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# Nonlinear Robust Hierarchical Control for Nonlinear Uncertain Systems

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A nonlinear robust control-system design framework predicated on a hierarchical switching controller architecture parameterized over a set of moving nominal system equilibria is developed. Specifically, using equilibria-dependent Lyapunov functions, a hierarchical nonlinear robust control strategy is developed that robustly stabilizes a given nonlinear system over a prescribed range of system uncertainty by robustly stabilizing a collection of nonlinear controlled uncertain subsystems. The robust switching nonlinear controller architecture is designed based on a generalized (lower semicontinuous) Lyapunov function obtained by minimizing a potential function over a given switching set induced by the parameterized nominal system equilibria. The proposed framework robustly stabilizes a compact positively invariant set of a given nonlinear uncertain dynamical system with structured parametric uncertainty. Finally, the efficacy of the proposed approach is demonstrated on a jet engine propulsion control problem with uncertain pressure-flow map data.

*Keywords:* Equilibria-dependent Lyapunov functions; Robust nonlinear control; Parametric uncertainty; Domains of attraction; Hierarchical switching control

## 1. INTRODUCTION

Since all physical systems are inherently nonlinear with system nonlinearities arising from numerous sources including, for example, friction

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(e.g., Coulomb, hysteresis), gyroscopic effects (e.g., rotational motion), kinematic effects (e.g., backlash), input constraints (e.g., saturation, deadband), and geometric constraints, plant nonlinearities must be accounted for in the control-system design process. However, since nonlinear systems can exhibit multiple equilibria, limit cycles, bifurcations, jump resonance phenomena, and chaos, general nonlinear system stabilization is notoriously difficult and remains an open problem. Control system designers have usually resorted to Lyapunov methods [1–3] in order to obtain stabilizing controllers for nonlinear systems. In particular, for smooth feedback, Lyapunov-based methods were inspired by Jurdjevic and Quinn [4] who give sufficient conditions for smooth stabilization based on the ability of constructing a Lyapunov function for the closed-loop system [5]. Unfortunately, however, there does not exist a unified procedure for finding a Lyapunov function candidate that will stabilize the closed-loop system for general nonlinear systems. This is further exacerbated when addressing robustness in uncertain nonlinear systems.

In a recent paper [6], a nonlinear control design framework predicated on a hierarchical switching controller architecture parameterized over a set of moving system equilibria was developed. In this paper we extend the results of [6] to address the problem of robust stabilization for nonlinear *uncertain* systems. Specifically, using *equilibria-dependent Lyapunov functions* or, instantaneous (with respect to a given *nominal* parameterized equilibrium manifold) Lyapunov functions, a hierarchical *robust* nonlinear control strategy is developed that stabilizes a compact positively invariant set of a nonlinear uncertain system using a supervisory robust switching controller that coordinates lower-level stabilizing subcontrollers (see Fig. 1). Each robust subcontroller can be nonlinear and thus local set point designs can be nonlinear. Furthermore, for each nominally parameterized equilibrium manifold, the collection of the robust subcontrollers provide guaranteed domains of attraction with nonempty intersections that cover the region of operation over the prescribed range of system uncertainty of the nonlinear uncertain system in the state space. A hierarchical robust switching nonlinear controller architecture is developed based on a *generalized* lower semicontinuous Lyapunov function obtained by minimizing a potential function, associated with each domain of attraction, over a given switching set induced by the parameterized nominal system

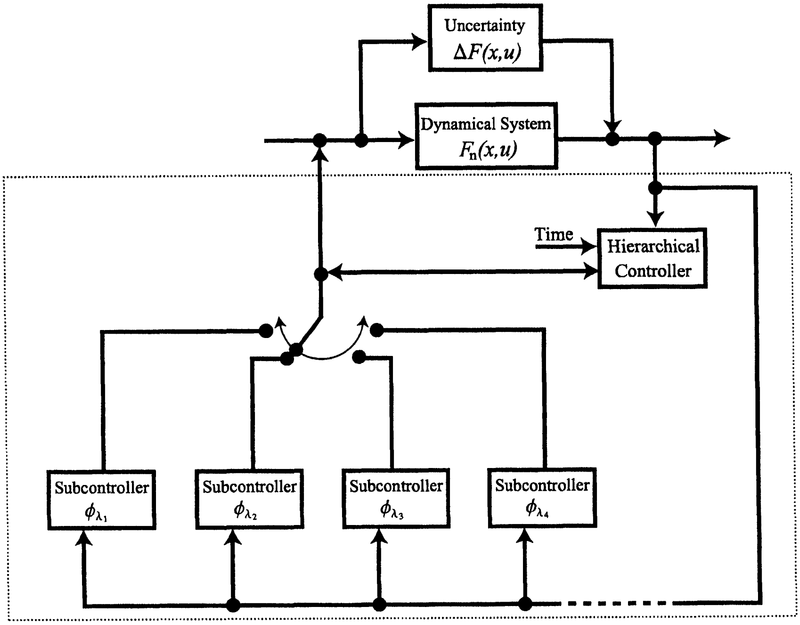


FIGURE 1 Robust switching controller architecture.

equilibria. The hierarchical robust switching nonlinear controller guarantees that the generalized Lyapunov function is nonincreasing along the closed-loop system trajectories for all parametric system uncertainty with strictly decreasing values at the switching points, establishing robust asymptotic stability of a compact positively invariant set. Furthermore, since the proposed robust switching nonlinear control strategy is predicated on a generalized Lyapunov framework with strictly decreasing values at the switching points, the possibility of a sliding mode is precluded. Hence, the proposed nonlinear robust stabilization framework avoids the undesirable effects of high-speed switching onto an invariant sliding manifold which is one of the main limitations of variable structure controllers.

The contents of the paper are as follows. In Section 2 we establish definitions, notation, and several key results used later in the paper. Then in Section 3, in order to address stability of uncertain closed-loop switching systems, we develop generalized Lyapunov and invariant set theorems for nonlinear uncertain closed-loop feedback dynamical systems wherein all regularity assumptions on the Lyapunov function and the closed-loop system dynamics are removed. In particular, local and global robust stability theorems are presented using generalized Lyapunov functions that are lower semicontinuous. Furthermore, generalized invariant set theorems are derived wherein closed-loop system trajectories converge to a union of largest invariant sets contained on the boundary of the intersections over finite intervals of the closure of generalized Lyapunov level surfaces. In Section 4 we concentrate on nonlinear robust stabilization of local set points over a set of parameterized nominal equilibria of the nonlinear uncertain system. In Section 5 a nonlinear connective stabilization framework predicated on a hierarchical robust switching controller architecture is developed. In Section 6, the proposed framework is used to design robust switching controllers to control the aerodynamic instabilities of rotating stall and surge in multi-mode axial flow compressor models with uncertain pressure-flow compressor performance characteristic maps. Finally, we draw some conclusions in Section 7.

## 2. MATHEMATICAL PRELIMINARIES

In this section we establish definitions, notation, and several key results used later in the paper. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^n$ denote the set of $n\times 1$ real column vectors, let $\mathbb{R}^{n\times m}$ denote the set of real $n\times m$ matrices, and let $(\cdot)^{\mathrm{T}}$ denote transpose. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $V'(x)$ for the Fréchet derivative of $V(\cdot)$ at $x$, and $A\geq 0$ (resp., $A>0$) to denote the fact that the Hermitian matrix $A$ is nonnegative (resp., positive) definite. For a subset $\mathcal{S}\subset\mathbb{R}^n$, we write $\partial\mathcal{S}, \overset{\circ}{\mathcal{S}}, \bar{\mathcal{S}}$ for the boundary, the interior, and the closure of $\mathcal{S}$ respectively. A set $\mathcal{S}\subseteq\mathbb{R}^n$, is *connected* if there does not exist open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathbb{R}^n$ such that $\mathcal{S}\subset\mathcal{O}_1\cup\mathcal{O}_2$, $\mathcal{S}\cap\mathcal{O}_1\neq\emptyset$, $\mathcal{S}\cap\mathcal{O}_2\neq\emptyset$, and $\mathcal{S}\cap\mathcal{O}_1\cap\mathcal{O}_2=\emptyset$. Recall that $\mathcal{S}$ is a connected subset of $\mathbb{R}$ if and only if $\mathcal{S}$ is either an interval or a single point. Finally, let $\mathrm{C}^0$ denote the set of continuous functions and $\mathrm{C}^n$ denote the set of functions with $n$-continuous derivatives.

In this paper we consider nonlinear controlled uncertain dynamical systems of the form

$$\dot{x}(t)=F(x(t),u(t)),\quad x(0)=x_0,\ F(\cdot,\cdot)\in\mathcal{F},\ t\in\mathcal{I}_{x_0},\tag{1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \in \mathcal{I}_{x_0}$, is the system state vector, $\mathcal{I}_{x_0} \subseteq \mathbb{R}$ is the maximal interval of existence of a solution $x(\cdot)$ of (1), $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$, $t \in \mathcal{I}_{x_0}$, is the control input, $\mathcal{U}$ is the set of all admissible controls such that $u(\cdot)$ is a measurable function with $0 \in \mathcal{U}$, and $\mathcal{F} \subset \{F: \mathcal{D} \times \mathcal{U} \to \mathbb{R}^n : F(\cdot,\cdot) \in \mathrm{C}^0\}$ denotes the class of uncertain nonlinear dynamics. Furthermore, we introduce the nominal controlled dynamical system

$$\dot{x}(t) = F_{\mathrm{n}}(x(t), u(t)), \quad x(0) = x_0, \ t \in \mathcal{I}_{x_0}, \tag{2}$$

where $F_{\mathrm{n}}(\cdot,\cdot) \in \mathcal{F}$ represents the nominal system dynamics.

DEFINITION 2.1 *The point $\bar{x} \in \mathcal{D}$ is an* equilibrium point *of* (1) *if there exists $\bar{u} \in \mathcal{U}$ such that $F(\bar{x}, \bar{u}) = 0$.*

In this paper we assume that given an equilibrium point $\bar{x} \in \mathcal{D}$ of (2) corresponding to $\bar{u} \in \mathcal{U}$ and a mapping $\varphi: \mathcal{D} \times \Lambda \to \mathcal{U}$, $\Lambda \subseteq \mathbb{R}^q$, $0 \in \Lambda$, such that $\varphi(\bar{x}, 0) = \bar{u}$, there exist neighborhoods $\mathcal{D}_{\mathrm{o}} \subset \mathcal{D}$ of $\bar{x}$ and $\Lambda_{\mathrm{o}} \subset \Lambda$ of 0, and a continuous function $\psi: \Lambda_{\mathrm{o}} \to \mathcal{D}_{\mathrm{o}}$ such that $\bar{x} = \psi(0)$, and, for every $\lambda \in \Lambda_{\mathrm{o}}$, $x_\lambda = \psi(\lambda)$ is an equilibrium point of (2); that is, $F_{\mathrm{n}}(\psi(\lambda), \varphi(\psi(\lambda), \lambda)) = 0$, $\lambda \in \Lambda_{\mathrm{o}}$. This is a necessary condition for nominal parametric stability with respect to $\Lambda_{\mathrm{o}}$ as defined in [7,8]. Note that the connected set $\Lambda \subseteq \mathbb{R}^q$ corresponds to a parameterization set with the function $\psi(\cdot)$ parameterizing the nominal system equilibria. In the special case where $q = m$ and $\varphi(x, \lambda) = \lambda$, it follows that the parameterized nominal system equilibria are given by the constant control $u(t) \equiv \lambda$. A parameterization that provides a local characterization of the nominal equilibrium manifold, including in neighborhoods of bifurcation points, is given in [9]. Alternatively, the well-known Implicit Function Theorem provides *sufficient* conditions for guaranteeing the existence of such a parameterization under the more restrictive condition of continuous differentiability of the mapping $\psi(\cdot)$.

THEOREM 2.1 [10] *Assume the function $\tilde{F}_{\mathrm{n}}(x, \lambda) \triangleq F_{\mathrm{n}}(x, \varphi(x, \lambda))$, $x \in \mathcal{D}$, $\lambda \in \Lambda$, is $\mathrm{C}^1$ at each point $(x, \lambda) \in \mathcal{D} \times \Lambda$. Suppose $\tilde{F}_{\mathrm{n}}(\bar{x}, 0) = 0$ for $(\bar{x}, 0) \in \mathcal{D} \times \Lambda$ and the Jacobian matrix $\partial \tilde{F}_{\mathrm{n}}(\bar{x}, 0)/\partial x$ is full rank. Then there exist open neighborhoods $\mathcal{D}_{\mathrm{o}} \subset \mathcal{D}$ of $\bar{x}$ and $\Lambda_{\mathrm{o}} \subset \Lambda$ of 0 such that $\tilde{F}_{\mathrm{n}}(x, \lambda) = 0$, $\lambda \in \Lambda_{\mathrm{o}}$, has a unique solution $x_\lambda \in \mathcal{D}_{\mathrm{o}}$. In particular, there exists a unique $\mathrm{C}^1$ mapping $\psi: \Lambda_{\mathrm{o}} \to \mathcal{D}_{\mathrm{o}}$ such that $x_\lambda = \psi(\lambda)$, $\lambda \in \Lambda_{\mathrm{o}}$, and $\bar{x} = \psi(0)$.*

Next, we consider nonlinear feedback controlled uncertain dynamical systems. A measurable mapping $\phi:\mathcal{D}\to\mathcal{U}$ satisfying $\phi(\bar{x})=\bar{u}$ is called a *control law*. Furthermore, if $u(t)=\phi(x(t))$, where $\phi(\cdot)$ is a control law and $x(t)$, $t\in\mathcal{I}_{x_0}$, satisfies (1), then $u(\cdot)$ is called a *feedback control law*. Here, we consider nonlinear closed-loop uncertain dynamical systems of the form

$$\dot{x}(t)=F(x(t),\phi(x(t))),\quad x(0)=x_0,\ F(\cdot,\cdot)\in\mathcal{F},\ t\in\mathcal{I}_{x_0}. \tag{3}$$

A function $x:\mathcal{I}_{x_0}\to\mathcal{D}$ is said to be a solution to (3) on the interval $\mathcal{I}_{x_0}\subseteq\mathbb{R}$ with initial condition $x(0)=x_0$, if $x(t)$ satisfies (3) for all $t\in\mathcal{I}_{x_0}$. Note that we do not assume any regularity condition on the function $\phi(\cdot)$. However, we do assume that for every $y\in\mathcal{D}$ there exists a unique solution $x(\cdot)$ of (3) defined on $\mathcal{I}_y$ satisfying $x(0)=y$. Furthermore, we assume that all the solutions $x(t)$, $t\in\mathcal{I}_{x_0}$, to (3) are continuous functions of the system initial conditions $x_0\in\mathcal{D}$, which, with the assumption of uniqueness of solutions, implies continuity of solutions $x(t)$, $t\in\mathcal{I}_{x_0}$, to (3) [10, p. 24].

*Remark 2.1* If $F(\cdot,\phi(\cdot))$, $F(\cdot,\cdot)\in\mathcal{F}$, is Lipschitz continuous on $\mathcal{D}$ then there exists a unique solution to (3). In this case, the semi-group property $s(t+\tau,x_0)=s(t,s(\tau,x_0))$, $t,\tau\in\mathcal{I}_{x_0}$, and the continuity of $s(t,\cdot)$ on $\mathcal{D}$, $t\in\mathcal{I}_{x_0}$, hold, where $s(\cdot,x_0)$ denotes the solution of the nonlinear feedback controlled uncertain dynamical system (3). Alternatively, uniqueness of solutions in time along with the continuity of $F(\cdot,\phi(\cdot))$ ensure that the solutions to (3) satisfy the semi-group property and are continuous functions of the initial condition $x_0\in\mathcal{D}$ even when $F(\cdot,\phi(\cdot))$ is not Lipschitz continuous on $\mathcal{D}$ (see [11, Theorem 4.3, p. 59]). More generally, $F(\cdot,\phi(\cdot))$ need not be continuous. In particular, if $F(\cdot,\phi(\cdot))$ is discontinuous but bounded and $x(\cdot)$ is the unique solution to (3) in the sense of Filippov [12], then the semi-group property along with the continuous dependence of solutions on initial conditions hold [12].

Next, we introduce several definitions and key results that are necessary for the main results of this paper.

DEFINITION 2.2 *Let $\mathcal{D}_c\subseteq\mathcal{D}$ and let $V:\mathcal{D}_c\to\mathbb{R}$. For $\alpha\in\mathbb{R}$, the set $V^{-1}(\alpha)\triangleq\{x\in\mathcal{D}_c\colon V(x)=\alpha\}$ is called the* $\alpha$-level set. *For $\alpha,\beta\in\mathbb{R}$,*

*$\alpha \leq \beta$, the set $V^{-1}([\alpha,\beta]) \triangleq \{x \in \mathcal{D}_c\colon \alpha \leq V(x) \leq \beta\}$ is called the* $[\alpha,\beta]$-sublevel set.

DEFINITION 2.3 *A set $\mathcal{M}^+ \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ (resp., $\mathcal{M}^-$) is a* positively (*resp.*, negatively) invariant *set for the nonlinear feedback controlled uncertain dynamical system* (3) *if $x_0 \in \mathcal{M}^+$ (resp., $\mathcal{M}^-$) implies that $[0,+\infty) \subseteq \mathcal{I}_{x_0}$ (resp., $(-\infty,0] \subseteq \mathcal{I}_{x_0}$) and $x(t) \in \mathcal{M}^+$ (resp., $\mathcal{M}^-$) for all $t \geq 0$ (resp., $t \leq 0$) and $F(\cdot,\cdot) \in \mathcal{F}$. A set $\mathcal{M} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ is an* invariant *set for the nonlinear feedback controlled uncertain dynamical system* (3) *if $x_0 \in \mathcal{M}$ implies that $\mathcal{I}_{x_0} = \mathbb{R}$ and $x(t) \in \mathcal{M}$ for all $t \in \mathbb{R}$ and $F(\cdot,\cdot) \in \mathcal{F}$.*

DEFINITION 2.4 *$p \in \bar{\mathcal{D}} \subseteq \mathbb{R}^n$ is a* positive limit point *of the trajectory $x(t)$, $t \in \mathcal{I}_{x_0}$, if $[0,+\infty) \subseteq \mathcal{I}_{x_0}$ and there exists a sequence $\{t_n\}_{n=0}^{\infty}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$, $t \in \mathcal{I}_{x_0}$, is the* positive limit set $\mathcal{P}_{x_0}^+$ *of $x(t)$, $t \in \mathcal{I}_{x_0}$.*

The following result on positive limit sets is fundamental and forms the basis for all the generalized robust stability and invariant set theorems developed in Section 3.

LEMMA 2.1 [3] *Suppose the forward solution $x(t)$, $t \geq 0$, to* (3) *corresponding to an initial condition $x(0) = x_0$ exists and is bounded. Then the positive limit set $\mathcal{P}_{x_0}^+$ of $x(t)$, $t \in \mathcal{I}_{x_0}$, is a nonempty, compact, connected invariant set. Furthermore, $x(t) \to \mathcal{P}_{x_0}^+$ as $t \to \infty$ for all $F(\cdot,\cdot) \in \mathcal{F}$.*

*Remark 2.2* It is important to note that Lemma 2.1 holds for time-invariant nonlinear feedback controlled dynamical systems (3) possessing unique solutions with solutions being continuous functions of the system initial conditions. More generally, Lemma 2.1 holds if $s(t+\tau,x_0) = s(t,s(\tau,x_0))$, $t,\tau \in \mathcal{I}_{x_0}$, and $s(\cdot,x_0)$ is a continuous function of $x_0 \in \mathcal{D}$.

