3) are

$$\xi_r \approx \beta\frac{1}{E_S}(\phi_r - \phi_S) \to 0, \qquad (4.2.4a)$$

$$\tau_r,\ \varepsilon_r \approx H_S(\phi_r - \phi_S) \to 0. \qquad (4.2.4b, c)$$

In turn, as $\xi_r \to (L_{ir}\xi_{ir})/L_r \to 0$,

$$\phi_{(r)} \approx \phi_S + (\beta L_r)^{-1}E_S(L_{ir}\xi_{ir}), \qquad (4.2.5a)$$

$$\tau_{(r)}, \varepsilon_{(r)} \approx (\beta L_r)^{-1}H_S E_S(L_{ir}\xi_{ir}), \qquad (4.2.5b, c)$$

and the solutions of the induction and first-reaction regions are seen to match.

As the maximum-temperature state is approached, $(\phi_M - \phi_r) \to 0$, and the behaviors of (4.2.3) are

$$\xi_r \approx \xi_r^\circ - [\beta^{1/2}\exp(-\beta G_S)]\xi_m^\circ \frac{\exp\{[\beta^{1/2}K_M(\phi_M - \phi_r)]^2\}}{[\beta^{1/2}K_M(\phi_M - \phi_r)]} \to \xi_r^\circ,$$

$$\text{with } \xi_m^\circ = \left[\frac{1}{2E_M K_M}\right] > 0, \tag{4.2.6a}$$

$$\tau_r,\, \varepsilon_r \approx J_M - H_M(\phi_M - \phi_r) \to J_M. \tag{4.2.6b, c}$$

Thus, as $\xi_r \to \xi_r^\circ - (L_{rm}\xi_{rm})/L_r \to \xi_r^\circ$, in anticipation that $L_M = [\Lambda\beta^{1/2}]^{-1} \to \infty$,

$$\phi_{(r)} \approx \phi_M - \beta^{-1/2}\frac{1}{K_M}\left[\log\left\{\frac{(L_{rm}\xi_{rm})}{(L_m\xi_m^\circ)}\right\}\right]^{1/2}, \tag{4.2.7a}$$

$$\tau_{(r)},\, \varepsilon_{(r)} \approx J_M - \beta^{-1/2}\frac{H_M}{K_M}\left[\log\left\{\frac{(L_{rm}\xi_{rm})}{(L_m\xi_m^\circ)}\right\}\right]^{1/2}. \tag{4.2.7b, c}$$

The relations for L_r and L_m show that $(L_m/L_r) = [\beta^{1/2}\exp(-\beta G_S)] \to 0$.

4.3 The Maximum-Temperature Region

For the maximum-temperature region, with its "center" being the maximum-temperature state,

$$\xi_m = \{(L_d\xi_d) - (L_m^*\xi_m^*)\}/L_m = \{\xi - (L_m^*\xi_m^*)\}/L_m,$$
$$\text{with } (L_m^*\xi_m^*) = (L_r\xi_r^\circ) \to \infty,\ L_m = [\Lambda\beta^{1/2}]^{-1} \to \infty; \tag{4.3.1}$$

$$\phi_d(\xi_d) \approx \phi_M + \delta_m\phi_m(\xi_m),$$
$$\tau_d(\xi_d) \approx \tau_M + \sigma_m\tau_m(\xi_m),$$
$$\varepsilon_d(\xi_d) \approx \varepsilon_M + \pi_m\varepsilon_m(\xi_m),$$
$$\text{with } \tau_M, \varepsilon_M = J_M,\quad \delta_m, \sigma_m, \pi_m = \lambda_m = \beta^{-1/2} \to 0. \tag{4.3.2}$$

Thus, for this region,

$$\tau_m(\phi_m),\ \varepsilon_m(\phi_m) \approx H_m\phi_m; \tag{4.3.3a, b}$$

$$\frac{d\xi_m}{d\phi_m} \approx \frac{1}{E_M}\exp\{(K_M\phi_m)^2\} : \xi_m(\phi_m) \approx 2\xi_m^\circ \int_0^{K_M\phi_m} \exp(t^2)\,dt. \tag{4.3.3c}$$

As $\phi_m \to 0$, (4.3.3c) yields

$$\xi_m \approx \frac{1}{E_M}\phi_m \to 0, \tag{4.3.4}$$

such that, as $\xi_m \to 0$,

$$\phi_{(m)} \approx \phi_M + \beta^{-1/2} E_M \xi_m, \tag{4.3.5a}$$

$$\tau_{(m)}, \varepsilon_{(m)} \approx J_M + \beta^{-1/2} H_M E_M \xi_m. \tag{4.3.5b, c}$$

As $|\phi_m| \to \infty$, the integral of (4.3.3c) produces

$$\xi_m \approx \xi_m^{\circ} \frac{\exp\{(K_M\phi_m)^2\}}{(K_M\phi_m)}, \tag{4.3.6}$$

such that $\xi_m \to \pm\infty$ as $\phi_m \to \pm\infty$. Upstream, for $\xi_m \to -(L_{rm}\xi_{rm})/L_m \to -\infty$, then,

$$\phi_{(m)} \approx \phi_M - \beta^{-1/2}\frac{1}{K_M}\left[\log\left\{\frac{(L_{rm}\xi_{rm})}{(L_m\xi_m^{\circ})}\right\}\right]^{1/2}, \tag{4.3.7a}$$

$$\tau_{(m)}, \varepsilon_{(m)} \approx J_M - \beta^{-1/2}\frac{H_M}{K_M}\left[\log\left\{\frac{(L_{rm}\xi_{rm})}{(L_m\xi_m^{\circ})}\right\}\right]^{1/2}, \tag{4.3.7b, c}$$