The following definition introduces three types of stability as well as attraction of (3) with respect to a compact positively invariant set.

DEFINITION 2.5 *Let $\mathcal{D}_0 \subset \mathcal{D}$ be a compact positively invariant set for the nonlinear feedback controlled uncertain dynamical system* (3). *$\mathcal{D}_0$ is* robustly Lyapunov stable *if for every open neighborhood $\mathcal{O}_1 \subseteq \mathcal{D}$ of $\mathcal{D}_0$, there exists an open neighborhood $\mathcal{O}_2 \subseteq \mathcal{O}_1$* of *$\mathcal{D}_0$ such that $x(t) \in \mathcal{O}_1$, $t \geq 0$, for all $x_0 \in \mathcal{O}_2$ and $F(\cdot,\cdot) \in \mathcal{F}$. $\mathcal{D}_0$ is* robustly attractive *if there* *exists an open neighborhood* $\mathcal{O}_3 \subseteq \mathcal{D}$ *of* $\mathcal{D}_0$ *such that* $\mathcal{P}_{x_0}^+ \subseteq \mathcal{D}_0$ *for all* $x_0 \in \mathcal{O}_3$ *and* $F(\cdot,\cdot) \in \mathcal{F}$. $\mathcal{D}_0$ *is* robustly asymptotically stable *if it is robustly Lyapunov stable and robustly attractive.* $\mathcal{D}_0$ *is* robustly globally asymptotically stable *if it is robustly Lyapunov stable and* $\mathcal{P}_{x_0}^+ \subseteq \mathcal{D}_0$ *for all* $x_0 \in \mathbb{R}^n$ *and* $F(\cdot,\cdot) \in \mathcal{F}$. *Finally,* $\mathcal{D}_0$ *is* unstable *if it is not robustly Lyapunov stable.*

Next, we give a set theoretic definition involving the domain, or region, of attraction of the compact positively invariant set $\mathcal{D}_c$ of (3).

DEFINITION 2.6 *Suppose the compact positively invariant set* $\mathcal{D}_0 \subset \mathcal{D}$ *of* (3) *is robustly attractive. Then the* domain of attraction $\mathcal{D}_A$ *of* $\mathcal{D}_0$ *is defined as*

$$\mathcal{D}_A \triangleq \{x_0 \in \mathcal{D}: \mathcal{P}_{x_0}^+ \subseteq \mathcal{D}_0\}. \tag{4}$$

Recall that $\mathcal{D}_A$ is an open, connected invariant set [13, Proposition 4.15, p. 88].

Next, we present a key theorem due to Weierstrass involving lower semicontinuous functions on compact sets. For the statement of the result the following definition is needed.

DEFINITION 2.7 *Let* $\mathcal{D}_c \subset \mathcal{D}$. *A function* $V: \mathcal{D}_c \to \mathbb{R}$ *is* lower semicontinuous *on* $\mathcal{D}_c$ *if for every sequence* $\{x_n\}_{n=0}^{\infty} \subset \mathcal{D}_c$ *such that* $\lim_{n\to\infty} x_n = x$, $V(x) \le \liminf_{n\to\infty} V(x_n)$.

THEOREM 2.2 [14] *Suppose* $\mathcal{D}_c \subset \mathcal{D}$ *is compact and* $V: \mathcal{D}_c \to \mathbb{R}$ *is lower semicontinuous. Then there exists* $x^* \in \mathcal{D}_c$ *such that* $V(x^*) \le V(x)$, $x \in \mathcal{D}_c$.

## 3. GENERALIZED ROBUST STABILITY THEOREMS FOR NONLINEAR UNCERTAIN FEEDBACK SYSTEMS

Most Lyapunov stability and invariant set theorems presented in the literature require that the Lyapunov function for a nonlinear dynamical system be a $C^1$ function with a negative-definite derivative (see [1,3,15–18] and the numerous references therein). However, even though in the case of discontinuous system dynamics with continuous motions standard Lyapunov theory is applicable, it might be simpler to construct discontinuous "Lyapunov" functions to establish closed-loop system stability. In particular, as mentioned in the introduction, the key step in obtaining a general nonlinear robust stabilization framework is to use several different robust controllers designed over several fixed operating points covering the system's operating range in the state space, and to switch between them over this range. Even though for each operating range one can construct a $C^1$ Lyapunov function, to show closed-loop system robust stability over the whole system operating envelope and prescribed range of system uncertainty for a given switching robust control strategy, a *generalized* Lyapunov function obtained by minimizing a potential function associated with domains of attraction for each operating range is constructed. As will be shown in Section 5 the generalized Lyapunov function is nonsmooth and noncontinuous. Hence, in this section we develop generalized Lyapunov and invariant set theorems for nonlinear feedback controlled uncertain dynamical systems wherein all regularity assumptions on the Lyapunov function and the closed-loop system dynamics are removed. The following result generalizes the Barbashin–Krasovskii–LaSalle invariant set theorems [3,17,19–21] to the case where the Lyapunov function is lower semicontinuous. For the remainder of the results of this paper define the notation

$$\mathcal{R}_\gamma \triangleq \bigcap_{c>\gamma} \overline{V^{-1}([\gamma, c])}, \tag{5}$$

for arbitrary $V : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\gamma \in \mathbb{R}$, and let $\mathcal{M}_\gamma$ denote the largest invariant set (with respect to (3)) contained in $\mathcal{R}_\gamma$.

THEOREM 3.1 *Consider the nonlinear feedback controlled uncertain dynamical system* (3), *let $x(t)$, $t \in \mathcal{I}_{x_0}$, denote the solution to* (3), *and let $\mathcal{D}_c \subset \mathcal{D}$ be a compact positively invariant set with respect to* (3). *Assume that there exists a lower semicontinuous function $V : \mathcal{D}_c \to \mathbb{R}$ such that $V(x(t)) \leq V(x(\tau))$, $0 \leq \tau \leq t$, for all $x_0 \in \mathcal{D}_c$. If $x_0 \in \mathcal{D}_c$, then $x(t) \to \mathcal{M} \triangleq \bigcup_{\gamma \in \mathbb{R}} \mathcal{M}_\gamma$ as $t \to \infty$ for all $F(\cdot, \cdot) \in \mathcal{F}$.*

*Proof* Let $x(t)$, $t \in \mathcal{I}_{x_0}$, be the solution to (3) with $x_0 \in \mathcal{D}_c$ so that $[0, +\infty) \subseteq \mathcal{I}_{x_0}$. Since $V(\cdot)$ is lower semicontinuous on the compact set $\mathcal{D}_c$, there exists $\beta \in \mathbb{R}$ such that $V(x) \geq \beta, x \in \mathcal{D}_c$. Hence, since $V(x(t))$, $t \geq 0$, is nonincreasing, $\gamma_{x_0} \triangleq \lim_{t\to\infty} V(x(t))$, $x_0 \in \mathcal{D}_c$, exists. Now, for all $p \in \mathcal{P}_{x_0}^+$ there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, such that $\lim_{n\to\infty} x(t_n) = p$. Next, since $V(x(t_n))$, $n \geq 0$, is nonincreasing it follows that for all $n \geq 0$, $\gamma_{x_0} \leq V(x(t_n)) \leq V(x(t_N))$, $n \geq N$, or, equivalently, since $\mathcal{D}_c$ is positively invariant, $x(t_n) \in V^{-1}([\gamma_{x_0}, V(x(t_N))])$, $n \geq N$. Now, since $\lim_{n\to\infty} x(t_n) = p$ it follows that $p \in \overline{V^{-1}([\gamma_{x_0}, V(x(t_n))])}$, $n \geq 0$. Furthermore, since $\lim_{n\to\infty} V(x(t_n)) = \gamma_{x_0}$ it follows that for every $c > \gamma_{x_0}$, there exists $n \geq 0$ such that $\gamma_{x_0} \leq V(x(t_n)) \leq c$ which implies that for every $c > \gamma_{x_0}$, $p \in \overline{V^{-1}([\gamma_{x_0}, c])}$. Hence, $p \in \mathcal{R}_{\gamma_{x_0}}$ which implies that $\mathcal{P}_{x_0}^+ \subseteq \mathcal{R}_{\gamma_{x_0}}$. Now, since $\mathcal{D}_c$ is compact and positively invariant it follows that the forward solution $x(t)$, $t \geq 0$, to (3) is bounded for all $x_0 \in \mathcal{D}_c$ and hence it follows from Lemma 2.1 that $\mathcal{P}_{x_0}^+$ is a nonempty, compact, connected invariant set which further implies that $\mathcal{P}_{x_0}^+$ is a subset of the largest invariant set contained in $\mathcal{R}_{\gamma_{x_0}}$, that is, $\mathcal{P}_{x_0}^+ \subseteq \mathcal{M}_{\gamma_{x_0}}$. Hence, for all $x_0 \in \mathcal{D}_c$, $\mathcal{P}_{x_0}^+ \subseteq \mathcal{M}$. Finally, since $x(t) \to \mathcal{P}_{x_0}^+$ as $t \to \infty$ it follows that $x(t) \to \mathcal{M}$ as $t \to \infty$ for all $F(\cdot,\cdot) \in \mathcal{F}$. □

*Remark 3.1* Note that since $V^{-1}([\gamma, c]) = \{x \in \mathcal{D}_c\colon V(x) \geq \gamma\} \cap \{x \in \mathcal{D}_c\colon V(x) \leq c\}$ and $\{x \in \mathcal{D}_c\colon V(x) \leq c\}$ is a closed set, it follows that $\hat{\mathcal{R}}_{\gamma,c} \subset \{x \in \mathcal{D}_c\colon V(x) < \gamma\}$, where $\hat{\mathcal{R}}_{\gamma,c} \triangleq \overline{V^{-1}([\gamma, c])} \backslash V^{-1}([\gamma, c])$, $c > \gamma$, for a fixed $\gamma \in \mathbb{R}$. Hence,

$$\mathcal{R}_\gamma = \bigcap_{c>\gamma} \left(V^{-1}([\gamma, c]) \cup \hat{\mathcal{R}}_{\gamma,c}\right) = V^{-1}(\gamma) \cup \hat{\mathcal{R}}_\gamma,$$

where $\hat{\mathcal{R}}_\gamma \triangleq \bigcap_{c>\gamma} \hat{\mathcal{R}}_{\gamma,c}$, is such that $V(x) < \gamma$, $x \in \hat{\mathcal{R}}_\gamma$. Finally, if $V(\cdot)$ is $C^0$ then $\hat{\mathcal{R}}_{\gamma,c} = \emptyset$, $\gamma \in \mathbb{R}$, $c > \gamma$, and hence $\mathcal{R}_\gamma = V^{-1}(\gamma)$.

*Remark 3.2* Note that if $V\colon \mathcal{D}_c \to \mathbb{R}$ is a lower semicontinuous function such that all the conditions of Theorem 3.1 are satisfied, then for every $x_0 \in \mathcal{D}_c$ there exists $\gamma_{x_0} \leq V(x_0)$ such that $\mathcal{P}_{x_0}^+ \subseteq \mathcal{M}_{\gamma_{x_0}} \subseteq \mathcal{M}$.

*Remark 3.3* It is important to note that as in standard Lyapunov and invariant set theorems involving $C^1$ functions, Theorem 3.1 allows one to characterize the invariant set $\mathcal{M}$ *without* knowledge of the closed-loop system trajectories $x(t)$, $t \in \mathcal{I}_{x_0}$. Similar remarks hold for the remainder of the theorems in this section.

Next, we sharpen the results of Theorem 3.1 by providing a refined construction of the invariant set $\mathcal{M}$. In particular, we show that the closed-loop system trajectories converge to a union of largest invariant sets contained on the boundary of the intersections over *finite* intervals of the closure of generalized Lyapunov level surfaces.

THEOREM 3.2 *Consider the nonlinear feedback uncertain controlled uncertain dynamical system* (3), *let* $\mathcal{D}_{\mathrm{c}}$ *and* $\mathcal{D}_0$ *be compact positively invariant sets with respect to* (3) *such that* $\mathcal{D}_0 \subset \mathcal{D}_{\mathrm{c}} \subset \mathcal{D}$, *and let* $x(t)$, $t \in \mathcal{I}_{x_0}$, *denote the solution to* (3) *corresponding to* $x_0 \in \mathcal{D}_{\mathrm{c}}$. *Assume that there exists a lower semicontinuous function* $V: \mathcal{D}_{\mathrm{c}} \to \mathbb{R}$ *such that*

$$V(x) = 0, \quad x \in \mathcal{D}_0, \tag{6}$$

$$V(x) > 0, \quad x \in \mathcal{D}_{\mathrm{c}}, \ x \notin \mathcal{D}_0, \tag{7}$$

$$V(x(t)) \leq V(x(\tau)), \quad 0 \leq \tau \leq t. \tag{8}$$

*Furthermore, assume that for all* $x_0 \in \mathcal{D}_{\mathrm{c}}$, $x_0 \notin \mathcal{D}_0$, *there exists an increasing unbounded sequence* $\{t_n\}_{n=0}^{\infty}$, *with* $t_0 = 0$, *such that*

$$V(x(t_{n+1})) < V(x(t_n)), \quad n = 0, 1, \ldots \tag{9}$$

*Then, either* $\mathcal{M}_\gamma \subset \hat{\mathcal{R}}_\gamma \triangleq \mathcal{R}_\gamma \backslash V^{-1}(\gamma)$, $\gamma > 0$, *or* $\mathcal{M}_\gamma = \emptyset$. *Furthermore, if* $x_0 \in \mathcal{D}_{\mathrm{c}}$, *then* $x(t) \to \hat{\mathcal{M}} \triangleq \bigcup_{\gamma \in \mathcal{G}} \mathcal{M}_\gamma$ *as* $t \to \infty$ *for all* $F(\cdot,\cdot) \in \mathcal{F}$, *where* $\mathcal{G} \triangleq \{\gamma \geq 0 : \mathcal{R}_\gamma \cap \mathcal{D}_0 \neq \emptyset\}$. *If, in addition,* $\mathcal{D}_0 \subset \mathring{\mathcal{D}}_{\mathrm{c}}$ *and* $V(\cdot)$ *is continuous on* $\mathcal{D}_0$, *then* $\mathcal{D}_0$ *is locally asymptotically stable for all* $F(\cdot,\cdot) \in \mathcal{F}$ *and* $\mathcal{D}_{\mathrm{c}}$ *is a subset of the domain of attraction.*

*Proof* Since $\mathcal{D}_{\mathrm{c}}$ is a compact positively invariant set, it follows that for all $x_0 \in \mathcal{D}_{\mathrm{c}}$, the forward solution $x(t)$, $t \geq 0$, to (3) is bounded. Hence, it follows from Lemma 2.1 that, for all $x_0 \in \mathcal{D}_{\mathrm{c}}$, $\mathcal{P}_{x_0}^{+}$ is a nonempty, compact, connected invariant set. Next, it follows from Theorem 3.1, Remark 3.2, and the fact that $V(\cdot)$ is positive-definite (with respect to $\mathcal{D}_{\mathrm{c}} \backslash \mathcal{D}_0$), that for every $x_0 \in \mathcal{D}_{\mathrm{c}}$ there exists $\gamma_{x_0} \geq 0$ such that $\mathcal{P}_{x_0}^{+} \subseteq \mathcal{M}_{\gamma_{x_0}} \subseteq \mathcal{R}_{\gamma_{x_0}}$. Now, given $x(0) \in V^{-1}(\gamma_{x_0}), \gamma_{x_0} > 0$, (9) implies that there exists $t_1 > 0$ such that $V(x(t_1)) < \gamma_{x_0}$ and $x(t_1) \notin V^{-1}(\gamma_{x_0})$. Hence, $V^{-1}(\gamma_{x_0}) \subset \mathcal{R}_{\gamma_{x_0}}$ does not contain any invariant set. Alternatively, if $x(0) \in \hat{\mathcal{R}}_{\gamma_{x_0}}$ then $V(x(0)) < \gamma_{x_0}$ and (9) implies that $x(t) \notin V^{-1}(\gamma_{x_0})$, $t \geq 0$. Hence, any invariant set contained in $\mathcal{R}_{\gamma_{x_0}}$ is a subset of

$\hat{\mathcal{R}}_{\gamma_{x_0}}$, which implies that $\mathcal{M}_{\gamma_{x_0}} \subset \hat{\mathcal{R}}_{\gamma_{x_0}}$, $\gamma_{x_0} > 0$. If $\hat{\gamma} > 0$ is such that $\hat{\gamma} \neq \gamma_{x_0}$, for all $x_0 \in \mathcal{D}_c$, then there does not exist $x_0 \in \mathcal{D}_c$ such that $\mathcal{P}^+_{x_0} \subseteq \mathcal{R}_{\hat{\gamma}}$ and hence $\mathcal{M}_{\hat{\gamma}} = \emptyset$. Now, *ad absurdum*, suppose $\mathcal{D}_0 \cap \mathcal{P}^+_{x_0} = \emptyset$. Since $V(\cdot)$ is lower semicontinuous it follows from Theorem 2.2 that there exists $\hat{x} \in \mathcal{P}^+_{x_0}$ such that $\alpha = V(\hat{x}) \leq V(x)$, $x \in \mathcal{P}^+_{x_0}$. Now, with $x(0) = \hat{x} \notin \mathcal{D}_0$ it follows from (9) that there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, such that $V(x(t_{n+1})) < V(x(t_n))$, $n = 0, 1, \ldots$, which implies that there exists $t > 0$ such that $V(x(t)) < \alpha$ which further implies that $x(t) \notin \mathcal{P}^+_{x_0}$ contradicting the fact that $\mathcal{P}^+_{x_0}$ is an invariant set. Hence, there exists $q \in \mathcal{D}_0$ such that $q \in \mathcal{P}^+_{x_0} \subseteq \mathcal{R}_{\gamma_{x_0}}$ which implies that $\mathcal{R}_{\gamma_{x_0}} \cap \mathcal{D}_0 \neq \emptyset$. Thus, $\gamma_{x_0} \in \mathcal{G}$ for all $x_0 \in \mathcal{D}_c$ which further implies that $\mathcal{P}^+_{x_0} \subseteq \hat{\mathcal{M}}$. Now, since $x(t) \to \mathcal{P}^+_{x_0} \subseteq \hat{\mathcal{M}}$ as $t \to \infty$ it follows that $x(t) \to \hat{\mathcal{M}}$ as $t \to \infty$ for all $F(\cdot,\cdot) \in \mathcal{F}$.