and the solutions of the first-reaction and maximum-temperature regions match. Similarly, downstream, for $\xi_m \to (L_{mq}\xi_{mq})/L_m \to \infty$,

$$\phi_{(m)} \approx \phi_M + \beta^{-1/2}\frac{1}{K_M}\left[\log\left\{\frac{(L_{mq}\xi_{mq})}{(L_m\xi_m^{\circ})}\right\}\right]^{1/2}, \tag{4.3.8a}$$

$$\tau_{(m)}, \varepsilon_{(m)} \approx J_M + \beta^{-1/2}\frac{H_M}{K_M}\left[\log\left\{\frac{(L_{mq}\xi_{mq})}{(L_m\xi_m^{\circ})}\right\}\right]^{1/2}. \tag{4.3.8b, c}$$

4.4 The Second-Reaction Region

For the second-reaction region, in which there is an acceleration of the flow from the maximum-temperature state to the hot-boundary state,

$$\xi_q = \{(L_d\xi_d) - (L_q^*\xi_q^*)\}/L_q = \{\xi - (L_q^*\xi_q^*)\}/L_q,$$

$$\text{with } (L_q^*\xi_q^*) = (L_r\xi_r^{\circ}) \to \infty,\ L_q = [\Lambda\beta\exp(-\beta G_\infty)]^{-1} \to \infty; \tag{4.4.1}$$

$$\phi_d(\xi_d) \approx \phi_q(\xi_q), \quad \tau_d(\xi_d) \approx \tau_q(\xi_q), \quad \varepsilon_d(\xi_d) \approx \varepsilon_q(\xi_q). \tag{4.4.2}$$

Note that, with $(G_S - G_\infty) > 0$,

$$(L_q/L_r) = \exp\{-\beta(G_S - G_\infty)\} \to 0. \tag{4.4.3}$$

The governing equations for this region are

$$\tau_q(\phi_q),\ \varepsilon_q(\phi_q) \approx J_d(\phi_q); \tag{4.4.4a, b}$$

$$\frac{d\xi_q}{d\phi_q} \approx \beta\frac{\exp\{-\beta[G_\infty - G_d(\phi_q)]\}}{E_d(\phi_q)}:$$

$$\xi_q(\phi_q) \approx \frac{\exp\{-\beta[G_\infty - G_d(\phi_q)]\}}{E_d(\phi_q)F_d(\phi_q)}. \tag{4.4.4c}$$

As the maximum-temperature state is approached, $(\phi_q - \phi_M) \to 0$, and the asymptotic behaviors of (4.4.4) are

$$\xi_q \approx [\beta^{1/2}\exp(-\beta G_\infty)]\xi_m^\circ \frac{\exp\{[\beta^{1/2}K_M(\phi_q - \phi_M)]^2\}}{[\beta^{1/2}K_M(\phi_q - \phi_M)]} \to 0, \tag{4.4.5a}$$

$$\tau_q,\ \varepsilon_q \approx J_M + H_M(\phi_q - \phi_M) \to J_M. \tag{4.4.5b, c}$$

In turn, as $\xi_q \to (L_{mq}\xi_{mq})/L_q \to 0$,

$$\phi_{(q)} \approx \phi_M + \beta^{-1/2}\frac{1}{K_M}\left[\log\left\{\frac{(L_{mq}\xi_{mq})}{(L_m\xi_m^\circ)}\right\}\right]^{1/2}, \tag{4.4.6a}$$

$$\tau_{(q)},\ \varepsilon_{(q)} \approx J_M + \beta^{-1/2}\frac{H_M}{K_M}\left[\log\left\{\frac{(L_{mq}\xi_{mq})}{(L_m\xi_m^\circ)}\right\}\right]^{1/2} \tag{4.4.6b, c}$$

and the solutions for the maximum-temperature and the second-reaction regions match.

As the hot-boundary state is approached, i.e., $(\phi_\infty - \phi_q) \to 0$, the behaviors of (4.4.4) are

$$\xi_h \approx \beta\xi_h^\circ \frac{\exp\{-[\beta F_\infty(\phi_\infty - \phi_q)]\}}{[\beta F_\infty(\phi_\infty - \phi_q)]} \to \infty, \quad \text{with } \xi_h^\circ = \left[\frac{1}{D_\infty}\right] > 0, \tag{4.4.7a}$$

$$\tau_q,\ \varepsilon_q \approx 1 - I_\infty(\phi_\infty - \phi_q)^2 \to 1. \tag{4.4.7b, c}$$

Thus, as $\xi_q \to (L_{qh}\xi_{qh})/L_q \to \infty$, in anticipation that $L_h = [\Lambda \times \exp(-\beta G_\infty)]^{-1} \to \infty$,

$$\phi_{(q)} \approx \phi_\infty - \beta^{-1}\frac{1}{F_\infty}\left[\log\left\{\frac{(L_h\xi_h^\circ)}{(L_{qh}\xi_{qh})}\right\}\right], \tag{4.4.8a}$$

$$\tau_{(q)},\ \varepsilon_{(q)} \approx 1 - \beta^{-2}\frac{I_\infty}{F_\infty^2}\left[\log\left\{\frac{(L_h\xi_h^\circ)}{(L_{qh}\xi_{qh})}\right\}\right]^2. \tag{4.4.8b, c}$$

4.5 The Hot-Boundary Region

In the hot-boundary region, in which the "final state" is the hot-boundary state, the variables are

$$\xi_h = \{(L_d\xi_d) - (L_h^*\xi_h^*)\}/L_h = \{\xi - (L_h^*\xi_h^*)\}/L_h,$$

$$\text{with } (L_h^*\xi_h^*) = (L_r\xi_r^\circ) \to \infty, \quad L_h = [\Lambda\exp(-\beta G_\infty)]^{-1} \to \infty; \tag{4.5.1}$$

$$\phi_d(\xi_d) \approx \phi_\infty - \delta_h\phi_h(\xi_h), \quad \tau_d(\xi_d) \approx 1 - \sigma_h\tau_h(\xi_h), \quad \varepsilon_d(\xi_d) \approx 1 - \pi_h\varepsilon_h(\xi_h),$$

$$\text{with } \delta_h = \beta^{-1} \to 0, \quad \sigma_h, \pi_h = \lambda_h = \beta^{-2} \to 0. \tag{4.5.2}$$

Note that

$$(L_h/L_q) = \beta \to \infty; \quad (L_h/L_r) = \beta\exp\{-\beta(G_S - G_\infty)\} \to 0. \tag{4.5.3}$$

The equations for this region are

$$\tau_h(\phi_h),\ \varepsilon_h(\phi_h) \approx I_\infty\phi_h^2; \tag{4.5.4a, b}$$

$$\frac{d\xi_h}{d\phi_h} \approx -\xi_h^\circ\frac{\exp\{-(F_\infty\phi_h)\}}{\phi_h}:$$

$$\xi_h(\phi_h) \approx \xi_h^\circ Ei(F_\infty\phi_h), \quad \text{with } Ei(z) = \int_z^\infty t^{-1}\exp(-t)\,dt. \tag{4.5.4c}$$

Upstream, as $\phi_h \to \infty$,

$$\xi_h \approx \xi_h^\circ\frac{\exp\{-(F_\infty\phi_h)\}}{(F_\infty\phi_h)} \to 0, \tag{4.5.5a}$$

$$\tau_h,\ \varepsilon_h \approx \Phi_\infty\phi_h^2 \to \infty. \tag{4.5.5b, c}$$

In the limit of $\Lambda \to 0$, $\beta \to \infty$, ξ_u, ϕ_u, τ_u, ε_u fixed, from (2.8) and (2.9), the governing equations in velocity space are

$$\frac{d\xi_u}{d\phi_u} \approx -\frac{1}{P_u(\phi_u)}; \tag{5.0.3a}$$

$$\frac{d\varepsilon_u}{d\phi_u} \approx -\frac{1}{P_u(\phi_u)} \frac{[\Lambda\beta^{-1}\exp(-\beta G_S)]}{\pi_u} \beta Q_u(\phi_u)\exp\{-\beta R_u(\phi_u)\}; \tag{5.0.3b}$$

$$\frac{d\tau_u}{d\phi_u} \approx -\frac{1}{P_u(\phi_u)}\left[\tau_u - \left(\frac{\pi_u}{\sigma_u}\right)\varepsilon_u\right]. \tag{5.0.3c}$$

Here,

$$P_u(\phi_u) = Z(\phi_u, 0) = \frac{(\gamma+1)}{2\gamma}\left[\frac{1}{\phi_u}\{(1-\phi_u)(\phi_u - \phi_S)\}\right]; \tag{5.0.4a}$$

$$\begin{aligned} Q_u(\phi_u) = X(\phi_u, 0) = \frac{1}{\phi_u}&\left[1 + \frac{1}{2}(\gamma-1)M_0^2(1-\phi_u^2)\right] \\ &\times \left[\frac{1}{2}(\gamma-1)M_0^2(1-\phi_u^2)\right]; \end{aligned} \tag{5.0.4b}$$

$$R_u(\phi_u) = [Y(\phi_u, 0) - G_S] = (1+G_S)\frac{[(1/2)(\gamma-1)M_0^2(\phi_u^2 - \phi_S^2)]}{[1+(1/2)(\gamma-1)M_0^2(1-\phi_u^2)]},$$

$$\begin{aligned} \frac{dR_u}{d\phi_u} &= S_u(\phi_u) \\ &= (1+G_S)\frac{[1+(1/2)(\gamma-1)M_0^2(1-\phi_S^2)][(\gamma-1)M_0^2\phi_u]}{[1+(1/2)(\gamma-1)M_0^2(1-\phi_u^2)]^2}. \end{aligned} \tag{5.0.4c}$$

As the cold-boundary state is approached, i.e. $(1-\phi_u) \to 0$, the functions of (5.0.4) have the following behaviors:

$$P_u(\phi_u) \sim \frac{1}{A}(1-\phi_u)(1+\cdots) \to 0, \quad \text{with } A = \left[\frac{2\gamma}{\gamma+1}\frac{1}{(1-\phi_S)}\right]; \tag{5.0.5a}$$

$$Q_u(\phi_u) \sim C_0(1-\phi_u)(1+\cdots) \to 0, \quad \text{with } C_0 = \left[(\gamma-1)M_0^2\right]; \tag{5.0.5b}$$

$$\begin{aligned} &R_u(\phi_u) \sim R_0 - S_0(1-\phi_u)(1+\cdots) \to R_0, \\ &\quad \text{with } R_0 = (1+G_S)[(1/2)(\gamma-1)M_0^2(1-\phi_S^2)], \\ &\qquad S_0 = (1+G_S)[(\gamma-1)M_0^2][1+(1/2)(\gamma-1)M_0^2(1-\phi_S^2)]. \end{aligned} \tag{5.0.5c}$$