Next, we show that if $V(\cdot)$ is continuous on $\mathcal{D}_0 \subset \mathring{\mathcal{D}}_c$, then the compact positively invariant set $\mathcal{D}_0$ of (3) is robustly Lyapunov stable. Let $\mathcal{O}_1 \subseteq \mathcal{D}_c$ be an open neighborhood of $\mathcal{D}_0$. Since $\partial\mathcal{O}_1$ is compact and $V(x)$, $x \in \mathcal{D}_c$, is lower semicontinuous, it follows from Theorem 2.2 that there exists $\alpha = \min_{x \in \partial\mathcal{O}_1} V(x)$. Note that $\alpha > 0$ since $\mathcal{D}_0 \cap \partial\mathcal{O}_1 = \emptyset$ and $V(x) > 0$, $x \in \mathcal{D}_c$, $x \notin \mathcal{D}_0$. Next, using the facts that $V(x) = 0$, $x \in \mathcal{D}_0$, and $V(\cdot)$ is continuous on $\mathcal{D}_0$, it follows that the set $\mathcal{O}_2 \triangleq \{x \in \mathcal{O}_1\colon V(x) < \alpha\}^{\circ}$ is not empty. Now, it follows from (8) that for all $x(0) \in \mathcal{O}_2$,

$$V(x(t)) \leq V(x(0)) < \alpha, \quad t \geq 0,$$

which, since $V(x) \geq \alpha$, $x \in \partial\mathcal{O}_1$, implies that $x(t) \notin \partial\mathcal{O}_1$, $t \geq 0$. Hence, for every open neighborhood $\mathcal{O}_1 \subseteq \mathcal{D}_c$ of $\mathcal{D}_0$, there exists an open neighborhood $\mathcal{O}_2 \subseteq \mathcal{O}_1$ of $\mathcal{D}_0$ such that, if $x(0) \in \mathcal{O}_2$, then $x(t) \in \mathcal{O}_1$, $t \geq 0$, which proves Lyapunov stability of the compact positively invariant set $\mathcal{D}_0$ of (3) for all $F(\cdot,\cdot) \in \mathcal{F}$. Finally, from the continuity of $V(\cdot)$ on $\mathcal{D}_0$ and the fact that $V(x) = 0$ for all $x \in \mathcal{D}_0$, it follows that $\mathcal{G} = \{0\}$ and $\hat{\mathcal{M}} \equiv \mathcal{M}_0$. Hence, $\mathcal{P}^+_{x_0} \subseteq \mathcal{D}_0$ for all $x_0 \in \mathcal{D}_c$ establishing local asymptotic stability of the compact positively invariant set $\mathcal{D}_0$ of (3) for all $F(\cdot,\cdot) \in \mathcal{F}$ with a subset of the domain of attraction given by $\mathcal{D}_c$. □

*Remark 3.4* If in Theorem 3.2 $\hat{\mathcal{M}} \subseteq \mathcal{D}_0$, then the compact positively invariant set $\mathcal{D}_0$ of (3) is attractive for all $F(\cdot,\cdot) \in \mathcal{F}$. If, in addition, $V(\cdot)$ is continuous on $\mathcal{D}_0 \subset \mathcal{D}_c$ then the compact positively invariant set $\mathcal{D}_0$ of (3) is locally asymptotically stable for all $F(\cdot,\cdot) \in \mathcal{F}$. In both cases, $\mathcal{D}_c$ is a subset of the domain of attraction.

A lower semicontinuous function $V(\cdot)$, with $V(\cdot)$ being continuous on $\mathcal{D}_0$, satisfying (6) and (7) is called a *generalized Lyapunov function candidate* for the nonlinear feedback controlled uncertain dynamical system (3). If, additionally, $V(\cdot)$ satisfies (8), $V(\cdot)$ is called a *generalized Lyapunov function* for the nonlinear feedback controlled uncertain dynamical system (3). Note that in the case where the function $V(\cdot)$ is $C^1$ on $\mathcal{D}_c$ in Theorem 3.2, it follows that $V(x(t)) \leq V(x(\tau))$, for all $t \geq \tau \geq 0$, is equivalent to $\dot{V}(x) \triangleq V'(x)F(x,\phi(x)) \leq 0$, $x \in \mathcal{D}_c$, $F(\cdot,\cdot) \in \mathcal{F}$. In this case conditions (6)–(8) in Theorem 3.2 specialize to the standard Lyapunov stability conditions [3,15,17].

Next, we present a generalized global invariant set theorem for guaranteeing global robust attraction and global asymptotic robust stability of a compact positively invariant set of a nonlinear feedback controlled uncertain dynamical system.

THEOREM 3.3 *Consider the nonlinear feedback controlled uncertain dynamical system* (3) *with* $\mathcal{D} = \mathbb{R}^n$ *and* $\mathcal{U} = \mathbb{R}^m$ *and let* $x(t)$, $t \in \mathcal{I}_{x_0}$, *denote the solution to* (3). *Assume that there exists a compact positively invariant set* $\mathcal{D}_0$ *with respect to* (3) *and a lower semicontinuous function* $V : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$V(x) = 0, \quad x \in \mathcal{D}_0, \tag{10}$$

$$V(x) > 0, \quad x \in \mathbb{R}^n, \ x \notin \mathcal{D}_0, \tag{11}$$

$$V(x(t)) \leq V(x(\tau)), \quad 0 \leq \tau \leq t, \tag{12}$$

$$V(x) \to \infty \quad \text{as } \|x\| \to \infty. \tag{13}$$

*Then for all* $x_0 \in \mathbb{R}^n$, $x(t) \to \mathcal{M} \triangleq \bigcup_{\gamma \geq 0} \mathcal{M}_\gamma$, *as* $t \to \infty$ *for all* $F(\cdot,\cdot) \in \mathcal{F}$. *If, in addition, for all* $x_0 \in \mathbb{R}^n$, $x_0 \notin \mathcal{D}_0$, *there exists an increasing unbounded sequence* $\{t_n\}_{n=0}^{\infty}$, *with* $t_0 = 0$, *such that*

$$V(x(t_{n+1})) < V(x(t_n)), \quad n = 0, 1, \ldots, \tag{14}$$

*then, either* $\mathcal{M}_\gamma \subset \hat{\mathcal{R}}_\gamma \triangleq \mathcal{R}_\gamma \backslash V^{-1}(\gamma)$, *or* $\mathcal{M}_\gamma = \emptyset$, $\gamma > 0$. *Furthermore,* $x(t) \to \hat{\mathcal{M}} \triangleq \bigcup_{\gamma \in \mathcal{G}} \mathcal{M}_\gamma$ *as* $t \to \infty$ *for all* $F(\cdot,\cdot) \in \mathcal{F}$, *where* $\mathcal{G} \triangleq \{\gamma \geq 0 \colon \mathcal{R}_\gamma \cap \mathcal{D}_0 \neq \emptyset\}$. *Finally, if* $V(\cdot)$ *is continuous on* $\mathcal{D}_0$ *then the compact positively invariant set* $\mathcal{D}_0$ *of* (3) *is robustly globally asymptotically stable.*

*Proof* Note that since $V(x) \to \infty$ as $\|x\| \to \infty$ it follows that for every $\beta > 0$ there exists $r > 0$ such that $V(x) > \beta$ for all $\|x\| > r$, or, equivalently, $V^{-1}([0,\beta]) \subseteq \{x\colon \|x\| \le r\}$ which implies that $V^{-1}([0,\beta])$ is bounded for all $\beta > 0$. Hence, for all $x_0 \in \mathbb{R}^n$, $V^{-1}([0,\beta_{x_0}])$ is bounded, where $\beta_{x_0} \triangleq V(x_0)$. Furthermore, since $V(\cdot)$ is a positive-definite lower semicontinuous function, it follows that $V^{-1}([0,\beta_{x_0}])$ is closed and, since $V(x(t))$, $t \ge 0$, is nonincreasing, $V^{-1}([0,\beta_{x_0}])$ is positively invariant. Hence, for every $x_0 \in \mathbb{R}^n$, $V^{-1}([0,\beta_{x_0}])$ is a compact positively invariant set. Now, with $\mathcal{D}_c = V^{-1}([0,\beta_{x_0}])$ it follows from Theorem 3.1 and Remark 3.2 that there exists $\gamma_{x_0} \in [0,\beta_{x_0}]$ such that $\mathcal{P}^+_{x_0} \subseteq \mathcal{M}_{\gamma_{x_0}} \subset \hat{\mathcal{R}}_{\gamma_{x_0}}$ which implies that $x(t) \to \mathcal{M}$ as $t \to \infty$ for all $F(\cdot,\cdot) \in \mathcal{F}$. If, in addition, for all $x_0 \in \mathbb{R}^n$, $x_0 \notin \mathcal{D}_0$, there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, such that (14) holds then it follows from Theorem 3.2 that $x(t) \to \hat{\mathcal{M}}$ as $t \to \infty$ for all $F(\cdot,\cdot) \in \mathcal{F}$.

Finally, if $V(\cdot)$ is continuous on $\mathcal{D}_0$ then robust Lyapunov stability follows as in the proof of Theorem 3.2. Furthermore, in this case, $\mathcal{G} = \{0\}$ which implies that $\hat{\mathcal{M}} = \mathcal{M}_0$. Hence, $\mathcal{P}^+_{x_0} \subseteq \mathcal{D}_0$ establishing global asymptotic robust stability of the compact positively invariant set $\mathcal{D}_0$ of (3). □

*Remark 3.5* If in Theorems 3.2 and 3.3 the function $V(\cdot)$ is $C^1$ on $\mathcal{D}_c$ and $\mathbb{R}^n$, respectively, $\mathcal{D}_0 \equiv \{0\}$, and $V'(x)F(x,\phi(x)) < 0$, $x \in \mathcal{D}_c$, $x \neq 0$, $F(\cdot,\cdot) \in \mathcal{F}$, then every increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, is such that $V(x(t_{n+1})) < V(x(t_n))$, $n = 0,1,\ldots$ In this case, Theorems 3.2 and 3.3 specialize to the standard Lyapunov stability theorems for local and global asymptotic robust stability, respectively, as applied to a closed-loop feedback controlled uncertain system.

It is important to note that even though the robust stability conditions appearing in Theorems 3.1–3.3 are system trajectory dependent, in Section 5 we present a hierarchical robust switching nonlinear controller guaranteeing nonlinear system stabilization over a prescribed range of system parametric uncertainty *without* requiring knowledge of the closed-loop system trajectories. Finally, we note that the concept of lower semicontinuous Lyapunov functions has been considered in the literature. Specifically, lower semicontinuous Lyapunov functions have been considered in [13,22] with [22] focusing on viability theory and differential inclusions. However, the present formulation provides *new* invariant set stability theorem generalizations characterizing system limit sets in terms of lower semicontinuous Lyapunov functions not considered in [13,22].

## 4. PARAMETERIZED NOMINAL SYSTEM EQUILIBRIA, SYSTEM ATTRACTORS, AND DOMAINS OF ATTRACTION

The nonlinear robust control design framework developed in this paper is predicated on a hierarchical robust switching nonlinear controller architecture parameterized over a set of nominal system equilibria. It is important to note that both the nominal dynamical system and the robust controller for each parameterized nominal equilibrium can be nonlinear and thus local set point designs are in general nonlinear. Hence, the nonlinear controlled uncertain dynamical system can be viewed as a collection of controlled uncertain subsystems with a hierarchical robust switching controller architecture. In this section we concentrate on robust nonlinear stabilization of compact positively invariant sets, parameterized in $\mathcal{D}$, of the nonlinear closed-loop uncertain subsystems. Specifically, we consider the nonlinear controlled uncertain dynamical system (1) with the origin being an equilibrium point of the nominal system corresponding to the control $u=0$, that is, $F_{\rm n}(0,0)=0$. Furthermore, we assume that given a mapping $\varphi:\mathcal{D}\times\Lambda\to\mathcal{U}$, $\varphi(0,0)=0$, there exists a continuous function $\psi:\Lambda_{\rm o}\to\mathcal{D}_{\rm o}$, where $\mathcal{D}_{\rm o}\subseteq\mathcal{D}$, $0\in\mathcal{D}_{\rm o}$, and $\Lambda_{\rm o}\subseteq\Lambda$, $0\in\Lambda_{\rm o}$, such that $F_{\rm n}(x_\lambda,\varphi(x_\lambda,\lambda))=0$ with $x_\lambda=\psi(\lambda)\in\mathcal{D}_{\rm o}$ for all $\lambda\in\Lambda_{\rm o}$. As discussed in Section 2, this is a necessary condition for nominal parametric stability with respect to $\Lambda_{\rm o}$ as defined in [7,8] while Theorem 2.1 provides sufficient conditions for guaranteeing the existence of such a parameterization.

Next, we consider a family of stabilizing feedback control laws for the *nominal* system given by

$$\Phi\triangleq\{\phi_\lambda:\mathcal{D}\to\mathcal{U}\colon\ \phi_\lambda\in{\rm C}^0,\phi_\lambda(x_\lambda)=\varphi(x_\lambda,\lambda),\ \lambda\in\Lambda_{\rm S}\},\quad \Lambda_{\rm S}\subseteq\Lambda_{\rm o},\tag{15}$$

such that, for $\phi_\lambda(\cdot)\in\Phi$, $\lambda\in\Lambda_{\rm S}$, the nonlinear closed-loop *nominal* system

$$\dot{x}(t)=F_{\rm n}(x(t),\phi_\lambda(x(t))),\quad x(0)=x_0,\ t\in\mathcal{I}_{x_0},\tag{16}$$

has an asymptotically stable equilibrium point $x_\lambda \in \mathcal{D}_o \subseteq \mathcal{D}$ with a corresponding Lyapunov function $V_\lambda(\cdot)$. Hence, in the terminology of [7,8], (16) is (nominally) parametrically asymptotically stable with respect to $\Lambda_S \subseteq \Lambda_o$. Here, we assume that for each $\lambda \in \Lambda_S$, the linear or nonlinear feedback controllers $\phi_\lambda(\cdot)$ are given. In particular, these controllers correspond to local set point designs and can be obtained using any appropriate standard linear or nonlinear stabilization scheme with a domain of attraction for each $\lambda \in \Lambda_S$. For example, appropriate nonlinear stabilization techniques such as feedback linearization [23], nonlinear $H_\infty$ control [24], constructive nonlinear control [25], optimal nonlinear control [26], and nonlinear regulation via state-dependent Riccati techniques [27], as well as linear-quadratic stabilization schemes based on locally approximated linearizations, can be used to design the controllers $\phi_\lambda(\cdot)$ for each $\lambda \in \Lambda_S$. It is important to note that even though $x_\lambda$, $\lambda \in \Lambda_S$, is an equilibrium point of the nominal system (2), in general, $x_\lambda$, $\lambda \in \Lambda_S$, is not an equilibrium point for the uncertain system (1). Hence, $V_\lambda(\cdot)$ is *not* a standard Lyapunov function for the nonlinear closed-loop uncertain system

$$\dot{x}(t) = F(x(t), \phi_\lambda(x(t))), \quad x(0) = x_0, \ F(\cdot,\cdot) \in \mathcal{F}, \ t \in \mathcal{I}_{x_0}. \tag{17}$$

However, under an additional assumption on the structure of the system uncertainty, it can be shown that $u = \phi_\lambda(x)$ is a robust control law that *robustly* asymptotically stabilizes a compact positively invariant set $\mathcal{N}_\lambda$, containing the nominal equilibrium point $x_\lambda$, $\lambda \in \Lambda_S$, with domain of attraction $\mathcal{D}_\lambda$. In this case, $V_\lambda(\cdot)$ serves as a Lypunov function of the uncertain system guaranteeing stability with respect to a compact positively invariant set. In particular, defining $\Delta F(x,u) \triangleq F(x,u) - F_n(x,u)$ and assuming that $V'_\lambda(x)\Delta F(x, \phi_\lambda(x)) < -V'_\lambda(x) \times F_n(x, \phi_\lambda(x))$ for all $x \in \mathcal{D}_\lambda$ such that $\|x - x_\lambda\| > r$, $r > 0$, it follows that $\phi_\lambda(\cdot)$ is a robustly stabilizing feedback controller of a compact positively invariant set $\mathcal{N}_\lambda$ of (17).

Next, given a stabilizing feedback robust controller $\phi_\lambda(\cdot)$ for each $\lambda \in \Lambda_S$, we provide a guaranteed subset of the domain of attraction $\mathcal{D}_\lambda$ of a compact positively invariant set $\mathcal{N}_\lambda$ for the nonlinear closed-loop *uncertain* system using Lyapunov stability theory.

THEOREM 4.1 [15] *Let $\lambda \in \Lambda_S$. Consider the nonlinear uncertain closed-loop system* (17) *with $\phi_\lambda(\cdot) \in \Phi$ and let $\mathcal{N}_\lambda$ be a compact positively* *invariant set of* (17). *Furthermore, let* $\mathcal{X}_\lambda \subset \mathcal{D}$ *be a compact neighborhood of* $\mathcal{N}_\lambda$. *Then* $\mathcal{N}_\lambda$ *is a robustly asymptotically stable set of* (17) *for all* $F(\cdot,\cdot) \in \mathcal{F}$, *if and only if there exists a* $C^0$ *function* $V_\lambda : \mathcal{X}_\lambda \to \mathbb{R}$, *with* $V_\lambda$ $C^1$ *on* $\mathcal{X}_\lambda \backslash \mathcal{N}_\lambda$, *such that*

$$V_\lambda(x) = 0, \quad x \in \mathcal{N}_\lambda, \tag{18}$$

$$V_\lambda(x) > 0, \quad x \in \mathcal{X}_\lambda \backslash \mathcal{N}_\lambda, \tag{19}$$

$$\dot{V}_\lambda(x) \triangleq V'_\lambda(x) F(x, \phi_\lambda(x)) < 0, \quad x \in \mathcal{X}_\lambda \backslash \mathcal{N}_\lambda, \ F(\cdot,\cdot) \in \mathcal{F}. \tag{20}$$

*In addition, a subset of the domain of attraction of* $\mathcal{N}_\lambda$ *is given by*

$$\mathcal{D}_\lambda \triangleq V_\lambda^{-1}([0, c_\lambda]), \tag{21}$$

*where* $c_\lambda \triangleq \max\{\beta > 0\colon\ V_\lambda^{-1}([0,\beta]) \subseteq \mathcal{X}_\lambda\}$.

*Remark 4.1* It follows from Theorem 4.1 that for all $x_0 \in \mathcal{D}_\lambda$ and each open set $\mathcal{O}$ such that $\mathcal{N}_\lambda \subset \mathcal{O} \subset \mathcal{D}_\lambda$, there exists a finite time $T > 0$ such that $x(t) \in \mathcal{O}$ for all $t \geq T$ and $F(\cdot,\cdot) \in \mathcal{F}$. Alternatively, Theorem 4.1 can be restated by requiring $V_\lambda(\cdot)$ to be a $C^1$ function on $\mathcal{X}_\lambda$ such that conditions (19) and (20) hold and $V_\lambda(x) \geq 0$, $x \in \mathcal{N}_\lambda$. In this case the compact positively invariant set $\mathcal{N}_\lambda$ is defined by $\mathcal{N}_\lambda \triangleq V_\lambda^{-1}([0, b_\lambda])$, where $b_\lambda \triangleq \inf\{\beta \geq 0\colon\ \dot{V}_\lambda(x) < 0,\ x \in V_\lambda^{-1}([\beta, c_\lambda])\}$.

Note that conditions (18)–(20) imply that $V_\lambda(x)$ is a Lyapunov function guaranteeing robust stability of a compact positively invariant set $\mathcal{N}_\lambda$ of the closed-loop uncertain system (17). However, Condition (20) is unverifiable since it is dependent on the uncertain system dynamics $F(\cdot,\cdot) \in \mathcal{F}$. This condition is implied by the conditions

$$V'_\lambda(x) F(x, \phi_\lambda(x)) \leq V'_\lambda(x) F_{\mathrm{n}}(x, \phi_\lambda(x)) + \Gamma_\lambda(x, \phi_\lambda(x)),$$
$$x \in \mathcal{X}_\lambda \backslash \mathcal{N}_\lambda, \ F(\cdot,\cdot) \in \mathcal{F}, \tag{22}$$

$$V'_\lambda(x) F_{\mathrm{n}}(x, \phi_\lambda(x)) + \Gamma_\lambda(x, \phi_\lambda(x)) < 0, \quad x \in \mathcal{X}_\lambda \backslash \mathcal{N}_\lambda, \tag{23}$$

where $\Gamma_\lambda : \mathcal{D}_\lambda \times \mathcal{U} \longrightarrow \mathbb{R}$, $\lambda \in \Lambda_{\mathrm{S}}$. It is important to note that condition (23) is a verifiable condition since it is independent of the uncertain system dynamics $F(\cdot,\cdot) \in \mathcal{F}$. To apply Theorem 4.1, we specify a bounding function $\Gamma_\lambda(\cdot,\cdot)$ for an uncertainty set $\mathcal{F}$ such that $\Gamma_\lambda(\cdot,\cdot)$ bounds $\mathcal{F}$. In this case conditions (22) and (23) are satisfied. Hence, if the $V_\lambda(x)$ satisfying (18), (19), and (23) can be determined, then robust stability of a compact positively invariant set $\mathcal{N}_\lambda$ of (17) is guaranteed. For further details see Section 6 and [28]. For the remainder of the paper we assume that the structure of the system uncertainty is such that there exists $\Gamma_\lambda(\cdot,\cdot)$ such that (22) and (23) hold.

We stress that the aim of Theorem 4.1 is not to make direct comparisons with existing methods for estimating domains of attraction, but rather in aiding to provide a streamlined presentation of the main results of Section 5 requiring estimates of domains of attraction for local set point designs. Since $\mathcal{D}_\lambda$ given in Theorem 4.1 gives an estimate of the domain of attraction using closed Lyapunov sublevel sets, it may be conservative. To reduce conservatism in estimating a subset of the domain of attraction several alternative methods can be used. For example, maximal Lyapunov functions [29], Zubov's method [15,30], ellipsoidal estimate mappings [31], Carlemann linearizations [32], computer generated Lyapunov functions [33], iterative Lyapunov function constructions [34], trajectory-reversing methods [3], and open Lyapunov sublevel sets [35], can be used to construct less conservative estimates of the domain of attraction.

## 5. ROBUST NONLINEAR SYSTEM STABILIZATION VIA A HIERARCHICAL SWITCHING CONTROLLER ARCHITECTURE

In this section we develop a nonlinear robust stabilization framework predicated on a hierarchical switching controller architecture parameterized over a set of moving *nominal* system equilibria. Specifically, using *equilibria-dependent Lyapunov functions* or, instantaneous (with respect to a given parameterized nominal equilibrium manifold) Lyapunov functions, a hierarchical nonlinear robust control strategy is developed that stabilizes a compact positively invariant set by robustly stabilizing a collection of nonlinear uncertain closed-loop subsystems while providing an explicit expression for a guaranteed domain of attraction. A switching nonlinear robust controller architecture is developed based on a *generalized* lower semicontinuous Lyapunov function obtained by minimizing a potential function, associated with the domain of attraction of each controlled uncertain subsystem, over a given switching set induced by the parameterized nominal system equilibria. In the case where one of the compact positively invariant sets parameterized by the nominal system equilibria is globally robustly asymptotically stable with a given robust subcontroller and a structural topological constraint is enforced on the switching set, the proposed nonlinear robust stabilization framework guarantees global asymptotic robust stability of a compact positively invariant set associated to any given parameterized nominal system equilibrium.

To state the main results of this section several definitions and a key assumption are needed. Recall that the set $\Lambda_S \subseteq \Lambda_o$, $0 \in \Lambda_S$, is such that for every $\lambda \in \Lambda_S$ there exists a robust feedback control law $\phi_\lambda(\cdot) \in \Phi$ such that a compact positively invariant neighborhood $\mathcal{N}_\lambda \subseteq \mathcal{D}$ of the nominal equilibrium point $x_\lambda \in \mathcal{D}_o$ of (16) is robustly asymptotically stable with an estimate of the domain of attraction given by $\mathcal{D}_\lambda$. Since $\mathcal{N}_\lambda$, $\lambda \in \Lambda_S$, is a positively invariant set, it follows from Theorem 4.1 that there exists a Lyapunov function $V_\lambda(\cdot)$ satisfying (18)–(20), and hence, without loss of generality, we can take $\mathcal{D}_\lambda$, $\lambda \in \Lambda_S$, given by (21). Furthermore, we assume that the set-valued map $\Psi : \Lambda_S \rightsquigarrow 2^{\mathcal{D}}$, where $2^{\mathcal{D}}$ denotes the collection of all subsets of $\mathcal{D}$, is such that $\mathcal{D}_\lambda = \Psi(\lambda)$, $\lambda \in \Lambda_S$, is continuous. Here, continuity of a set-valued map is defined in the sense of [22, p. 56] and has the property that the limit of a sequence of a continuous set-valued map is the value of the map at the limit of the sequence. In particular, since $\mathcal{D}_\lambda$, $\lambda \in \Lambda_S$, is given by (21), the continuity of the set-valued map $\Psi(\cdot)$ is guaranteed provided that $V_\lambda(x)$, $x \in \mathcal{D}_\lambda$, and $c_\lambda$ are continuous functions of the parameter $\lambda \in \Lambda_S$. Next, let $\mathcal{S} \subseteq \Lambda_S$, $0 \in \mathcal{S}$, denote a *switching set* such that the following key assumption is satisfied.

ASSUMPTION 5.1 *The switching set $\mathcal{S} \subseteq \Lambda_S$ is such that the following properties hold:*

(i) *There exists a continuous positive-definite function $p : \mathcal{S} \to \mathbb{R}$ such that for all $\lambda \in \mathcal{S}$, $\lambda \neq 0$, there exists $\lambda_1 \in \mathcal{S}$ such that*

$$p(\lambda_1) < p(\lambda), \quad \mathcal{N}_\lambda \subset \overset{\circ}{\mathcal{D}}_{\lambda_1}. \tag{24}$$

(ii) *If $\lambda, \lambda_1 \in \mathcal{S}$, $\lambda \neq \lambda_1$, is such that $p(\lambda) = p(\lambda_1)$, then $\mathcal{D}_\lambda \cap \mathcal{D}_{\lambda_1} = \emptyset$.*

Note that Assumption 5.1 assumes the existence of a positive-definite *potential function* $p(\lambda)$, for all $\lambda$ in the switching set $\mathcal{S}$. It follows that, for each $\lambda \in \mathcal{S}$, there exists an equilibrium point $x_\lambda$ with an associated domain of attraction $\mathcal{D}_\lambda$, and potential value $p(\lambda)$. Hence, every domain of attraction has an associated value of the potential function such that, according to (ii), domains of attraction corresponding to different local set point designs intersect each other only if their corresponding potentials are different. In particular, given $\mathcal{D}_\lambda$, $\lambda \in \mathcal{S}\backslash\{0\}$, it is always possible to find at least one intersecting domain of attraction $\mathcal{D}_{\lambda_1}$, $\lambda_1 \in \mathcal{S}$, such that the potential function decreases and $\overset{\circ}{\mathcal{D}}_{\lambda_1}$ contains $\mathcal{N}_\lambda$. This guarantees that if a forward trajectory $x(t)$, $t \geq 0$, of the controlled uncertain system approaches $\mathcal{N}_\lambda$, then there exists a finite time $T > 0$ such that the trajectory enters $\mathcal{D}_{\lambda_1}$. Finally, it is important to note that the switching set $\mathcal{S}$ is arbitrary. In particular, we do not assume that $\mathcal{S}$ is countable or countably infinite. For example, the switching set $\mathcal{S}$ can have a hybrid topological structure involving isolated points and closed sets homeomorphic to intervals on the real line.

Next, we show that Assumption 5.1 implies that every level set of the potential function $p(\cdot)$ is either empty or consists of only isolated points. Furthermore, in a neighborhood of $\lambda = 0$ every level set of $p(\cdot)$ consists of at most *one* isolated point. For the statement of this result, let $\mathcal{B}_\lambda$, $\lambda \in \mathcal{U}_0$, denote the largest open ball centered at $x_\lambda$ and contained in $\mathcal{D}_\lambda$, that is $\mathcal{B}_\lambda \triangleq \{x \in \mathcal{D}\colon \|x - x_\lambda\| < r_\lambda\}$, where $r_\lambda \triangleq \min_{x \in \partial\mathcal{D}_\lambda} \|x - x_\lambda\|$.

PROPOSITION 5.1 *Let* $\mathcal{S} \subseteq \Lambda_{\mathrm{S}}$ *be such that Assumption* 5.1 *holds. Then for every* $\alpha > 0$, $p^{-1}(\alpha)$ *is either empty or consists of only isolated points. Furthermore, there exists* $\beta > 0$ *such that for every* $\alpha < \beta$, $p^{-1}(\alpha)$ *consists of at most one isolated point.*

*Proof* Suppose, *ad absurdum*, that there exists $\hat{\lambda} \in p^{-1}(\alpha)$, $\alpha > 0$, such that $\hat{\lambda}$ is not an isolated point in $p^{-1}(\alpha)$. Now, let $\hat{\mathcal{N}} \subset p^{-1}(\alpha)$ be a neighborhood of $\hat{\lambda}$ and note that, by continuity of $\psi(\cdot)$ and the fact that $\hat{\lambda} \in p^{-1}(\alpha)$ is not an isolated point, for every $\epsilon > 0$, there exist $\lambda_1, \lambda_2 \in \hat{\mathcal{N}}$ such that $\|x_{\lambda_1} - x_{\lambda_2}\| < \epsilon$, and $p(\lambda_1) = p(\lambda_2) = \alpha$. Now, choosing $\epsilon < r_{\lambda_1} + r_{\lambda_2}$ yields $\mathcal{D}_{\lambda_1} \cap \mathcal{D}_{\lambda_2} \neq \emptyset$ contradicting (ii) of Assumption 5.1. Hence, if $p^{-1}(\alpha)$, $\alpha > 0$, is non-empty, it must consist of only isolated points.

Next, suppose, *ad absurdum*, that for all $\delta > 0$ there exist two isolated points $\lambda_1, \lambda_2 \in \hat{\mathcal{N}}_\delta \triangleq \{\lambda \in \mathcal{S}: \|\lambda\| < \delta\}$ such that $p(\lambda_1) = p(\lambda_2)$. Now, repeating the above arguments leads to a contradiction. Hence, there exists $\hat{\delta} > 0$ such that if $\lambda_1 \in \hat{\mathcal{N}}_{\hat{\delta}}$, then $\hat{\mathcal{N}}_{\hat{\delta}} \cap p^{-1}(p(\lambda_1)) = \{\lambda_1\}$. Now, since $p(\cdot)$ is continuous and positive definite, it follows that there exists $\beta > 0$ such that $p^{-1}(\alpha) \subseteq \hat{\mathcal{N}}_{\hat{\delta}}$, $\alpha < \beta$, and hence $p^{-1}(\alpha)$, $\alpha < \beta$, consists of at most one isolated point. □

Note that Proposition 5.1 implies that, if $p^{-1}(\alpha)$, $\alpha > 0$, is bounded, then there exists a finite distance between isolated points contained in $p^{-1}(\alpha)$ which consists of at most a *finite* number of isolated points. Finally, since in a neighborhood of $\lambda = 0$ every level set of $p(\cdot)$ consists of at most one isolated point, a particular topology for $\mathcal{S}$, in a neighborhood of the $\lambda = 0$, is homeomorphic to the interval $[0, a]$, $a > 0$, with $0 \in \mathcal{S}$ corresponding to $0 \in \mathbb{R}$.

Now, for every $x \in \mathcal{D}_c \triangleq \bigcup_{\lambda \in \mathcal{S}} \mathcal{D}_\lambda$, define the *viable switching set* $\mathcal{V}_\mathcal{S}(x) \triangleq \{\lambda \in \mathcal{S}: x \in \mathcal{D}_\lambda\}$, which contains all $\lambda \in \mathcal{S}$ such that $x \in \mathcal{D}_\lambda$. Note that if we consider a sequence $\{\lambda_n\}_{n=1}^\infty \subset \mathcal{V}_\mathcal{S}(x)$, that is, $x \in \mathcal{D}_{\lambda_n}$, such that $\lim_{n\to\infty} \lambda_n = \bar{\lambda}$, it follows from the continuity of the set-valued map $\Psi(\cdot)$ that $x \in \mathcal{D}_{\bar{\lambda}}$. Thus, $\bar{\lambda} \in \mathcal{V}_\mathcal{S}(x)$ which implies that $\mathcal{V}_\mathcal{S}(x)$ is a nonempty closed set since it contains all of its accumulation points.

Next, we introduce the *switching function* $\lambda_\mathcal{S}(x)$, $x \in \mathcal{D}_c$, such that the following definitions hold

$$V(x) \triangleq p(\lambda_\mathcal{S}(x)), \quad \lambda_\mathcal{S}(x) \triangleq \arg\min\{p(\lambda): \lambda \in \mathcal{V}_\mathcal{S}(x)\}, \ x \in \mathcal{D}_c. \quad (25)$$

In particular, $\lambda_\mathcal{S}(x)$, $x \in \mathcal{D}_c$, corresponds to the value at which $p(\lambda)$ is minimized wherein $\lambda$ belongs to the viable switching set. The following proposition shows that "min" in (25) is attained and hence $V(x)$ is well defined.

PROPOSITION 5.2 *Let $\mathcal{S} \subseteq \Lambda_S$ and let $p: \mathcal{S} \to \mathbb{R}$ be a continuous positive-definite function such that Assumption* 5.1 *holds. Then, for all $x \in \mathcal{D}_c$, there exists a unique $\lambda_\mathcal{S}(x) \in \mathcal{V}_\mathcal{S}(x)$ such that $p(\lambda_\mathcal{S}(x)) = \min\{p(\lambda): \lambda \in \mathcal{V}_\mathcal{S}(x)\}$.*

*Proof* Existence follows from the fact that $p(\cdot)$ is lower bounded and $\mathcal{V}_\mathcal{S}(x)$, $x \in \mathcal{D}_c$, is a nonempty closed set. Now, to prove uniqueness suppose, *ad absurdum*, $\lambda_\mathcal{S}(x)$ is not unique. In this case, there exist

$\lambda_1, \lambda_2 \in \mathcal{S}, \lambda_1 \neq \lambda_2$, such that $p(\lambda_1) = p(\lambda_2)$ and $x \in \mathcal{D}_{\lambda_1} \cap \mathcal{D}_{\lambda_2} \neq \emptyset$ which contradicts (ii) in Assumption 5.1. □

The next result shows that $V(\cdot)$ given by (25) is a generalized Lyapunov function candidate, that is, $V(\cdot)$ is lower semicontinuous on $\mathcal{D}_c$.

THEOREM 5.1 *Let $\mathcal{S} \subseteq \Lambda_S$ be such that Assumption 5.1 holds. Then the function $V(x) = p(\lambda_{\mathcal{S}}(x))$, $x \in \mathcal{D}_c$, is lower semicontinuous on $\mathcal{D}_c$ and continuous on $\overset{\circ}{\mathcal{D}}_{\lambda_{\mathcal{S}}(x)}$.*

*Proof* Let the sequence $\{x_n\}_{n=0}^{\infty} \subset \mathcal{D}_c$ be such that $\lim_{n\to\infty} x_n = \hat{x}$ and define $\hat{\lambda} \triangleq \liminf_{n\to\infty} \lambda_{\mathcal{S}}(x_n)$. Here we assume without loss of generality that $\{\lambda_{\mathcal{S}}(x_n)\}_{n=0}^{\infty}$ converges to $\hat{\lambda}$; if this is not the case, it is always possible to construct a subsequence having this property. Since $p(\cdot)$ is continuous (and hence $p(\lim_{n\to\infty} \lambda_{\mathcal{S}}(x_n)) = \lim_{n\to\infty} p(\lambda_{\mathcal{S}}(x_n)))$, it suffices to show that $V(\hat{x}) \leq p(\hat{\lambda})$. Suppose, *ad absurdum*, that $V(\hat{x}) > p(\hat{\lambda})$. In this case, there exists a positive integer $n_0$ such that $V(\hat{x}) > V(x_n)$, $n \geq n_0$. Now, since by definition $\lambda_{\mathcal{S}}(\hat{x})$ minimizes $p(\lambda)$ for $\lambda \in \mathcal{V}_{\mathcal{S}}(\hat{x})$, it follows that $V(\hat{x}) \leq p(\lambda)$, $\lambda \in \mathcal{V}_{\mathcal{S}}(\hat{x})$. Hence, since $V(\hat{x}) > V(x_n) = p(\lambda_{\mathcal{S}}(x_n))$, $n \geq n_0$, it follows that $\lambda_{\mathcal{S}}(x_n) \notin \mathcal{V}_{\mathcal{S}}(\hat{x})$ and $\hat{x} \notin \mathcal{D}_{\lambda_{\mathcal{S}}(x_n)}$, $n \geq n_0$. Now, define the closed set $\hat{\mathcal{D}} \triangleq \bigcup_{n=n_0}^{\infty} \mathcal{D}_{\lambda_{\mathcal{S}}(x_n)}$ such that $\{x_n\}_{n=n_0}^{\infty} \subset \hat{\mathcal{D}}$. Since $\hat{\mathcal{D}}$ is closed, it follows that $\hat{x} \in \hat{\mathcal{D}}$ which implies that there exist $n_1 \geq n_0$ such that $\hat{x} \in \mathcal{D}_{\lambda_{\mathcal{S}}(x_{n_1})}$ which is a contradiction.

To show that $V(x)$ is continuous on $\overset{\circ}{\mathcal{D}}_{\lambda_{\mathcal{S}}(\hat{x})}$ it need only be shown that $V(\hat{x})$ is upper semicontinuous on $\overset{\circ}{\mathcal{D}}_{\lambda_{\mathcal{S}}(\hat{x})}$, or, equivalently, $V(\hat{x}) \geq p(\hat{\lambda})$. Since $\lim_{n\to\infty} x_n = \hat{x}$ and $\hat{x} \in \overset{\circ}{\mathcal{D}}_{\lambda_{\mathcal{S}}(\hat{x})}$, there exists a positive integer $n_2$ such that $x_n \in \mathcal{D}_{\lambda_{\mathcal{S}}(\hat{x})}$, $n \geq n_2$. Hence, $\lambda_{\mathcal{S}}(\hat{x}) \in \mathcal{V}_{\mathcal{S}}(x_n)$ and $V(\hat{x}) \geq V(x_n)$, $n \geq n_2$, which implies that $V(\hat{x}) \geq p(\hat{\lambda})$. □

Next, we show that with the hierarchical nonlinear robust feedback control strategy $u = \phi_{\lambda_{\mathcal{S}}(x)}(x)$, $x \in \mathcal{D}_c$, $V(\cdot)$ given by (25) is a generalized Lyapunov function for the nonlinear feedback controlled uncertain dynamical system (3). The controller notation $\phi_{\lambda_{\mathcal{S}}(x)}(x)$ denotes a switching nonlinear robust feedback controller where the switching function $\lambda_{\mathcal{S}}(x)$, $x \in \mathcal{D}_c$, is such that definition (25) of the generalized Lyapunov function $V(x)$, $x \in \mathcal{D}_c$, holds for a given potential function $p(\cdot)$ and switching set $\mathcal{S}$ satisfying Assumption 5.1. Furthermore, note that since $\phi_{\lambda_{\mathcal{S}}(x)}(x)$ is defined for $x \in \mathcal{D}_c$, it follows that the solution

$x(\cdot)$ to (3) with $x_0 \in \mathcal{D}_c$ and $u = \phi_{\lambda_{\mathcal{S}}(x)}(x)$ is defined for all values of $t \in \mathcal{I}_{x_0}$ such that $x(t) \in \mathcal{D}_c$. However, as will be shown, since $\mathcal{D}_c$ is a positively invariant set, $[0, +\infty) \subseteq \mathcal{I}_{x_0}$, while if $x_0 \in \mathcal{D}_c$ is such that $x(t)$, $t < 0$, is always contained in $\mathcal{D}_c$, then $\mathcal{I}_{x_0} = \mathbb{R}$. Finally, note that since the solution $x(t)$, $t \in \mathcal{I}_{x_0}$, to (3) with $x_0 \in \mathcal{D}_c$ and $u = \phi_{\lambda_{\mathcal{S}}(x)}(x)$ is continuous, it follows from Theorem 5.1 that $V(x(t))$, $t \in \mathcal{I}_{x_0}$, is right continuous. Hence, using the continuity of $p(\cdot)$ and the definition of $V(x)$, $x \in \mathcal{D}_c$, it follows that $\lambda_{\mathcal{S}}(x(t))$, $t \in \mathcal{I}_{x_0}$, is also right continuous. Now, the continuity of $F(\cdot,\cdot) \in \mathcal{F}$ and $\phi_\lambda(\cdot)$, $\lambda \in \Lambda_S$, imply that $F(x(t), \phi_{\lambda_{\mathcal{S}}(x(t))}(x(t)))$, $F(\cdot,\cdot) \in \mathcal{F}$, $t \in \mathcal{I}_{x_0}$, is right continuous.