As the shock state is approached, $(\phi_u - \phi_S) \to 0$, and

$$P_u(\phi_u) \sim \frac{1}{B}(\phi_u - \phi_S)(1 + \cdots) \to 0, \quad \text{with } B = \left[\frac{2\gamma}{\gamma+1}\frac{\phi_S}{(1-\phi_S)}\right]; \tag{5.0.6a}$$

$$Q_u(\phi_u) \sim Q_S(1 + \cdots) \to Q_S,$$
$$\text{with } Q_S = \frac{1}{\phi_S}\left[1 + \frac{1}{2}(\gamma - 1)M_0^2(1 - \phi_S^2)\right]^{\omega}\left[\frac{1}{2}(\gamma - 1)M_0^2(1 - \phi_S^2)\right]; \tag{5.0.6b}$$

$$R_u(\phi_u) \sim S_S(\phi_u - \phi_S)(1 + \cdots) \to 0,$$
$$\text{with } S_S = (1 + G_S)\frac{[(\gamma - 1)M_0^2\phi_S]}{[1 + (1/2)(\gamma - 1)M_0^2(1 - \phi_S^2)]}. \tag{5.0.6c}$$

5.1 The Principal Dynamic Region

The principal dynamic region effectively spans the domain from the cold-boundary state to the shock state. The variables for this region are

$$\xi_p = \xi_u - \{(L_u^*\xi_u^*) - (L_p^*\xi_p^*)\} = \xi + (L_p^*\xi_p^*),$$
$$\text{with } (L_p^*\xi_p^*) = B\log(\beta L_r) \to \infty; \tag{5.1.1}$$

$$\phi_u(\xi_u) \approx \phi_p(\xi_p), \quad \sigma_u\tau_u(\xi_u) \approx \sigma_p\tau_p(\xi_p), \quad \pi_u\varepsilon_u(\xi_u) \approx \pi_p\varepsilon_p(\xi_p),$$
$$\text{with } \sigma_p = (\beta^{1+B}L_r)^{-1} \to 0, \quad \pi_p = (\beta^2 L_r)^{-1} \to 0, \tag{5.1.2}$$

such that $(\pi_p/\sigma_p) = \beta^{-(1-B)} \to 0$, since $B < 1$. The equations for this region take the forms

$$\frac{d\xi_p}{d\phi_p} \approx -\frac{1}{P_u(\phi_p)}; \tag{5.1.3a}$$

$$\frac{d\varepsilon_p}{d\phi_p} \approx -\beta\frac{Q_u(\phi_p)}{P_u(\phi_p)}\exp\{-\beta R_u(\phi_p)\}; \tag{5.1.3b}$$

$$\frac{d\tau_p}{d\phi_p} \approx -\frac{\tau_p}{P_u(\phi_p)}. \tag{5.1.3c}$$

Their solutions are determined to be

$$\xi_p(\phi_p) \approx \log\left\{\frac{(1-\phi_p)^A}{(\phi_p-\phi_S)^B}\right\}, \tag{5.1.4a}$$

$$\tau_p(\phi_p) \approx \tau_p^\circ\left\{\frac{(1-\phi_p)^A}{(\phi_p-\phi_S)^B}\right\}, \quad \text{with } \tau_p^\circ = \left[\frac{B\Gamma(B)Q_S}{(1-\phi_S)^A S_S^B}\right], \tag{5.1.4b}$$

$$\varepsilon_p(\phi_p) \approx \frac{Q_u(\phi_p)}{P_u(\phi_p)S_u(\phi_p)}\exp\{-\beta R_u(\phi_p)\}. \tag{5.1.4c}$$

Upstream, toward the cold-boundary state, as $(1-\phi_p)\to 0$,

$$\xi_p \approx \log\left\{\left[\frac{(1-\phi_p)}{\phi_{pc}^\circ}\right]^A\right\} \to -\infty, \quad \text{with } \phi_{pc}^\circ = (1-\phi_S)^{B/A}, \tag{5.1.5a}$$

$$\tau_p \approx \tau_p^\circ\left[\frac{(1-\phi_p)}{\phi_{pc}^\circ}\right]^A \to 0, \tag{5.1.5b}$$

$$\varepsilon_p \approx \varepsilon_{pc}^\circ \exp(-\beta R_0)\exp\{\beta S_0(1-\phi_p)\} \to \varepsilon_{pc}^\circ \exp(-\beta R_0),$$
$$\text{with } \varepsilon_{pc}^\circ = \left[\frac{AC_0}{S_0}\right]. \tag{5.1.5c}$$

In turn, as $\xi_p \to \{(L_{cp}\xi_{cp}) + (L_p^*\xi_p^*)\} \to -\infty$,

$$\phi_{(p)} \approx 1 - [\exp\{(L_p^*\xi_p^*)/A\}]\phi_{pc}^\circ \exp\{(L_{cp}\xi_{cp})/A\}$$
$$= 1 - (\beta L_r)^{B/A}(1-\phi_S)^{B/A}\exp\{(L_{cp}\xi_{cp})/A\} \equiv 1 - \Phi_p, \tag{5.1.6a}$$

$$\tau_{(p)} \approx [\sigma_p \exp\{(L_p^*\xi_p^*)\}]\tau_p^\circ \exp\{(L_{cp}\xi_{cp})\}$$
$$= (\beta L_r^{1-B})^{-1}\left[\frac{B\Gamma(B)Q_S}{(1-\phi_S)^A S_S^B}\right]\exp(L_{cp}\xi_{cp}), \tag{5.1.6b}$$

$$\varepsilon_{(p)} \approx [\pi_p \exp(-\beta R_0)]\varepsilon_{pc}^\circ \exp(\beta S_0\Phi_p)$$
$$= [(\beta^2 L_r)^{-1}\exp(-\beta R_0)]\left[\frac{AC_0}{S_0}\right]\exp(\beta S_0\Phi_p). \tag{5.1.6c}$$