THEOREM 5.2 *Consider the nonlinear controlled uncertain dynamical system* (1) *with* $F_n(0,0) = 0$, *and assume there exists a continuous function* $\psi : \Lambda_o \to \mathcal{D}_o$, $0 \in \Lambda_o$, *parameterizing a nominal equilibrium manifold of* (2), *such that* $x_\lambda = \psi(\lambda)$, $\lambda \in \Lambda_o$. *Furthermore, assume that there exists a* $C^0$ *feedback control law* $\phi_\lambda(\cdot)$, $\lambda \in \Lambda_S \subseteq \Lambda_o$ *with* $0 \in \Lambda_S$, *that locally stabilizes a compact positively invariant neighborhood* $\mathcal{N}_\lambda$ *of* $x_\lambda$ *for all* $F(\cdot,\cdot) \in \mathcal{F}$ *with a domain of attraction estimate* $\mathcal{D}_\lambda$ *and let* $\mathcal{S} \subseteq \Lambda_S$, $0 \in \mathcal{S}$, *be such that Assumption* 5.1 *holds. If* $\lambda_{\mathcal{S}}(x)$, $x \in \mathcal{D}_c$, *is such that* $V(x)$, $x \in \mathcal{D}_c$, *given by* (25) *holds and* $x(t)$, $t \in \mathcal{I}_{x_0}$, *is the solution to* (1) *with* $x(0) = x_0 \in \mathcal{D}_c$ *and robust feedback control law*

$$u = \phi_{\lambda_{\mathcal{S}}(x)}(x), \quad x \in \mathcal{D}_c, \tag{26}$$

*then* $\mathcal{D}_c$ *is robustly positively invariant and* $V(x(t))$, $t \geq 0$, *is nonincreasing. Furthermore, for all* $t_1, t_2 \geq 0$, $V(x(t)) = V(x(t_1))$, $t \in [t_1, t_2]$, *if and only if* $\lambda_{\mathcal{S}}(x(t)) = \lambda_{\mathcal{S}}(x(t_1))$, $t \in [t_1, t_2]$. *Finally, for all* $t \in \mathcal{I}_{x_0}$ *such that* $\lambda_{\mathcal{S}}(x(t)) \neq 0$, *there exists a finite time* $T > 0$ *such that* $V(x(t+T)) < V(x(t))$.

*Proof* First, note that $x \in \partial\mathcal{D}_c$ implies $x \in \partial\mathcal{D}_{\lambda_{\mathcal{S}}(x)}$. Since $\phi_{\lambda_{\mathcal{S}}(x)}$ robustly stabilizes $\mathcal{N}_{\lambda_{\mathcal{S}}(x)}$ with domain of attraction $\mathcal{D}_{\lambda_{\mathcal{S}}(x)}$, it follows that, for all $x \in \partial\mathcal{D}_c$ and $F(\cdot,\cdot) \in \mathcal{F}$, the flow of $F(x, \phi_{\lambda_{\mathcal{S}}(x)})$ is directed towards the interior of $\mathcal{D}_{\lambda_{\mathcal{S}}(x)}$ and consequently towards the interior of $\mathcal{D}_c$, which proves positive invariance of $\mathcal{D}_c$. Next, let $x(t)$, $t \in \mathcal{I}_{x_0}$, satisfy (1) with $u(t) = \phi_{\lambda_t}(x(t))$, where $\lambda_t \triangleq \lambda_{\mathcal{S}}(x(t))$, and let, for an arbitrary time $t_1 \geq 0$, the feedback control law $u = \phi_{\lambda_{t_1}}(x)$ robustly asymptotically stabilize the compact positively invariant neighborhood $\mathcal{N}_{\lambda_{t_1}}$ of $x_{\lambda_{t_1}}$ with domain of attraction $\mathcal{D}_{\lambda_{t_1}}$. Now, it follows from

(i) of Assumption 5.1 that $x(t_1) \in \mathcal{D}_{\lambda_{t_1}} \backslash \mathcal{N}_{\lambda_{t_1}}$ and hence, by Theorem 4.1, there exists a $C^1$ Lyapunov function $V_{\lambda_{t_1}}(\cdot)$ such that $\dot{V}_{\lambda_{t_1}}(x(t)) \stackrel{\Delta}{=} V'_{\lambda_{t_1}}(x(t))F(x(t), \phi_{\lambda_{\mathcal{S}(x(t))}}(x(t)))$, $t \geq 0$, and $\dot{V}_{\lambda_{t_1}}(x(t_1)) = V'_{\lambda_{t_1}}(x(t_1))$ $F(x(t_1), \phi_{\lambda_{t_1}}(x(t_1))) < 0$, $F(\cdot,\cdot) \in \mathcal{F}$. Next, since $F(x(t), \phi_{\lambda_{\mathcal{S}(x(t))}}(x(t)))$, $F(\cdot,\cdot) \in \mathcal{F}$, $t \in \mathcal{I}_{x_0}$, is right continuous, it follows that there exists $\delta > 0$ such that $\dot{V}_{\lambda_{t_1}}(x(t)) < 0$, $t \in [t_1, t_1 + \delta]$, which implies that $V_{\lambda_{t_1}}(x(t)) < V_{\lambda_{t_1}}(x(t_1))$, $t \in [t_1, t_1 + \delta]$. Hence, $x(t) \in \mathcal{D}_{\lambda_{t_1}} \backslash \mathcal{N}_{\lambda_{t_1}} \subset \mathcal{D}_c$, $t \in [t_1, t_1 + \delta]$, and $\lambda_{t_1} \in \mathcal{V}_{\mathcal{S}}(x(t))$, $t \in [t_1, t_1 + \delta]$, which implies that $V(x(t)) \leq V(x(t_1)) = p(\lambda_{t_1})$, $t \in [t_1, t_1 + \delta]$. Now, since $t_1 \geq 0$ is arbitrary, it follows that $V(x(t))$, $t \geq 0$, is a nonincreasing function along the forward trajectories $x(t)$, $t \geq 0$, of (1) with $u(t) = \phi_{\lambda_t}(x(t))$.

Next, assume that $x(t) \in \mathcal{D}_{\lambda_{t_1}}$ and $V(x(t)) = V(x(t_1))$, $t \in [t_1, t_1 + \hat{\delta}]$. Now, suppose, *ad absurdum*, that $\lambda_t$, $t \in [t_1, t_1 + \hat{\delta}]$, is not constant, that is, there exists $t_2 \in [t_1, t_1 + \hat{\delta}]$ such that $\lambda_{t_1} \neq \lambda_{t_2}$. In this case $x(t_2) \in \mathcal{D}_{\lambda_{t_1}} \cap \mathcal{D}_{\lambda_{t_2}}$ and $p(\lambda_{t_1}) = p(\lambda_{t_2})$, which contradicts (ii) of Assumption 5.1. Hence, it follows that if $V(x(t)) = V(x(t_2))$, $t \in [t_1, t_1 + \hat{\delta}]$, then $\lambda_t = \lambda_{t_1}$, $t \in [t_1, t_1 + \hat{\delta}]$. Conversely, if for $t_1, t_2 \geq 0$, $\lambda_t = \lambda_{t_1}$, $t \in [t_1, t_2]$, then $V(x(t)) = V(x(t_1))$, $t \in [t_1, t_2]$, is immediate.

Finally, for an arbitrary $t_1 \geq 0$, suppose, *ad absurdum*, that $V(x(t)) = V(x(t_1)) \neq 0$, $t \geq t_1$, or, equivalently, $\lambda_t = \lambda_{t_1} \in \mathcal{S} \backslash \{0\}$, $t \geq t_1$. Then the feedback control law $\phi_{\lambda_t}(\cdot) = \phi_{\lambda_{t_1}}(\cdot)$ robustly asymptotically stabilizes the compact positively invariant set $\mathcal{N}_{\lambda_{t_1}}$. In this case, it follows from Assumption 5.1 that there exists $\lambda_1 \neq \lambda_{t_1}$ such that $p(\lambda_1) < V(x(t_1))$ and $\mathcal{N}_{\lambda_{t_1}} \in \mathring{\mathcal{D}}_{\lambda_1}$, which implies that there exists $0 < \alpha < c_\lambda$ such that $\mathcal{N}_{\lambda_{t_1}} \subset V^{-1}_{\lambda_{t_1}}([0, \alpha]) \subseteq \mathcal{D}_{\lambda_1}$. Hence, it follows from Remark 4.1 that $x(t)$ approaches the level set $V^{-1}_{\lambda_{t_1}}(\alpha)$ in a finite time $T > 0$ so that $V(x(t_1 + T)) \leq p(\lambda_1) < V(x(t_1))$, which contradicts the original supposition. □

Next, we show that the hierarchical robust switching nonlinear controller (26) guarantees that the generalized Lyapunov function (25) is nonincreasing along the closed-loop system trajectories with strictly decreasing values *only* at the switching times which occur when the closed-loop system trajectory enters a new domain of attraction with an associated lower potential value.

COROLLARY 5.1 *Consider the nonlinear controlled dynamical system* (1) *with* $F_n(0,0) = 0$ *and assume the hypothesis of Theorem* 5.2 *hold. Then* $V(x(t))$, $t \geq 0$, *is strictly decreasing only at the switching times which* *occur when the trajectory* $x(t)$, $t \in \mathcal{I}_{x_0}$, *enters a new domain of attraction with an associated lower potential value.*

*Proof* First, we consider the case where $x(t_2) \in \mathring{\mathcal{D}}_{\lambda_{t_2}}$, with $\lambda_{t_2} \triangleq \lambda_{\mathcal{S}}(x(t_2))$ and $t_2 > 0$. It follows from the continuity of the closed-loop system trajectories $x(\cdot)$ that there exists $t_1 < t_2$ such that $x(t_1) \in \mathcal{D}_{\lambda_{t_2}}$, which implies that $\lambda_{t_2} \in \mathcal{V}_{\mathcal{S}}(x(t_1))$ and $V(x(t_1)) \leq V(x(t_2))$. Since $V(x(t))$, $t \geq 0$, is a nonincreasing function of time, it follows that $V(x(t)) = V(x(t_2))$, $t \in [t_1, t_2]$. Alternatively, assume that $x(t_2) \in \partial\mathcal{D}_{\lambda_{t_2}}$, and suppose, *ad absurdum*, that there exists $t_1 < t_2$ such that $x(t_1) \in \mathcal{D}_{\lambda_{t_2}}$. Then $\lambda_t = \lambda_{t_2}$, $t \in [t_1, t_2]$, and, since $V_{\lambda_{t_2}}(x)$, $x \in \mathcal{D}_{\lambda_{t_2}}$, attains its maximum at $x(t_2) \in \partial\mathcal{D}_{\lambda_{t_2}}$, it follows that $V_{\lambda_{t_2}}(x(t)) \leq V_{\lambda_{t_2}}(x(t_2))$, $t \in [t_1, t_2]$, which contradicts the fact that $V_{\lambda_{t_2}}(x(t))$, $t \geq 0$, is a decreasing function of time. Hence, $x(t) \notin \mathcal{D}_{\lambda_{t_2}}$ and $V(x(t)) < V(x(t_2))$, for all $t < t_2$. □

Finally, we present the main result of this section. Specifically, we show that the hierarchical robust switching nonlinear controller given by (26) guarantees that the closed-loop system trajectories converge to a union of largest invariant sets contained on the boundary of intersections over finite intervals of the closure of the generalized Lyapunov level surfaces. In addition, if the switching set $\mathcal{S}$ is homeomorphic to an interval on the real line and/or consists of only isolated points, then the hierarchical switching nonlinear controller establishes robust asymptotic stability of the compact positively invariant set $\mathcal{N}_0$.

THEOREM 5.3 *Consider the nonlinear controlled uncertain dynamical system* (1) *with* $F_{\mathrm{n}}(0,0) = 0$ *and assume there exists a continuous function* $\psi : \Lambda_{\mathrm{o}} \to \mathcal{D}_{\mathrm{o}}$, $0 \in \Lambda_{\mathrm{o}}$, *parameterizing a nominal equilibrium manifold of* (2), *such that* $x_\lambda = \psi(\lambda)$, $\lambda \in \Lambda_{\mathrm{o}}$. *Furthermore, assume that there exists a* $\mathrm{C}^0$ *feedback control law* $\phi_\lambda(\cdot)$, $\lambda \in \Lambda_{\mathrm{S}} \subseteq \Lambda_{\mathrm{o}}$ *with* $0 \in \Lambda_{\mathrm{S}}$, *that locally stabilizes a compact positively invariant neighborhood* $\mathcal{N}_\lambda$ *of* $x_\lambda$ *for all* $F(\cdot,\cdot) \in \mathcal{F}$ *with a domain of attraction estimate* $\mathcal{D}_\lambda$ *and let* $\mathcal{S} \subseteq \Lambda_{\mathrm{S}}, 0 \in \mathcal{S}$, *be such that Assumption* 5.1 *holds. In addition, assume* $\lambda_{\mathcal{S}}(x)$, $x \in \mathcal{D}_{\mathrm{c}}$, *is such that* $V(x)$, $x \in \mathcal{D}_{\mathrm{c}}$, *given by* (25) *holds, and, for* $x_0 \in \mathcal{D}_{\mathrm{c}}$, $x(t)$, $t \in \mathcal{I}_{x_0}$, *is the solution to* (3) *with the robust feedback control law*

$$u = \phi_{\lambda_{\mathcal{S}}(x)}(x), \quad x \in \mathcal{D}_{\mathrm{c}}. \tag{27}$$

*If* $x_0 \in \mathcal{D}_c$, *then* $x(t) \to \hat{\mathcal{M}} \triangleq \bigcup_{\gamma \in \mathcal{G}} \mathcal{M}_\gamma$ *as* $t \to \infty$ *for all* $F(\cdot, \cdot) \in \mathcal{F}$, *where* $\mathcal{G} \triangleq \{\gamma \geq 0 : \mathcal{R}_\gamma \cap \mathcal{D}_0 \neq \emptyset\}$. *If, in addition,* $\mathcal{S}_0 \triangleq \{\lambda \in \mathcal{S}: \mathcal{D}_\lambda \cap \mathcal{D}_0 \neq \emptyset\}$ *is homeomorphic to* $[0, a]$, $a > 0$, *with* $0 \in \mathcal{S}_0$ *corresponding to* $0 \in \mathbb{R}$, *or* $\mathcal{S}_0$ *consists of only isolated points, then the compact positively invariant set* $\mathcal{N}_0$ *is locally robustly asymptotically stable with an estimate of domain of attraction given by* $\mathcal{D}_c$. *Finally, if* $\mathcal{D} = \mathbb{R}^n$ *and there exists* $\hat{\lambda} \in \mathcal{S}$ *such that the feedback control law* $\phi_{\hat{\lambda}}(\cdot)$ *globally robustly asymptotically stabilizes the compact positively invariant set* $\mathcal{N}_{\hat{\lambda}}$, *then the above results are global.*

*Proof* The result follows from Theorems 3.2, 5.1, and 5.2. Specifically, Theorem 5.2 implies that if $x(\hat{t}) \in \mathcal{D}_0$ for an arbitrary $\hat{t} \geq 0$, then $V(x(t)) = V(x(\hat{t})) = 0$, $t \geq \hat{t}$. Hence, $\lambda_{\mathcal{S}}(x(t)) = 0$, $t \geq \hat{t}$, and the feedback control law $u = \phi_0(x)$ robustly asymptotically stabilizes $\mathcal{N}_0$ with an estimate of the domain of attraction given by $\mathcal{D}_0$. In this case, $\mathcal{D}_0$ is a compact positively invariant set of (3) with the robust feedback control law (27). Next, it follows from Theorems 5.1 and 5.2 that $V(\cdot)$ is a generalized Lyapunov function defined on $\mathcal{D}_c$. Now, it follows from Theorem 3.2 that, for all $x_0 \in \mathcal{D}_c$, $x(t) \to \hat{\mathcal{M}}$ as $t \to \infty$ for all $F(\cdot, \cdot) \in \mathcal{F}$.

Next, if $\mathcal{S}_0$ is homeomorphic to $[0, a]$, $a > 0$, with $0 \in \mathcal{S}_0$ corresponding to $0 \in \mathbb{R}$, so that Assumption 5.1 is satisfied, it follows from the continuity of the set-valued map $\Psi(\cdot)$ restricted to $\mathcal{S}_0$ that $V(\cdot)$ is continuous on $\mathcal{D}_0 = \mathcal{R}_0$, hence Theorem 3.2 guarantees that $\hat{\mathcal{M}} \equiv \mathcal{M}_0$. Alternatively, if $\mathcal{S}_0$ consists of only isolated points with finite pairwise distance, it follows that $\mathcal{G}$ consists of the isolated values of $p(\cdot)$ evaluated on the elements of $\mathcal{S}_0$. Hence, since $\hat{\mathcal{R}}_\gamma \triangleq \mathcal{R}_\gamma \backslash V^{-1}(\gamma)$, $\gamma \in \mathcal{G}$, is bounded, $V(\cdot)$ can only assume a finite number of distinct values on $\hat{\mathcal{R}}_\gamma$, $\gamma \in \mathcal{G}$, including the zero value. Now, it follows from Theorems 3.2 and 5.2, respectively, that $\mathcal{M}_\gamma \subset \hat{\mathcal{R}}_\gamma$ and, for all $x_0 \in \mathcal{D}_c \backslash \mathcal{D}_0$, there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, such that $V(x(t_{n+1})) < V(x(t))$, $n = 0, 1, \ldots$ Thus, no forward trajectories can be entirely contained in $\mathcal{R}_\gamma$, $\gamma \in \mathcal{G} \backslash \{0\}$. Hence, $\mathcal{M}_\gamma = \emptyset$, $\gamma \in \mathcal{G} \backslash \{0\}$, and $\hat{\mathcal{M}} \equiv \mathcal{M}_0$. In both the aforementioned cases, since $\mathcal{N}_0$ is a compact positively invariant set with an estimate of the domain of attraction given by $\mathcal{D}_0$ and $\mathcal{M}_0$ is the largest invariant set in $\mathcal{D}_0$, it follows that $\mathcal{M}_0 \subseteq \mathcal{N}_0$. Hence, $x(t) \to \mathcal{N}_0$ as $t \to \infty$ for all $F(\cdot, \cdot) \in \mathcal{F}$ establishing local robust asymptotic stability with an estimate of domain of attraction given by $\mathcal{D}_c$.