From (5.1.6), it is seen that $\phi_{(p)} \to 1$, $\tau_{(p)} \to 0$ as $\xi_p \to -\infty$; however, $\varepsilon_{(p)} \not\to 0$ as $\xi_p \to -\infty$. Thus, the solutions of the principal dynamic region satisfy only two of the three upstream boundary conditions for the detonation boundary-value problem (cf. (2.10a)). All three of these boundary conditions are satisfied through the introduction of the cold-boundary region, which is analyzed in Section 5.2.

Downstream, toward the shock state, as $(\phi_p - \phi_S) \to 0$,

$$\xi_p \approx \log\left\{\left[\frac{\phi^\circ_{pv}}{(\phi_p - \phi_S)}\right]^B\right\} \to \infty, \quad \text{with } \phi^\circ_{pv} = (1-\phi_S)^{A/B}, \tag{5.1.7a}$$

$$\tau_p \approx \tau^\circ_p \left[\frac{\phi^\circ_{pv}}{(\phi_p - \phi_S)}\right]^B \to \infty, \tag{5.1.7b}$$

$$\varepsilon_p \approx \varepsilon^\circ_{pv} \frac{\exp\{-\beta[S_S(\phi_p - \phi_S)]\}}{[S_S(\phi_p - \phi_S)]} \to \infty, \quad \text{with } \varepsilon^\circ_{pv} = BQ_S. \tag{5.1.7c}$$

In turn, as $\xi_p \to \{(L_{pv}\xi_{pv}) + (L^*_p\xi^*_p)\} \to \infty$,

$$\begin{aligned}
\phi_{(p)} &\approx \phi_S + [\exp\{-(L^*_p\xi^*_p)/B\}]\phi^\circ_{pv}\exp\{-(L_{pv}\xi_{pv})B\} \\
&= \phi_S + (\beta L_r)^{-1}(1-\phi_S)^{A/B}\exp\{-(L_{pv}\xi_{pv})/B\} \\
&\equiv \phi_S + \Psi_p, &(5.1.8a)\\
\tau_{(p)} &\approx [\sigma_p \exp\{(L^*_p\xi^*_p)\}]\tau^\circ_p \exp\{(L_{pv}\xi_{pv})\} \\
&= (\beta L_r^{1-B})^{-1}\left[\frac{B\Gamma(B)Q_S}{(1-\phi_S)^A S_S^B}\right]\exp(L_{pv}\xi_{pv}), &(5.1.8b)\\
\varepsilon_{(p)} &\approx [\beta\pi_p]\varepsilon^\circ_{pv}\frac{\exp\{-(\beta S_S\Psi_p)\}}{(\beta S_S\Psi_p)} \\
&= (\beta L_r)^{-1}BQ_S\frac{\exp\{-(\beta S_S\Psi_p)\}}{(\beta S_S\Psi_p)}. &(5.1.8c)
\end{aligned}$$

5.2 The Cold-Boundary Region

In the cold-boundary region, in which the "initial state" is the cold-boundary state, the variables are taken to be

$$\begin{aligned}
\xi_c = \xi_u - \{(L^*_u\xi^*_u) - (L^*_c\xi^*_c)\} &= \xi + (L^*_c\xi^*_c), \\
\text{with } (L^*_c\xi^*_c) &= \{B\log L_r + (A+B)\log\beta\} \\
&= \{A\log\beta + B\log(\beta L_r)\} \to \infty; &(5.2.1)
\end{aligned}$$

$$\phi_u(\xi_u) \approx 1 - \delta_c \phi_c(\xi_c), \quad \sigma_u \tau_u(\xi_u) \approx \sigma_c \tau_c(\xi_c), \quad \pi_u \varepsilon_u(\xi_u) \approx \pi_c \varepsilon_c(\xi_c),$$
$$\text{with } \delta_c = \beta^{-1} \to 0, \quad \sigma_c = (\beta^{1+A+B} L_r)^{-1} \to 0,$$
$$\pi_c = [(\beta^2 L_r)^{-1} \exp(-\beta R_0)] \to 0. \tag{5.2.2}$$

Note that $(\sigma_c/\delta_c), (\pi_c/\sigma_c) \to 0$; further, $(\sigma_c/\sigma_p) = \beta^{-A} \to 0$, and $(\pi_c/\pi_p) = \exp(-\beta R_0) \to 0$. The cold-boundary-region equations of motion in velocity-space are determined to be

$$\frac{d\xi_c}{d\phi_c} \approx \frac{A}{\phi_c}; \tag{5.2.3a}$$
$$\frac{d\varepsilon_c}{d\phi_c} \approx AC_0 \exp(S_0 \phi_c); \tag{5.2.3b}$$
$$\frac{d\tau_c}{d\phi_c} \approx \frac{A\tau_c}{\phi_c}. \tag{5.2.3c}$$