Finally, let $\mathcal{D} = \mathbb{R}^n$ and assume $\mathcal{S}_g \triangleq \{\lambda \in \mathcal{S}: \mathcal{D}_\lambda \equiv \mathbb{R}^n\}$ is not empty. In particular, if $\hat{\lambda} \in \mathcal{S}_g$, then the feedback control law $\phi_{\hat{\lambda}}(\cdot)$ globally robustly stabilizes $\mathcal{N}_{\hat{\lambda}}$. Now, Assumption 5.1 implies that if $\hat{\lambda}_1, \hat{\lambda}_2 \in \mathcal{S}_g$, $\hat{\lambda}_1 \neq \hat{\lambda}_2$, then $p(\lambda_1) \neq p(\lambda_2)$. Furthermore, since $\mathcal{D}_{\hat{\lambda}_1} \equiv \mathcal{D}_{\hat{\lambda}_2} \equiv \mathbb{R}^n$, we obtain that $\hat{\lambda}_1, \hat{\lambda}_2 \in \mathcal{V}_\mathcal{S}(x)$ for all $x \in \mathbb{R}^n$. Next, assume without loss of generality that $p(\hat{\lambda}_1) < p(\hat{\lambda}_2)$, and note that since $\lambda_\mathcal{S}(x)$ minimizes $p(\cdot)$ over $\mathcal{V}_\mathcal{S}(x)$, we obtain that $\lambda_\mathcal{S}(x) \neq \lambda_2$, $x \in \mathbb{R}^n$. It follows that $V(x) \leq \min\{p(\hat{\lambda}): \hat{\lambda} \in \mathcal{S}_g\}$ for all $x \in \mathbb{R}^n$ and only the (unique) value $\lambda \in \mathcal{S}_g$ that *minimizes* $p(\cdot)$ over $\mathcal{S}_g$ is assumed by the switching function $\lambda_\mathcal{S}(\cdot)$, so that all the other elements of $\mathcal{S}_g$ can be discarded from $\mathcal{S}$. Hence, without loss of generality, assume that there exists a unique $\hat{\lambda} \in \mathcal{S}$ such that $\phi_{\hat{\lambda}}(\cdot)$ globally robustly stabilizes $\mathcal{N}_{\hat{\lambda}}$. Now, define $\hat{\mathcal{S}} \triangleq \{\lambda \in \mathcal{S}: p(\lambda) < p(\hat{\lambda})\}$ and $\hat{\mathcal{D}}_c \triangleq \bigcup_{\lambda \in \hat{\mathcal{S}}} \mathcal{D}_\lambda$ which is a compact positively invariant set. Hence, if $x_0 \in \hat{\mathcal{D}}_c$, it follows from the first part of the theorem that $\hat{\mathcal{M}}$ is a local attractor and, if $\mathcal{S}_0$ is homeomorphic to an interval on $\mathbb{R}$ or consists of only isolated points, $\mathcal{N}_0$ is robustly asymptotically stable, with (in both cases) an estimate of the domain of attraction given by $\hat{\mathcal{D}}_c$. Now, global robust attraction to $\hat{\mathcal{M}}$ as well as global robust asymptotic stability of $\mathcal{N}_0$ is immediate by noting that if $x_0 \notin \hat{\mathcal{D}}_c$, then the forward trajectory of (3) approaches $\hat{\mathcal{D}}_c$ in a finite time for all $F(\cdot,\cdot) \in \mathcal{F}$. If, in fact, $x \notin \hat{\mathcal{D}}_c$, then $\lambda_\mathcal{S}(x) = \hat{\lambda}$ which implies that for all $x \notin \hat{\mathcal{D}}_c$ the feedback robust control law (27) stabilizes $\mathcal{N}_{\hat{\lambda}}$ and, by Assumption 5.1, $\mathcal{N}_{\hat{\lambda}} \in \mathring{\hat{\mathcal{D}}}_c$. In this case, it follows from Remark 4.1 that for all $x_0 \notin \hat{\mathcal{D}}_c$ there exists a finite time $T > 0$ such that $x(T) \in \hat{\mathcal{D}}_c$. Hence, global robust attraction as well as global robust asymptotic stability of the compact positively invariant set $\mathcal{N}_0$ is established for the respective cases. □

*Remark 5.1* The switching set $\mathcal{S}$ is quite general in the sense that it can have a hybrid topological structure involving isolated points and closed sets homeomorphic to intervals on the real line. In the special case where the switching set $\mathcal{S}$ consists of only isolated points, the hierarchical robust switching control strategy given by (27) is piecewise continuous. Alternatively, in the special case where the switching set $\mathcal{S}$ is homeomorphic to an interval on $\mathbb{R}$, the hierarchical robust switching control strategy given by (27) is *not* necessarily continuous.

*Remark 5.2* In the case where the switching set $\mathcal{S}$ is homeomorphic to an interval on $\mathbb{R}$ and a robust stabilizing controller $\phi_0(\cdot)$ for $\mathcal{N}_0$

cannot be obtained, that is, $c_0 = \beta_0$, $\mathcal{N}_0 \subset \mathcal{D}_c$ still holds. Hence, Theorem 5.3 guarantees attraction to $\mathcal{N}_0$ if $\partial\mathcal{N}_0 \cap \partial\mathcal{D}_c \neq \emptyset$. Alternatively, if $\mathcal{N}_0 \subset \overset{\circ}{\mathcal{D}}_c$, then $\mathcal{N}_0$ is robustly asymptotically stable.

It is important to note that since the hierarchical robust switching nonlinear controller $u = \phi_{\lambda_S(x)}(x)$, $x \in \mathcal{D}_c$, is constructed such that the switching function $\lambda_S(x)$, $x \in \mathcal{D}_c$, assures that $V(x)$, $x \in \mathcal{D}_c$, defined by (25) is a generalized Lyapunov function with strictly decreasing values at the switching points, the possibility of a sliding mode is precluded with the proposed robust control scheme. In particular, Theorem 5.2 guarantees that the closed-loop state trajectories cross the boundary of adjacent regions of attraction in the state space in an inward direction. Thus, the closed-loop state trajectories *enter* the lower potential-valued domain of attraction before subsequent switching can occur. Hence, the proposed robust nonlinear stabilization framework avoids the undesirable effects of high-speed switching onto an invariant sliding manifold.

Finally, to elucidate the hierarchical robust switching nonlinear controller presented in this section, we present an algorithm that outlines the key steps in constructing the feedback controller.

ALGORITHM 5.1 *To construct the robust hierarchical switching feedback control $\phi_{\lambda_S(x(t))}(x(t))$, $t \geq 0$, perform the following steps:*

*Step* 1 *Construct the nominal equilibrium manifold of* (2) *using $u = \varphi(x, \lambda)$, where $\varphi(\cdot, \cdot)$ is an arbitrary function of $\lambda \in \Lambda_o$. Use $F_n(x, \varphi(x, \lambda)) = 0$ to explicitly define the mapping $\psi(\cdot)$ such that $x_\lambda = \psi(\lambda)$, $\lambda \in \Lambda_o$, is a nominal equilibrium point of* (2) *corresponding to the parameter value $\lambda$. We note that the above parameterization can be constructed using the approaches given in* [7–9].

*Step* 2 *Construct the set $\Lambda_S \subseteq \Lambda_o$ such that, for each $\lambda \in \Lambda_S$, there exists a compact positively invariant set $\mathcal{N}_\lambda$ containing the nominal equilibrium point $x_\lambda$. Furthermore, for each $\lambda \in \Lambda_S$, construct an asymptotically stabilizing controller $\phi_\lambda(\cdot)$ for the positively invariant set $\mathcal{N}_\lambda$ with an associated domain of attraction $\mathcal{D}_\lambda$ corresponding to the level set $c_\lambda$ and Lyapunov function $V_\lambda(\cdot)$. Here, the controllers $\phi_\lambda(\cdot)$, $\lambda \in \Lambda_S$, can be obtained using any appropriate standard linear or nonlinear stabilization scheme.*

*Step* 3 *Construct the switching set* $\mathcal{S} \subseteq \Lambda_S$ *and a potential function* $p : \mathcal{S} \to \mathbb{R}^+$ *such that Assumption* 5.1 *is satisfied. In particular:*

(a) *If* $\lambda \in \mathcal{S}$ *is an isolated point of* $\mathcal{S}$ *with corresponding compact positively invariant set* $\mathcal{N}_\lambda$, *then there exists* $\lambda_1 \in \mathcal{S}$ *such that* $p(\lambda_1) < p(\lambda)$, $\mathcal{N}_\lambda \subset \overset{\circ}{\mathcal{D}}_{\lambda_1}$.

(b) *If* $\lambda \in \mathcal{S}$ *is an accumulation point of* $\mathcal{S}$ *then Step* 3(a) *is automatically satisfied if* $p(\cdot)$ *does not achieve a local minimum at* $\lambda$.

(c) *If* $\lambda, \lambda_1 \in \mathcal{S}$, $\lambda \neq \lambda_1$, *is such that* $p(\lambda) = p(\lambda_1)$, *then* $\mathcal{D}_\lambda \cap \mathcal{D}_{\lambda_1} = \emptyset$.

*Step* 4 *Given the state space point* $x(t)$ *at* $t \geq 0$, *search for solutions to* $V_\lambda(x(t)) = c_\lambda$, $\lambda \in \mathcal{S}$.

(a) *If no solution exists,* $\lambda_{\mathcal{S}}(x(t))$ *is unchanged.*

(b) *If one solution* $\lambda_1$ *exists and* $p(\lambda_1) < p(\lambda)$ *then switch* $\lambda_{\mathcal{S}}(x(t))$ *to* $\lambda_1$.

(c) *If more than one solution exists, repeat Step* 4(b) *with* $\lambda_1$ *replaced by the solution that minimizes* $p(\cdot)$. *Note that multiple solutions can be avoided by modifying the* $c_\lambda$*'s.*

*Step* 5 *Construct the hierarchical robust switching feedback controller* $\phi_{\lambda_{\mathcal{S}}(x(t))}(x(t))$ *where* $\lambda_{\mathcal{S}}(x(t))$, $x \in \mathcal{D}_c$, *constructed in Step* 4 *is such that* (25) *holds.*

Note that the existence of a switching set $\mathcal{S}$ and a potential function $p(\cdot)$ such that Step 3 is satisfied, can be guaranteed by modifying the first part Step 4 as follows:

*Step* 4′ *Given the state space point* $x(t)$ *at* $t = t_k \triangleq k\Delta T$, *where* $\Delta T > 0$ *and* $k = 0, 1, \ldots$, *search for the solutions of* $V_\lambda(x(t_k)) \leq c_\lambda$, $\lambda \in \Lambda_S$.

In this case, the switching set $\mathcal{S} \subseteq \Lambda_S$ need not be explicitly defined and is computed online.

## 6. HIERARCHICAL ROBUST CONTROL FOR PROPULSION SYSTEMS

In this section we apply the hierarchical robust switching nonlinear control framework to the control of rotating stall and surge in jet engine compression systems with uncertain compressor pressure-flow maps.

Rotating stall is an inherently two-dimensional local compression system oscillation which is characterized by regions of flow that rotate at a fraction of the compressor rotor speed while surge is a one-dimensional axisymmetric global compression system oscillation which involves axial flow oscillations and in some case even axial flow reversal and can damage engine components and cause flame-out to occur. For maximum compressor performance, operating conditions require that the pressure rise in the compressor correspond to the maximum pressure operating point on the stable axisymmetric branch of the compressor pressure-flow map for a given throttle (control) opening [36]. However, unavoidable discrepancies between compression systems models and real-world compression systems can result in degradation of control-system performance including stability. In particular, as shown in [28], feedback controllers that do not account for the presence of uncertainty in the compressor-flow map can have adverse effects on compression system performance by driving the compression system to a stalled equilibrium or a surge limit cycle.

To capture post-stall transients in axial flow compression systems we use an $n_{\mathrm{m}}$-mode Galerkin approximation model for the nonlinear partial differential equation characterizing the disturbance velocity potential at the compressor inlet proposed by Moore and Greitzer [36]. This model is given by [37]

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = A_S\phi + D_S^{-1}\psi_{\mathrm{C}}(\phi) - e\Psi, \quad \frac{\mathrm{d}\Psi}{\mathrm{d}\xi} = \frac{1}{4B^2 l_{\mathrm{C}}}\left(\frac{e^{\mathrm{T}}\phi}{n_{\mathrm{f}}} - \gamma_{\mathrm{th}}\sqrt{\Psi}\right), \tag{28}$$

where $\phi = [\phi_1\phi_2\cdots\phi_{n_{\mathrm{f}}}]^{\mathrm{T}}$ is a vector of $n_{\mathrm{f}} \triangleq 2n_{\mathrm{m}} + 1$ axial flow coefficients measured around the compressor inlet annulus, $\Psi$ is the normalized total-to-static pressure rise, $\xi$ is a nondimensional time, $\gamma_{\mathrm{th}}$ is a parameter proportional to the control throttle opening, $e \triangleq [1\ 1\ \cdots\ 1]^{\mathrm{T}} \in \mathbb{R}^{n_{\mathrm{f}}}$, $l_{\mathrm{C}}$ is the characteristic length of the compressor, $B$ is a nondimensional compliance parameter, $\psi_{\mathrm{C}}(\phi) \triangleq [\psi_{\mathrm{C}}(\phi_1)\ \psi_{\mathrm{C}}(\phi_2)\cdots\psi_{\mathrm{C}}(\phi_{n_{\mathrm{f}}})]^{\mathrm{T}} \in \mathbb{R}^{n_{\mathrm{f}}}$ is the vector compressor characteristic map, and the system matrices $A_S, D_S \in \mathbb{R}^{n_{\mathrm{f}}\times n_{\mathrm{f}}}$ are functions of the compressor geometry and mode number. For complete details of the model see [37]. The compliance parameter $B$ is a function of the compressor rotor speed and the system plenum size. For large values of $B$ a surge limit cycle can occur while rotating stall can occur for any value of $B$. The standard nominal model considered in literature [36] for the compressor pressure-flow characteristic map $\psi_{\mathrm{C}}^{\mathrm{nom}}(\phi_i)$, $i=1,\ldots,n_{\mathrm{f}}$, is a cubic function given by

$$\psi_{\mathrm{C}}^{\mathrm{nom}}(\phi_i) = \psi_{\mathrm{C}_0} + H\left[1 + \frac{3}{2}\left(\frac{\phi_i}{W} - 1\right) - \frac{1}{2}\left(\frac{\phi_i}{W} - 1\right)^3\right], \quad i = 1,\ldots,n_{\mathrm{f}}, \tag{29}$$

where $\psi_{\mathrm{C}_0}$, $H$, and $W$ are parameters that can be used to shape the nominal compressor characteristic map. In actual compressor data [38,39] however, the compressor characteristic map exhibits a noncubic morphology that can drive the compression system to deep hysteresis during rotating stall. Hence, to account for compressor performance pressure-flow map uncertainties we assume that

$$\psi_{\mathrm{C}}(\phi_i) \triangleq \psi_{\mathrm{C}}^{\mathrm{nom}}(\phi_i) + \delta\psi(\phi_i), \quad i = 1,\ldots,n_{\mathrm{f}}, \tag{30}$$

where $\delta\psi(\phi_i)$ is an uncertain perturbation of the nominal compressor characteristic map $\psi_{\mathrm{C}}^{\mathrm{nom}}(\phi_i)$, $i=1,\ldots,n_{\mathrm{f}}$.

In the case where $n_{\mathrm{m}}=1$ and $\delta\psi(\cdot)=0$, the above model collapses to the low-order three-state Moore–Greitzer model which has been extensively used by researchers in the literature to develop control design approaches for controlling axial flow compression systems. However, a fundamental shortcoming of the low-order, three-state Moore–Greitzer model and, as a consequence, the control design methodologies based on the model, is the fact that only a one-mode expansion of the disturbance velocity potential in the compression system is considered. Since the second- and higher-order disturbance velocity potential harmonics strongly interact with the first harmonic during stall inception, they must be accounted for in the control-system design process. This is clearly shown in [37] where a globally stabilizing backstepping controller predicated on the one-mode Moore–Greitzer model [40] drives the compression system to a stalled condition in the cases where two modes are used in the simulation. This clearly shows that a multi-mode model that accounts for higher mode interactions with the first mode is necessary for achieving control objectives during stall inception. The multi-mode modeling problem is further exacerbated when addressing compressor pressure-flow map uncertainty.

In order to address the above challenges for controlling uncertain multi-mode axial compression system models, we use the hierarchical robust switching nonlinear control framework developed in the paper. Here, the locus of the parameterized equilibrium points on which the equilibria-dependent Lyapunov functions are predicated on, are characterized by the axisymmetric stable pressure-flow equilibrium branch of the nominal system for a continuum of mass flow through the throttle. For this development define the shifted flow and pressure state variables $x_{\mathrm{f}} \triangleq \phi/W - 2e$ and $x_{\mathrm{p}} \triangleq (\Psi - \psi_{\mathrm{C}_0})/H - 2$, so that the maximum pressure point on the nominal compressor characteristic map is translated to the origin. In this case, the translated nonlinear uncertain system is given by

$$\dot{x}_{\mathrm{f}}(t) = Ax_{\mathrm{f}}(t) + P^{-1}\left[\psi_{\mathrm{sC}}^{\mathrm{nom}}(x_{\mathrm{f}}(t)) + \Delta\psi_{\mathrm{s}}(x_{\mathrm{f}}(t))\right] - ex_{\mathrm{p}}(t), \quad (31)$$

$$\dot{x}_{\mathrm{p}}(t) = \frac{1}{\beta^2}\left(\frac{e^{\mathrm{T}}x_{\mathrm{f}}(t)}{n_{\mathrm{f}}} - u(t)\right), \quad (32)$$

where

$$A \triangleq \frac{Wl_{\mathrm{C}}}{H}A_S, \quad P \triangleq \frac{1}{l_{\mathrm{C}}}D_S, \quad \beta \triangleq \frac{2BH}{W}, \quad u \triangleq \frac{\gamma_{\mathrm{th}}\sqrt{\Psi}}{W} - 2, \quad (33)$$

$$\psi_{\mathrm{sC}}^{\mathrm{nom}}(x_{\mathrm{f}}) \triangleq \left[\psi_{\mathrm{sC}}^{\mathrm{nom}}(x_{\mathrm{f}1}) \ \ldots \ \psi_{\mathrm{sC}}^{\mathrm{nom}}(x_{\mathrm{f}n_{\mathrm{f}}})\right]^{\mathrm{T}},$$

$$\Delta\psi_{\mathrm{s}}(x_{\mathrm{f}}) \triangleq [\delta\psi_{\mathrm{s}}(x_{\mathrm{f}1}) \ \ldots \ \delta\psi_{\mathrm{s}}(x_{\mathrm{f}n_{\mathrm{f}}})]^{\mathrm{T}}, \quad (34)$$

and $(\dot{\ })$ represents differentiation with respect to nondimensional scaled time $t \triangleq (H/Wl_{\mathrm{C}})\xi$.

Next, it follows from (30) that the actual compressor characteristic $\psi_{\mathrm{sC}}(x_{\mathrm{f}i})$, $i = 1, \ldots, n_{\mathrm{f}}$, is given by

$$\psi_{\mathrm{sC}}(x_{\mathrm{f}i}) = \psi_{\mathrm{sC}}^{\mathrm{nom}}(x_{\mathrm{f}i}) + \delta\psi_{\mathrm{s}}(x_{\mathrm{f}i}), \quad i = 1, \ldots, n_{\mathrm{f}}, \quad (35)$$

where $\psi_{\mathrm{sC}}^{\mathrm{nom}}(x_{\mathrm{f}i}) = -\frac{3}{2}x_{\mathrm{f}i}^2 - \frac{1}{2}x_{\mathrm{f}i}^3$ is the nominal compressor characteristic and $\delta\psi_{\mathrm{s}}(x_{\mathrm{f}i})$, $i = 1, \ldots, n_{\mathrm{f}}$, is an uncertain perturbation of the nominal characteristic $\psi_{\mathrm{sC}}^{\mathrm{nom}}(x_{\mathrm{f}i})$, $i=1,\ldots,n_{\mathrm{f}}$. Here, we assume

$$\begin{aligned}\delta\psi_{\mathrm{s}}(\cdot) \in \mathbf{\Delta} \triangleq \{\delta\psi_{\mathrm{s}}:\mathbb{R}\to\mathbb{R}\colon [\delta\psi_{\mathrm{s}}(y)-m_1(y)]\\ \times[\delta\psi_{\mathrm{s}}(y)-m_2(y)]\le 0,\ y\in\mathbb{R}\},\end{aligned}\tag{36}$$

where $m_1, m_2:\mathbb{R}\to\mathbb{R}$ are given arbitrary bounding functions.

Carrying out Step 1 of Algorithm 5.1, let $q=m=1$ and $\varphi(x_{\mathrm{f}}, x_{\mathrm{p}}, \lambda)=\lambda$ so that the system equilibria are parameterized by the constant control $u(t)=\lambda$. In this case, (31) and (32) with $\Delta\psi_{\mathrm{s}}(x_{\mathrm{f}}(t))\equiv 0$ have an equilibrium point at $(x_{\mathrm{f}\lambda}, x_{\mathrm{p}\lambda})$, where

$$x_{\mathrm{f}\lambda} \triangleq \lambda e, \qquad x_{\mathrm{p}\lambda} \triangleq \psi_{\mathrm{sC}}^{\mathrm{nom}}(\lambda) = -\tfrac{3}{2}\lambda^2 - \tfrac{1}{2}\lambda^3. \tag{37}$$

Next, we carry out Step 2 of Algorithm 5.1. Specifically, for the uncertain compression system (31) and (32), we show that there exists $\lambda>0$ and a robust control law such that a neighborhood $\mathcal{N}_\lambda$ of the nominal equilibrium point $(x_{\mathrm{f}\lambda}, x_{\mathrm{p}\lambda})$ is locally robustly asymptotically stable with domain of attraction $\mathcal{D}_\lambda$. Specifically, consider the equilibrium-dependent Lyapunov function candidate predicated on the nominal pressure-flow axisymmetric stable equilibria given by

$$V_\lambda(x_{\mathrm{f}}, x_{\mathrm{p}}) = \frac{1}{2n_{\mathrm{f}}}(x_{\mathrm{f}}-x_{\mathrm{f}\lambda})^{\mathrm{T}}P(x_{\mathrm{f}}-x_{\mathrm{f}\lambda}) + \frac{1}{2}\beta^2[x_{\mathrm{p}}-x_{\mathrm{p}\lambda}]^2, \tag{38}$$

with Lyapunov derivative

$$\begin{aligned}\dot V_\lambda(x_{\mathrm{f}}, x_{\mathrm{p}}) &= \frac{1}{n_{\mathrm{f}}}(x_{\mathrm{f}}-\lambda e)^{\mathrm{T}}P\big[Ax_{\mathrm{f}} + P^{-1}\big(\psi_{\mathrm{sC}}^{\mathrm{nom}}(x_{\mathrm{f}}) + \Delta\psi_{\mathrm{s}}(x_{\mathrm{f}})\big) - ex_{\mathrm{p}}\big]\\ &\quad + [x_{\mathrm{p}} - \psi_{\mathrm{sC}}^{\mathrm{nom}}(\lambda)]\left(\frac{e^{\mathrm{T}}x_{\mathrm{f}}}{n_{\mathrm{f}}} - u\right)\\ &= -\frac{1}{n_{\mathrm{f}}}(x_{\mathrm{f}}-\lambda e)^{\mathrm{T}}[\psi_{\mathrm{sC}}^{\mathrm{nom}}(\lambda)e - \psi_{\mathrm{sC}}^{\mathrm{nom}}(x_{\mathrm{f}}) - \Delta\psi_{\mathrm{s}}(x_{\mathrm{f}})]\\ &\quad - h_\lambda(x_{\mathrm{f}}, x_{\mathrm{p}})[x_{\mathrm{p}} - \psi_{\mathrm{sC}}^{\mathrm{nom}}(\lambda)],\end{aligned}\tag{39}$$

where $u(x_{\mathrm{f}}, x_{\mathrm{p}}) = u_\lambda(x_{\mathrm{f}}, x_{\mathrm{p}}) \triangleq \lambda + h_\lambda(x_{\mathrm{f}}, x_{\mathrm{p}})$ and $h_\lambda:\mathbb{R}^{n_{\mathrm{f}}}\times\mathbb{R}\to\mathbb{R}$ is such that $h_\lambda(x_{\mathrm{f}\lambda}, x_{\mathrm{p}\lambda})=0$. Now, it follows from Theorem 4.1 that requiring $\dot{V}_\lambda(x_\mathrm{f}, x_\mathrm{p}) < 0$, $(x_\mathrm{f}, x_\mathrm{p}) \in \mathcal{D}_\lambda \backslash \mathcal{N}_\lambda$, guarantees local robust stability of the compact positively invariant set $\mathcal{N}_\lambda$ for all $\delta\psi_\mathrm{s}(\cdot) \in \mathbf{\Delta}$. However, (39) is dependent on the system uncertainty and needs to be checked for all $\delta\psi_\mathrm{s}(x_{\mathrm{f}i}) \in \mathbf{\Delta}$, $i = 1, \ldots, n_\mathrm{f}$, and hence is unverifiable. To obtain verifiable conditions for robust stability we utilize Conditions (22) and (23) and introduce an equilibrium-dependent bounding function $\Gamma_\lambda(\cdot)$ for the uncertainty set $\mathbf{\Delta}$ such that $\Gamma_\lambda(\cdot)$ bounds $\mathbf{\Delta}$. Specifically, define $\Gamma_\lambda : \mathbb{R}^{n_\mathrm{f}} \to \mathbb{R}$ by

$$\begin{aligned}\Gamma_\lambda(x_\mathrm{f}) &\triangleq \tfrac{1}{4}[m_2(x_\mathrm{f}) - m_1(x_\mathrm{f})]^\mathrm{T}[m_2(x_\mathrm{f}) - m_1(x_\mathrm{f})] + \tfrac{1}{4}(x_\mathrm{f} - \lambda e)^\mathrm{T}(x_\mathrm{f} - \lambda e) \\ &\quad + \tfrac{1}{2}(x_\mathrm{f} - \lambda e)^\mathrm{T}[m_1(x_\mathrm{f}) + m_2(x_\mathrm{f})], \qquad (40)\end{aligned}$$

where $m_1(x_\mathrm{f}) \triangleq [m_1(x_{\mathrm{f}1}) \ldots m_1(x_{\mathrm{f}n_\mathrm{f}})]^\mathrm{T}$ and $m_2(x_\mathrm{f}) \triangleq [m_2(x_{\mathrm{f}1}) \ldots m_2(x_{\mathrm{f}n_\mathrm{f}})]^\mathrm{T}$. Now, note that if $\delta\psi(\cdot) \in \mathbf{\Delta}$ then

$$\begin{aligned}0 &\le \tfrac{1}{4}[m_1(x_\mathrm{f}) + m_2(x_\mathrm{f}) + x_\mathrm{f} - \lambda e - 2\Delta\psi_\mathrm{s}(x_\mathrm{f})]^\mathrm{T} \\ &\quad \times [m_1(x_\mathrm{f}) + m_2(x_\mathrm{f}) + x_\mathrm{f} - \lambda e - 2\Delta\psi_\mathrm{s}(x_\mathrm{f})] \\ &\quad - [\Delta\psi_\mathrm{s}(x_\mathrm{f}) - m_1(x_\mathrm{f})]^\mathrm{T}[\Delta\psi_\mathrm{s}(x_\mathrm{f}) - m_2(x_\mathrm{f})] \\ &= \tfrac{1}{4}[m_2(x_\mathrm{f}) - m_1(x_\mathrm{f})]^\mathrm{T}[m_2(x_\mathrm{f}) - m_1(x_\mathrm{f})] + \tfrac{1}{4}(x_\mathrm{f} - \lambda e)^\mathrm{T}(x_\mathrm{f} - \lambda e) \\ &\quad + \tfrac{1}{2}(x_\mathrm{f} - \lambda e)^\mathrm{T}[m_1(x_\mathrm{f}) + m_2(x_\mathrm{f})] - (x_\mathrm{f} - \lambda e)^\mathrm{T}\Delta\psi_\mathrm{s}(x_\mathrm{f}),\end{aligned}$$

and hence $(x_\mathrm{f} - \lambda e)^\mathrm{T}\Delta\psi(x_\mathrm{f}) \le \Gamma_\lambda(x_\mathrm{f})$, $\delta\psi(x_{\mathrm{f}i}) \in \mathbf{\Delta}$, $i = 1, \ldots, n_\mathrm{f}$. Now, requiring

$$\begin{aligned}&-\frac{1}{n_\mathrm{f}}(x_\mathrm{f} - \lambda e)^\mathrm{T}[\psi_{\mathrm{sC}}^{\mathrm{nom}}(\lambda)e - \psi_{\mathrm{sC}}^{\mathrm{nom}}(x_\mathrm{f})] - h_\lambda(x_\mathrm{f}, x_\mathrm{p})[x_\mathrm{p} - \psi_{\mathrm{sC}}^{\mathrm{nom}}(\lambda)] \\ &+\frac{1}{n_\mathrm{f}}\Gamma_\lambda(x_\mathrm{f}) < 0, \quad (x_\mathrm{f}, x_\mathrm{p}) \in \mathcal{D}_\lambda \backslash \mathcal{N}_\lambda, \qquad (41)\end{aligned}$$

it follows from (39) that $\dot{V}_\lambda(x_\mathrm{f}, x_\mathrm{p}) < 0$, $(x_\mathrm{f}, x_\mathrm{p}) \in \mathcal{D}_\lambda \backslash \mathcal{N}_\lambda$, so that all assumptions of Theorem 4.1 are satisfied.

Next, for simplicity of exposition we set $m_1(\cdot) = -m_2(\cdot) = m(\cdot)$, where $m : \mathbb{R} \to \mathbb{R}$ is a given arbitrary function. In this case, it follows from (37) and (41) that

$$\dot{V}_\lambda(x_\mathrm{f}, x_\mathrm{p}) \leq$$
$$-\frac{1}{2n_\mathrm{f}}\sum_{i=1}^{n_\mathrm{f}}\left\{(x_{\mathrm{f}i}-\lambda)^2\left[(x_{\mathrm{f}i})^2+(\lambda+3)x_{\mathrm{f}i}+\lambda(\lambda+3)-\frac{1}{2}\right]-2m^2(x_{\mathrm{f}i})\right\}$$
$$-h_\lambda(x_\mathrm{f}, x_\mathrm{p})[x_\mathrm{p}-\psi_{\mathrm{sC}}^{\mathrm{nom}}(\lambda)]<0, \quad (x_\mathrm{f}, x_\mathrm{p}) \in \mathcal{D}_\lambda \backslash \mathcal{N}_\lambda. \tag{42}$$

Now, a sufficient condition guaranteeing $\dot{V}_\lambda(x_\mathrm{f}, x_\mathrm{p})<0$, $(x_\mathrm{f}, x_\mathrm{p}) \in \mathcal{D}_\lambda \backslash \mathcal{N}_\lambda$, is given by

$$\frac{1}{2n_\mathrm{f}}\sum_{i=1}^{n_\mathrm{f}}(x_{\mathrm{f}i}-\lambda)^2 p_{1\lambda}(x_{\mathrm{f}i})>0, \quad (x_\mathrm{f}, x_\mathrm{p}) \in \mathcal{D}_\lambda, \tag{43}$$

$$\frac{1}{2n_\mathrm{f}}\sum_{i=1}^{n_\mathrm{f}}(x_{\mathrm{f}i}-\lambda)^2 p_{2\lambda}(x_{\mathrm{f}i})+h_\lambda(x_\mathrm{f}, x_\mathrm{p})[x_\mathrm{p}-\psi_{\mathrm{sC}}^{\mathrm{nom}}(\lambda)]$$
$$>\frac{1}{n_\mathrm{f}}\sum_{i=1}^{n_\mathrm{f}}m^2(x_{\mathrm{f}i}), \quad (x_\mathrm{f}, x_\mathrm{p}) \notin \mathcal{N}_\lambda, \tag{44}$$

where $p_{1\lambda}(x_{\mathrm{f}i}) \triangleq a_{1\lambda}(x_{\mathrm{f}i})^2+b_{1\lambda}x_{\mathrm{f}i}+c_{1\lambda}$ and $p_{2\lambda}(x_{\mathrm{f}i}) \triangleq a_{2\lambda}(x_{\mathrm{f}i})^2+b_{2\lambda}x_{\mathrm{f}i}+c_{2\lambda}$ are such that

$$p_{1\lambda}(x_{\mathrm{f}i})+p_{2\lambda}(x_{\mathrm{f}i})=(x_{\mathrm{f}i})^2+(\lambda+3)x_{\mathrm{f}i}+\lambda(\lambda+3)-\tfrac{1}{2}. \tag{45}$$

Note that (43) is satisfied in a domain $\mathcal{D}_\lambda \neq \emptyset$ only if there exists $d_\lambda>0$ such that $p_{1\lambda}(x_{\mathrm{f}i})>0$, $-d_\lambda<x_{\mathrm{f}i}-\lambda<d_\lambda$, $i=1,\ldots,n_\mathrm{f}$, and, in order to satisfy (44), we require that $p_{2\lambda}(x_{\mathrm{f}i})>0$, $i=1,\ldots,n_\mathrm{f}$. Hence, we require that $p_{1\lambda}(\lambda)>0$ and $p_{2\lambda}(\lambda)>0$. A particular choice of $h_\lambda(\cdot,\cdot)$ satisfying (44) is given by

$$h_\lambda(x_\mathrm{f}, x_\mathrm{p}) \triangleq w[x_\mathrm{p}-\psi_{\mathrm{sC}}^{\mathrm{nom}}(\lambda)]\,p(x_\mathrm{f}-\lambda e), \tag{46}$$

where $w:\mathbb{R}\to\mathbb{R}$ is such that $xw(x)>0$, $x\neq 0$, and $p:\mathbb{R}^{n_\mathrm{f}}\to\mathbb{R}$ is positive definite. However, note that for $x_\mathrm{f}=\lambda e$ it is not possible to satisfy (44) and hence by continuity there exists a neighborhood of this point where (42) *cannot* be satisfied. Thus, we construct a robust control law such that a neighborhood $\mathcal{N}_\lambda$ of the equilibrium point $(x_\mathrm{f}, x_\mathrm{p})$ is robustly stabilized with a given domain of attraction.

Next, note that it follows from (45) that for all $0 < \lambda \leq \sqrt{7/6} - 1$, $p_{1\lambda}(\lambda) + p_{2\lambda}(\lambda) \leq 0$ and hence the necessary conditions $p_{1\lambda}(\lambda) > 0$ and $p_{2\lambda}(\lambda) > 0$ for satisfying (43) and (44) are violated. Furthermore, if $\lambda > \sqrt{14/3} - 1$, then $p_{1\lambda}(x_{fi}) + p_{2\lambda}(x_{fi}) > 0$, $i = 1, \ldots, n_f$, which implies that it is always possible to choose $p_{1\lambda}(\cdot)$ such that $p_{1\lambda}(x_{fi}) > 0$, $i = 1, \ldots, n_f$. More generally, there exists $\lambda_0 \geq \sqrt{7/6} - 1$ and $\lambda_{\text{global}} > \sqrt{14/3} - 1$ such that $\mathcal{D}_{\lambda_0}$ collapses to the equilibrium point and $\mathcal{D}_{\lambda_{\text{global}}}$ coincides with the whole state space. Note that $\lambda_0$ and $\lambda_{\text{global}}$ are dependent on the particular choice of the coefficients $a_{1\lambda}$, $b_{1\lambda}$, $c_{1\lambda}$, $a_{2\lambda}$, $b_{2\lambda}$, and $c_{2\lambda}$.

Next, with $u(x_f, x_p) = u_\lambda(x_f, x_p)$, we provide an estimate of the domain of attraction for (31) and (32). In particular, define

$$\mathcal{D}_\lambda \triangleq \begin{cases} \{(x_f, x_p)\colon V_\lambda(x_f, x_p) \leq k_{1\lambda}\}, & \lambda_0 \leq \lambda \leq \lambda_{\text{global}}, \\ \mathbb{R}^{n_f} \times \mathbb{R}, & \lambda > \lambda_{\text{global}}, \end{cases} \tag{47}$$

$$\mathcal{N}_\lambda \triangleq \{(x_f, x_p)\colon V_\lambda(x_f, x_p) \leq k_{2\lambda}\}, \quad \lambda \geq \lambda_0, \tag{48}$$

where

$$k_{1\lambda} \triangleq \frac{\mu}{2n_f} d_\lambda^2, \quad \mu \triangleq \left(\max_i \{P_{ii}^{-1}\}\right)^{-1}, \tag{49}$$

and

$$k_{2\lambda} \triangleq \max_{(x_f, x_p) \in \mathcal{D}_\lambda} \frac{1}{2n_f}(x_f - x_{f\lambda})^{\mathrm{T}} P(x_f - x_{f\lambda}) + \frac{1}{2}\beta^2 [x_p - x_{p\lambda}]^2, \tag{50}$$

subject to

$$\frac{1}{2n_f}\sum_{i=1}^{n_f}(x_{fi} - \lambda)^2 p_{2\lambda}(x_{fi}) + h_\lambda(x_f, x_p)[x_p - \psi_{sC}^{\text{nom}}(\lambda)] = \frac{1}{n_f}\sum_{i=1}^{n_f} m^2(x_{fi}). \tag{51}$$

The Lyapunov level surfaces $V_\lambda(x_f, x_p) = k_{1\lambda}$ and $V_\lambda(x_f, x_p) = k_{2\lambda}$ are constructed such that the intersection of the boundary of $\mathcal{D}_\lambda$ with the plane $x_p = x_{p\lambda}$ is a closed surface contained in the region $\{x_f\colon -d_\lambda < x_{fi} - \lambda < d_\lambda,\ i = 1, \ldots, n_f\}$ and $\mathcal{N}_\lambda$ contains the region where (44) is *not* satisfied, so that $\dot{V}_\lambda(x_f, x_p) < 0$ for all $(x_f, x_p) \in \mathcal{D}_\lambda \backslash \mathcal{N}_\lambda$. Note that for $\lambda_0 \le \lambda \le \lambda_{\text{global}}$, $k_{1\lambda} \ge 0$ and $k_{2\lambda} \ge 0$. Furthermore, since $V_\lambda(x_f, x_p)$ is continuous and radially unbounded $\mathcal{N}_\lambda$ and $\mathcal{D}_\lambda$ are compact sets for $\lambda \in [\lambda_0, \lambda_{\text{global}}]$, and hence positively invariant. Thus, if the state space trajectories of (31) and (32) enter $\mathcal{D}_\lambda$, then $\mathcal{N}_\lambda$ serves as an attractor. Now, to ensure that $\mathcal{N}_\lambda \subset \mathcal{D}_\lambda$ we require that $k_{1\lambda} > k_{2\lambda}$. A typical plot for the level set values $k_{1\lambda}$ and $k_{2\lambda}$ as functions of $\lambda$ is shown in Fig. 2. Note that there exists $\lambda_{\min}$ such that $k_{1\lambda_{\min}} = k_{2\lambda_{\min}}$ and hence $\mathcal{D}_{\lambda_{\min}} = \mathcal{N}_{\lambda_{\min}}$. Hence, requiring $\lambda > \lambda_{\min}$ assures the necessary condition that $\mathcal{N}_\lambda \subset \mathcal{D}_\lambda$.