The solutions of these equations, satisfying the boundary conditions $\xi_c \to -\infty$, $\tau_c, \varepsilon_c \to 0$ as $\phi_c \to 0$, are

$$\xi_c(\phi_c) \approx \log\left\{\left(\frac{\phi_c}{\phi_c^\circ}\right)^A\right\}, \quad \text{with } \phi_c^\circ = \phi_{pc}^\circ = (1 - \phi_S)^{B/A}, \tag{5.2.4a}$$
$$\tau_c(\phi_c) \approx \tau_c^\circ \left(\frac{\phi_c}{\phi_c^\circ}\right)^A, \quad \text{with } \tau_c^\circ = \tau_p^\circ = \left[\frac{B\Gamma(B)Q_S}{(1 - \phi_S)^A S_S^B}\right], \tag{5.2.4b}$$
$$\varepsilon_c(\phi_c) \approx \varepsilon_c^\circ[\exp(S_0\phi_c) - 1], \quad \text{with } \varepsilon_c^\circ = \varepsilon_{pc}^\circ = \left[\frac{AC_0}{S_0}\right]. \tag{5.2.4c}$$

Upstream, as $\phi_c \to 0$, these solutions yield

$$\xi_c \approx \log\left\{\left(\frac{\phi_c}{\phi_c^\circ}\right)^A\right\} \to -\infty, \tag{5.2.5a}$$
$$\tau_c \approx \tau_c^\circ \left(\frac{\phi_c}{\phi_c^\circ}\right)^A \to 0, \tag{5.2.5b}$$
$$\varepsilon_c \approx \varepsilon_c^\circ (S_0 \phi_c) \to 0. \tag{5.2.5c}$$

In turn, as $\xi_c \to -\infty$,

$$\phi_{(c)} \approx 1 - \delta_c \phi_c^\circ \exp(\xi_c/A)$$
$$= 1 - \beta^{-1}(1 - \phi_S)^{B/A} \exp(\xi_c/A) \to 1 \text{ (exponentially)}, \tag{5.2.6a}$$

$$\begin{aligned}\tau_{(c)} &\approx \sigma_c \tau_c^{\circ} \exp(\xi_c)\\ &= (\beta^{1+A+B} L_r)^{-1} \left[\frac{B\Gamma(B) Q_S}{(1-\phi_S)^A S_S^B}\right] \exp(\xi_c) \to 0 \text{ (exponentially)},\end{aligned} \tag{5.2.6b}$$

$$\begin{aligned}\varepsilon_{(c)} &\approx \pi_c \varepsilon_c^{\circ} [S_0 \phi_c^{\circ} \exp(\xi_c/A)]\\ &= [(\beta^2 L_r)^{-1} \exp(-\beta R_0)][A C_0 (1-\phi_S)^{B/A}]\\ &\quad \times \exp(\xi_c/A) \to 0 \text{ (exponentially)}.\end{aligned} \tag{5.2.6c}$$

Thus, it is seen that all three of the upstream boundary conditions of the detonation boundary-value problem are satisfied.

Downstream, as $\phi_c \to \infty$, the solutions of (5.2.4) yield

$$\xi_c \approx \log\left\{\left(\frac{\phi_c}{\phi_c^{\circ}}\right)^A\right\} \to \infty, \tag{5.2.7a}$$

$$\tau_c \approx \tau_c^{\circ}\left(\frac{\phi_c}{\phi_c^{\circ}}\right)^A \to \infty, \tag{5.2.7b}$$

$$\varepsilon_c \approx \varepsilon_c^{\circ} \exp(S_0 \phi_c) \to \infty. \tag{5.2.7c}$$

Thus, as $\xi_c \to \{(L_c^* \xi_c^*) + (L_{cp} \xi_{cp})\} \to \infty$, it follows that

$$\begin{aligned}\phi_{(c)} &\approx 1 - [\delta_c \exp\{(L_c^* \xi_c^*)/A\}] \phi_c^{\circ} \exp\{(L_{cp}\xi_{cp})/A\}\\ &= 1 - (\beta L_r)^{B/A} (1-\phi_S)^{B/A} \exp\{(L_{cp}\xi_{cp})/A\}\\ &\equiv 1 - \beta^{-1}\Phi_c = 1 - \Phi_p,\end{aligned} \tag{5.2.8a}$$

$$\begin{aligned}\tau_{(c)} &\approx [\sigma_c \exp(L_c^* \xi_c^*)] \tau_c^{\circ} \exp(L_{cp}\xi_{cp})\\ &= (\beta L_r^{1-B})^{-1} \left[\frac{B\Gamma(B) Q_S}{(1-\phi_S)^A S_S^B}\right] \exp(L_{cp}\xi_{cp}),\end{aligned} \tag{5.2.8b}$$

$$\begin{aligned}\varepsilon_{(c)} &\approx \pi_c \varepsilon_c^{\circ} \exp(S_0 \Phi_c)\\ &= [(\beta^2 L_r)^{-1} \exp(-\beta R_0)] \left[\frac{A C_0}{S_0}\right] \exp(\beta S_0 \Phi_p).\end{aligned} \tag{5.2.8c}$$

The cold-boundary-region and principal-dynamic-region solutions are seen to match from (5.2.8) and (5.1.6).

5.3 The Incipient-Reaction Region

In the incipient-reaction region, for which the "terminal state" is the shock state, the variables are

$$\xi_v = \xi_u - \{(L_u^*\xi_u^*) - (L_v^*\xi_v^*)\} = \xi + (L_v^*\xi_v^*),$$
$$\text{with } (L_v^*\xi_v^*) = B\log L_r \to \infty; \tag{5.3.1}$$
$$\phi_u(\xi_u) \approx \phi_S + \delta_v\phi_v(\xi_v), \quad \sigma_u\tau_u(\xi_u) \approx \sigma_v\tau_v(\xi_v), \quad \pi_u\varepsilon_u(\xi_u) \approx \pi_v\varepsilon_v(\xi_v),$$
$$\text{with } \delta_v = \beta^{-1} \to 0, \quad \sigma_v, \pi_v = \lambda_v = (\beta L_r)^{-1} \to 0. \tag{5.3.2}$$

The equations for this region can be written as

$$\frac{d\xi_v}{d\phi_v} \approx -\frac{B}{\phi_v}; \tag{5.3.3a}$$
$$\frac{d\varepsilon_v}{d\phi_v} \approx -\frac{B}{\phi_v} Q_S \exp\{-(S_S\phi_v)\}; \tag{5.3.3b}$$
$$\frac{d\tau_v}{d\phi_v} \approx -\frac{B}{\phi_v}(\tau_v - \varepsilon_v). \tag{5.3.3c}$$

The solutions of these equations are

$$\xi_v(\phi_v) \approx \log\left\{\left(\frac{\phi_v^\circ}{\phi_v}\right)^B\right\}, \quad \text{with } \phi_v^\circ = \phi_{pv}^\circ = (1-\phi_S)^{A/B}, \tag{5.3.4a}$$

$$\varepsilon_v(\phi_v) \approx \varepsilon_v^\circ Ei(S_S\phi_v), \quad \tau_v(\phi_v) \approx \varepsilon_v^\circ\left[Ei(S_S\phi_v) + \frac{\gamma(S_S\phi_v; B)}{(S_S\phi_v)^B}\right],$$
$$\text{with } Ei(z) = \int_z^\infty t^{-1}\exp(-t)\,dt, \quad \gamma(z;k) = \int_0^z t^{k-1}\exp(-t)\,dt,$$
$$\varepsilon_v^\circ = \varepsilon_{pv}^\circ = BQ_S. \tag{5.3.4b, c}$$

Upstream, as $\phi_v \to \infty$, these solutions have the following asymptotic behaviors:

$$\xi_v \approx \log\left\{\left(\frac{\phi_v^\circ}{\phi_v}\right)^B\right\} \to -\infty, \tag{5.3.5a}$$

$$\tau_v \approx \varepsilon_v^\circ \frac{\Gamma(B)}{(S_S\phi_v)^B} = \tau_v^\circ\left(\frac{\phi_v^\circ}{\phi_v}\right)^B \to 0, \quad \text{with } \tau_v^\circ = \tau_p^\circ = \left[\frac{B\Gamma(B)Q_S}{(1-\phi_S)^A S_S^B}\right], \tag{5.3.5b}$$

$$\varepsilon_v \approx \varepsilon_v^\circ \frac{\exp\{-(S_S\phi_v)\}}{(S_v\phi_v)} \to 0. \tag{5.3.5c}$$

Thus, as $\xi_v \to \{(L_{pv}\xi_{pv}) + (L_v^*\xi_v^*)\} \to -\infty$,

$$\begin{aligned}\phi_{(v)} &\approx \phi_S + [\delta_v \exp\{-(L_v^*\xi_v^*)/B\}]\phi_v^\circ \exp\{-(L_{pv}\xi_{pv})/B\} \\ &= \phi_S + (\beta L_r)^{-1}(1-\phi_S)^{A/B}\exp\{-(L_{pv}\xi_{pv})/B\} \\ &\equiv \phi_S + \beta^{-1}\Psi_v = \phi_S + \Psi_p, \end{aligned}\tag{5.3.6a}$$

$$\begin{aligned}\tau_{(v)} &\approx \lambda_v\varepsilon_v^\circ \frac{\Gamma(B)}{(S_S\Psi_v)^B} \approx \sigma_v\tau_v^\circ\left(\frac{\phi_v^\circ}{\Psi_v}\right)^B \\ &= (\beta L_r^{1-B})^{-1}\left[\frac{B\Gamma(B)Q_S}{(1-\phi_S)^A S_S^B}\right]\exp(L_{pv}\xi_{pv}), \end{aligned}\tag{5.3.6b}$$

$$\begin{aligned}\varepsilon_{(v)} &\approx \lambda_v\varepsilon_v^\circ \frac{\exp\{-(S_S\Psi_v)\}}{(S_S\Psi_v)} \\ &= (\beta L_r)^{-1}BQ_S\frac{\exp\{-(\beta S_S\Psi_p)\}}{(\beta S_S\Psi_p)}. \end{aligned}\tag{5.3.6c}$$

From (5.3.6) and (5.1.8), it is seen that the solutions for the principal dynamic and incipient-reaction regions match.

Downstream, as $\phi_v \to 0$, the solutions of (5.3.4) have the behaviors:

$$\xi_v \approx \log\left\{\left(\frac{\phi_v^\circ}{\phi_v}\right)^B\right\} \to \infty, \tag{5.3.7a}$$

$$\tau_v,\ \varepsilon_v \approx \varepsilon_v^\circ \log\left\{\left(\frac{1}{\phi_v}\right)\right\} \to \infty. \tag{5.3.7b, c}$$

Thus, as $\xi_v \to \{(L_{vt}\xi_{vt}) + (L_v^*\xi_v^*)\} \to \infty$,

$$\begin{aligned}\phi_{(v)} &\approx \phi_S + [\delta_v \exp\{-(L_v^*\xi_v^*)/B\}]\phi_v^\circ \exp\{-(L_{vt}\xi_{vt})/B\} \\ &= \phi_S + (\beta L_r)^{-1}(1-\phi_S)^{A/B}\exp\{-(L_{vt}\xi_{vt})/B\}, \end{aligned}\tag{5.3.8a}$$

$$\begin{aligned}\tau_{(v)},\ \varepsilon_{(v)} &\approx \lambda_v\frac{\varepsilon_v^\circ}{B}\{(L_v^*\xi_v^*) + (L_{vt}\xi_{vt})\} \\ &= (\beta L_r)^{-1}Q_S\{(B\log L_r) + (L_{vt}\xi_{vt})\}. \end{aligned}\tag{5.3.8b, c}$$