The coefficients of the two parabolas $p_{1\lambda}(\cdot)$ and $p_{2\lambda}(\cdot)$ must be such that (45) is satisfied along with the above stated necessary conditions. This leaves some degree of freedom in the choice of the coefficients $a_{1\lambda}$, $b_{1\lambda}$, $c_{1\lambda}$, $a_{2\lambda}$, $b_{2\lambda}$, and $c_{2\lambda}$, which can be used to maximize the domain of attraction $\mathcal{D}_\lambda$ and minimize the attractor $\mathcal{N}_\lambda$. This leads to the following optimization problem for each $\lambda$:

$$\max_{a_{1\lambda}, b_{1\lambda}, c_{1\lambda}, a_{2\lambda}, b_{2\lambda}, c_{2\lambda}} \left( \lambda^2 - \frac{c_{1\lambda}}{a_{1\lambda}} \right), \tag{52}$$

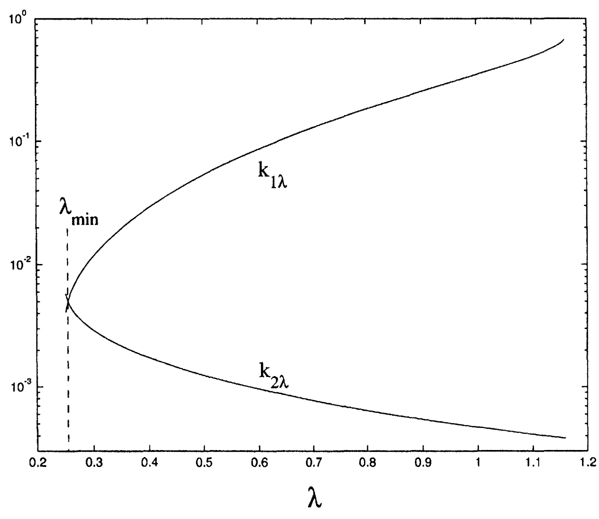


FIGURE 2 Level set values $k_{1\lambda}$ and $k_{2\lambda}$ as functions of $\lambda$.

subject to

$$a_{1\lambda} + a_{2\lambda} = 1, \tag{53}$$

$$b_{1\lambda} + b_{2\lambda} = \lambda + 3, \tag{54}$$

$$c_{1\lambda} + c_{2\lambda} = \lambda(\lambda + 3) - \tfrac{1}{2}, \tag{55}$$

$$(q_\lambda - \lambda)^2(a_{2\lambda}q_\lambda^2 + b_{2\lambda}q_\lambda + c_{2\lambda}) = 2n_f\kappa^2, \tag{56}$$

$$2a_{1\lambda}\lambda + b_{1\lambda} = 0, \tag{57}$$

$$a_{1\lambda} < 0, \tag{58}$$

$$b_{1\lambda}^2 - 4a_{1\lambda}c_{1\lambda} > 0, \tag{59}$$

$$b_{2\lambda}^2 - 4a_{2\lambda}c_{2\lambda} < 0, \tag{60}$$

where $q_\lambda \triangleq \left(2a_{2\lambda}\lambda - 3b_{2\lambda} - \sqrt{(2a_{2\lambda}\lambda - 3b_{2\lambda})^2 - 16a_{2\lambda}(2c_{2\lambda} - b_{2\lambda}\lambda)}\right)\Big/ 8a_{2\lambda}$ and $m(x_{fi})$ is chosen to be a constant value $\kappa \in \mathbb{R}$, $i = 1, \ldots, n_f$. Note that, under the assumption that $p_{1\lambda}(\cdot)$ achieves a maximum at $\lambda$, the objective function given by (52) corresponds to maximizing $d_\lambda^2$. Furthermore, conditions (53)–(55) are obtained by equating the coefficients of equal powers in (45). Condition (56) guarantees that $(x_{fi} - \lambda)^2 p_{2\lambda}(x_{fi})$, $i = 1, \ldots, n_f$, is a convex function for all $x_{fi}$ so that $\mathcal{N}_\lambda$ is minimized, while conditions (57)–(59) guarantee that $p_{1\lambda}(\cdot)$ achieves a maximum at $\lambda$ and $p_{1\lambda}(\lambda) > 0$. Finally, (60) guarantees that $p_{2\lambda}(\cdot) > 0$.

To carry out Step 3 of Algorithm 5.1, we consider two topologies for the switching set $\mathcal{S}$; namely an isolated point topology and a hybrid topology. For $\mathcal{S}$ consisting of countably finite isolated points let $\mathcal{S} = \{\lambda_0, \ldots, \lambda_q\}$ be such that $\lambda_{\min} < \lambda_q < \cdots < \lambda_1 \le \lambda_{\text{global}}$, $\lambda_0 > \lambda_{\text{global}}$, and $\mathcal{N}_{\lambda_{i+1}} \subset \overset{\circ}{\mathcal{D}}_{\lambda_i}$, $i \in \{0, \ldots, q-1\}$, and let $p(\lambda) = \lambda$, $\lambda \in \mathcal{S}$. To guarantee that $p(\cdot)$ satisfies Assumption 5.1 construct $\lambda_k$, $k = 0, 1, \ldots, q$, online by considering the smallest solution to the equation $V_{\lambda_k}(x(t_k)) = c_{\lambda_k}$, $t_k \triangleq k\Delta T$, where $\Delta T > 0$ and $k = 0, 1, \ldots, q$, and define $\mathcal{S} \triangleq \{\lambda_k\}_{k=0}^q$. Now, with the robust feedback switching control law $u = \phi_{\lambda_{\mathcal{S}}(x_f, x_p)}(x_f, x_p)$, where $\lambda_{\mathcal{S}}(x_f, x_p)$ is obtained as described in Step 4 of Algorithm 5.1, it follows from Theorem 5.3 that the compact positively invariant set $\mathcal{N}_{\lambda_q}$ is globally asymptotically stable for all $\delta\psi_s(\cdot) \in \mathbf{\Delta}$. Furthermore, note that $\lambda_{\mathcal{S}}(x(t))$, $t \ge 0$, is piecewise constant and hence the robust feedback switching control law $u = \phi_{\lambda_{\mathcal{S}}(x_{\mathrm{f}},x_{\mathrm{p}})}(x_{\mathrm{f}}, x_{\mathrm{p}})$ is piecewise continuous.

For $\mathcal{S}$ consisting of a hybrid topology let $\mathcal{S} = [\lambda_{\min}, \lambda_{\mathrm{global}}] \cup \{\bar{\lambda}\}$, where $\bar{\lambda} > \lambda_{\mathrm{global}}$ is such that $\mathcal{N}_{\bar{\lambda}} \in \overset{\circ}{\mathcal{D}}_{\hat{\lambda}}$, for at least one $\hat{\lambda} \in [\lambda_{\min}, \lambda_{\mathrm{global}}]$, and let $p(\lambda) = \lambda$, $\lambda \in \mathcal{S}$. Since $p(\cdot)$ does not have a local minimum in $\mathcal{S}$ (other than the origin) and every $\lambda \in [\lambda_{\min}, \lambda_{\mathrm{global}}]$ is an accumulation point for $\mathcal{S}$, we are guaranteed, by Step 3(b) of Algorithm 5.1, that Assumption 5.1 is satisfied. Now global robust asymptotic stability of $\mathcal{N}_{\lambda_{\min}}$ for all $\delta\psi_{\mathrm{s}}(\cdot) \in \mathbf{\Delta}$ is guaranteed by Theorem 5.3 with the feedback control law $u = \phi_{\lambda_{\mathcal{S}}(x_{\mathrm{f}},x_{\mathrm{p}})}(x_{\mathrm{f}}, x_{\mathrm{p}})$, where $\lambda_{\mathcal{S}}(x_{\mathrm{f}}, x_{\mathrm{p}})$ is obtained as described in Step 4 of Algorithm 5.1. In particular, if $(x_{\mathrm{f}}(0), x_{\mathrm{p}}(0)) \in \hat{\mathcal{D}} \triangleq \bigcup_{\lambda \in [\lambda_{\min}, \lambda_{\mathrm{global}}]} \mathcal{D}_\lambda$ then $\lambda_{\mathcal{S}}(x(t))$, $t \geq 0$, is a continuous function. Alternatively, if $(x_{\mathrm{f}}(0), x_{\mathrm{p}}(0)) \notin \hat{\mathcal{D}}$ then $\lambda_{\mathcal{S}}(x(t)) = \bar{\lambda}$, $t \in [0, \hat{t})$, where $\hat{t} > 0$ is such that $(x_{\mathrm{f}}(\hat{t}), x_{\mathrm{p}}(\hat{t})) \in \partial\hat{\mathcal{D}}$. In this case, $\lambda_{\mathcal{S}}(x(t))$, $t \geq 0$, is continuous modulo one discontinuity at $t = \hat{t}$. Note that since $\mathcal{N}_{\lambda_{\min}} \equiv \mathcal{D}_{\lambda_{\min}}$, $\mathcal{N}_{\lambda_{\min}}$ is a global attractor but not Lyapunov stable (see Remark 5.2 for details).

It is important to note that the proposed robust switching nonlinear controller framework can be incorporated to address practical actuator limitations such as control amplitude and rate saturation constraints. Specifically, since $\lambda_t \triangleq \lambda_{\mathcal{S}}(x_{\mathrm{f}}(t), x_{\mathrm{p}}(t))$ is proportional to the throttle opening (actuator) and since the dynamics of $\lambda_t$ indirectly characterize the fastest admissible rate at which the control throttle can open while maintaining stability of the controlled system, it follows that by constraining how fast $\lambda_t$ can change on the nominal equilibrium branch effectively places a rate constraint on the throttle opening. This corresponds to the case where the switching rate of the nonlinear controller is decreased so that the trajectory $(x_{\mathrm{f}}(t), x_{\mathrm{p}}(t))$, $t \geq 0$, is allowed to enter $\mathcal{D}_{\lambda_t}$. Additionally, amplitude saturation constraints and state constraints can also be enforced by simply choosing $\lambda_{\max} < \lambda_{\mathrm{global}}$ such that $\mathcal{D}_{\max} \triangleq \bigcup_{\lambda_{\min} \leq \lambda \leq \lambda_{\max}} \mathcal{D}_\lambda$ is contained in the region where the system is constrained to operate. In this case, the hierarchical robust switching nonlinear controller guarantees local robust asymptotic stability of $\mathcal{N}_{\lambda_{\min}}$ with an estimate of the domain of attraction given by $\mathcal{D}_{\max}$. Of course, in practice it is sufficient to implement controllers with adequate domains of attraction and *a priori* saturation constraint guarantees rather than implementing global controllers without realistic actuator limitations.

To show the efficacy of the proposed control approach, we consider a two-mode compressor model so that the state space model given by (28), or, equivalently, (31) and (32) is of sixth order. Furthermore, we recall from [37] that the axial flow variables $\phi_i$, $i=1,\ldots,n_{\mathrm{f}}$, are explicitly related to the system squared stall mode amplitudes $J_n$, $n=1,\ldots,n_{\mathrm{m}}$, while the averaged circumferential flow in the compressor is given by $\Phi \triangleq e^{\mathrm{T}}\phi/n_{\mathrm{f}}$. Using the parameter values $l_{\mathrm{C}}=6$, $H=0.32$, $W=0.18$, $\psi_{\mathrm{C}_0}=0.23$, and $B=0.1$, and the initial condition $(J_1, J_2, \Phi, \Psi)=(0.15, 0, 0.36, 0.87)$, corresponding to a perturbation in the first-mode disturbance velocity potential, the proposed robustly globally stabilizing controller and the nonrobust equilibria-dependent controller developed in [37] were used to compare the closed-loop system response. Here we model the uncertain perturbation to the nominal pressure-flow compressor characteristic map by

$$\delta\psi(x_{\mathrm{f}i}) = 0.1\cos[10(x_{\mathrm{f}i}-1)], \quad i=1,\ldots,5. \tag{61}$$

Figure 3 shows the nominal ($\psi_{\mathrm{C}}^{\mathrm{nom}}(\phi)$) and actual ($\psi_{\mathrm{C}}(\phi)$) pressure-flow compressor characteristic maps for $\kappa=0.1$. For this value of $\kappa$ the optimization problem outlined above for maximizing the domain

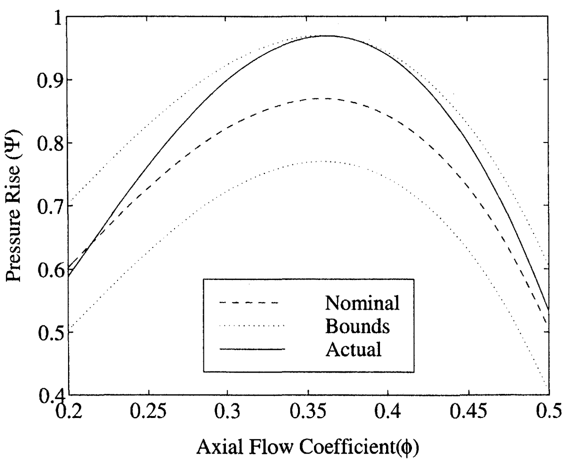


FIGURE 3 Actual and nominal compressor characteristics.

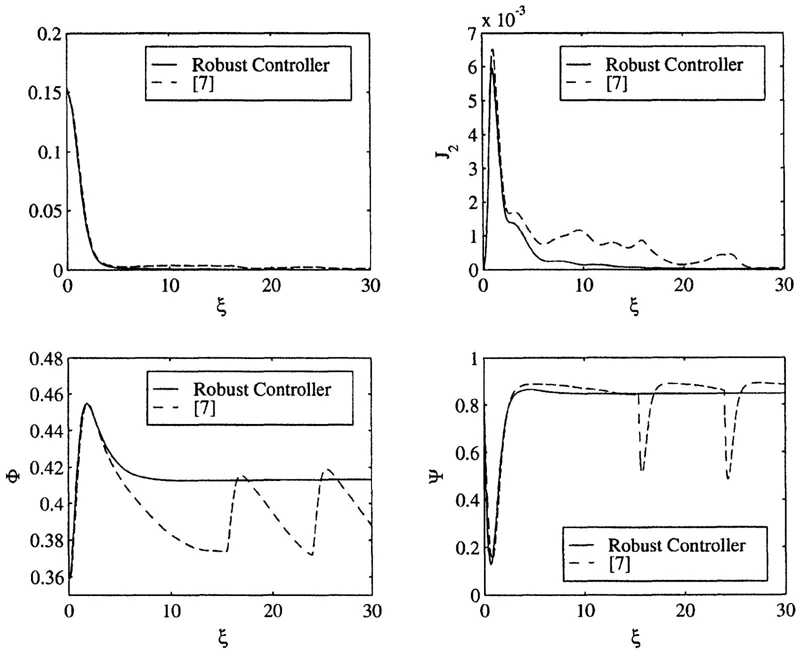


FIGURE 4 Controlled squared stall amplitudes, flow, and pressure versus time.

of attraction $\mathcal{D}_\lambda$ and minimizing the attractor $\mathcal{N}_\lambda$ yields

$$\lambda_{\min} = 0.2547, \quad \lambda_{\text{global}} = 1.1604, \quad d_{\lambda_{\min}} = 0.2236, \quad k_{\lambda_{\min}} = 0.0050.$$

Finally, we use $u(x_{\mathrm{f}}(t), x_{\mathrm{p}}(t)) = \lambda + h_\lambda(x_{\mathrm{f}}(t), x_{\mathrm{p}}(t))$, where $h_\lambda(x_{\mathrm{f}}(t), x_{\mathrm{p}}(t)) = x_{\mathrm{p}}(t) - \psi_{\mathrm{sC}}^{\mathrm{nom}}(\lambda)$.

Figure 4 shows the controlled responses for the squared stall cell amplitudes $J_1$ and $J_2$, the compressor flow $\Phi$, and the pressure rise $\Psi$ for both designs. This comparison illustrates that the robust controller globally stabilizes the axisymmetric operating point corresponding to $(J_1, J_2, \Phi, \Psi) = (0, 0, 0.4133, 0.8471)$. Alternatively, the nonrobust controller proposed in [37] drives the system to a limit-cycle instability induced by the control action. Finally, Fig. 5 shows the throttle opening versus time of the proposed robust controller.

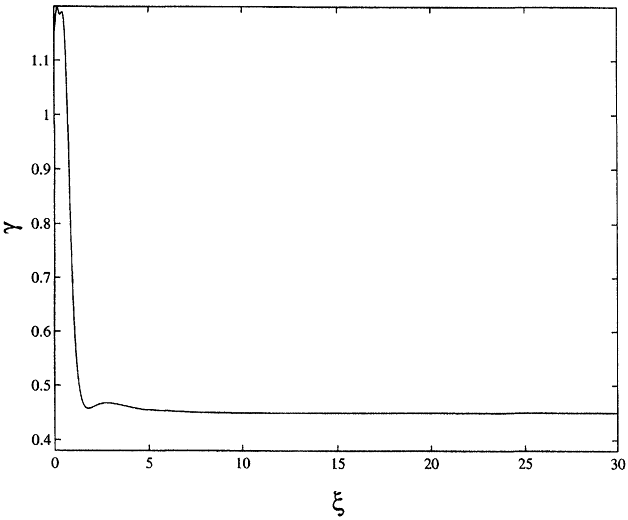


FIGURE 5 Throttle opening versus time.

## 7. CONCLUSION

A nonlinear robust control-system design framework predicated on a hierarchical switching controller architecture parameterized over a set of nominal system equilibria was developed. Specifically, a hierarchical robust switching nonlinear control strategy is constructed to stabilize a given uncertain nonlinear system by robustly stabilizing a collection of nonlinear controlled subsystems. The switching nonlinear controller architecture is designed based on a generalized Lyapunov function obtained by minimizing a potential function over a given switching set induced by the parameterized nominal system equilibria. The proposed framework was used to design robust switching controllers to control jet engine compression systems with uncertain pressure-flow map data.

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