6 THE TRANSITION ZONE

For the transition zone, in which the diffusion and chemical-reaction effects are of equal magnitude, the appropriate representations for the spatial coordinate and flow variables are

$$\begin{aligned}
&\xi_t = \xi; && (6.1)\\
&\phi(\xi) \approx \phi_S + \delta_t^*\phi_t^* + \delta_t\phi_t(\xi_t), \quad \tau(\xi) \approx \sigma_t^*\tau_t^* + \sigma_t\tau_t(\xi_t), &&\\
&\quad \varepsilon(\xi) \approx \pi_t^*\varepsilon_t^* + \pi_t\varepsilon_t(\xi_t), &&\\
&\qquad \text{with } \delta_t^*, \sigma_t^*, \pi_t^* = \lambda_t^* = [(\beta L_r)^{-1}(B\log L_r)] \to 0, &&\\
&\qquad\quad \delta_t, \sigma_t, \pi_t = \lambda_t = (\beta L_r)^{-1} \to 0, && (6.2)
\end{aligned}$$

such that $(\lambda_t/\lambda_t^*) = (B\log L_r)^{-1} \to 0$. The physical-space governing equations for this transition zone are

$$\tau_t^*,\ \varepsilon_t^* \approx H_S\phi_t^*, \quad \text{with } \phi_t^* = \frac{Q_S}{H_S}, \text{ i.e., } \tau_t^*,\ \varepsilon_t^* = j_t^* = Q_S; \qquad (6.3\text{a, b})$$

$$\frac{d\phi_t}{d\xi_t} \approx \frac{1}{B}\left(\phi_t - \frac{1}{H_S}\tau_t\right); \quad \frac{d\varepsilon_t}{d\xi_t} \approx Q_S; \quad \frac{d\tau_t}{d\xi_t} \approx (\tau_t - \varepsilon_t). \qquad (6.4\text{a, b, c})$$

The solutions of (6.4), for $\phi_t, \tau_t, \varepsilon_t \to 0$ as $\xi_t \to 0$, are

$$\begin{aligned}
&\phi_t(\xi_t) \approx -\phi_t^\circ[1 - \exp\{-(\xi_t/B)\}] + \phi_t^*\xi_t, &&\\
&\quad \text{with } \phi_t^\circ = \phi_v^\circ = (1 - \phi_S)^{A/B}, && (6.5\text{a})\\
&\tau_t(\xi_t),\ \varepsilon_t(\xi_t) \approx j_t^*\xi_t. && (6.5\text{b, c})
\end{aligned}$$

As $\xi_t \to (L_{vt}\xi_{vt}) \to -\infty$, these solutions yield

$$\begin{aligned}
\phi_{(t)} &\approx \phi_S + \lambda_t^*\phi_t^* + \lambda_t\phi_t^\circ \exp\{-(L_{vt}\xi_{vt})/B\}\\
&\approx \phi_S + (\beta L_r)^{-1}(1 - \phi_S)^{A/B}\exp\{-(L_{vt}\xi_{vt})/B\}\\
&\quad \text{for } (B\log L_r)\exp\{(L_{vt}\xi_{vt})/B\} \to 0, \qquad (6.6\text{a})\\
\tau_{(t)},\ \varepsilon_{(t)} &\approx \lambda_t^* j_t^* + \lambda_t j_t^*(L_{vt}\xi_{vt})\\
&= (\beta L_r)^{-1}Q_S\{(B\log L_r) + (L_{vt}\xi_{vt})\}. \qquad (6.6\text{b, c})
\end{aligned}$$

A comparison of (6.6) and (5.3.8) indicates that the downstream solutions of the upstream zone match to the upstream solutions of this transition zone.

At $\xi_t = 0$,

$$\phi_{(t)} \approx \phi_S + \lambda_t^* \phi_t^* = \phi_S + [(\beta L_r)^{-1}(B \log L_r)] \frac{Q_S}{H_S}, \tag{6.7a}$$

$$\tau_{(t)}, \varepsilon_{(t)} \approx \lambda_t^* j_t^* = [(\beta L_r)^{-1}(B \log L_r)] Q_S. \tag{6.7b, c}$$

As $\xi_t \to (L_{ti}\xi_t) \to \infty$, the transition-zone solutions can be written as

$$\begin{aligned} \phi_{(t)} &\approx \phi_S + \lambda_t^* \phi_t^* + \lambda_t \phi_t^* (L_{ti}\xi_{ti}) \\ &= \phi_S + (\beta L_r)^{-1} \frac{Q_S}{H_S} \{(B \log L_r) + (L_{ti}\xi_{ti})\}, \end{aligned} \tag{6.8a}$$

$$\begin{aligned} \tau_{(t)}, \varepsilon_{(t)} &\approx \lambda_t^* j_t^* + \lambda_t j_t^* (L_{ti}\xi_{ti}) \\ &= (\beta L_r)^{-1} Q_S \{(B \log L_r) + (L_{ti}\xi_{ti})\}. \end{aligned} \tag{6.8b, c}$$

A comparison of (6.8) and (4.1.5) indicates that the upstream solutions of the downstream zone match to the downstream solutions of this transition zone.

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