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Generalized $\mathcal{S}$-Procedure and Finite Frequency KYP Lemma

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The contribution of this paper is twofold. First we give a generalization of the $\mathcal{S}$-procedure which has been proven useful for robustness analysis of control systems. We then apply the generalized $\mathcal{S}$-procedure to derive an extension of the Kalman–Yakubovich–Popov lemma that converts a frequency domain condition within a finite interval to a linear matrix inequality condition suitable for numerical computations.

Keywords: Control systems; $\mathcal{S}$-procedure; Positive-real lemma

1 INTRODUCTION

Consider the following condition given by multiple inequalities:

$$\zeta^*\Theta\zeta < 0, \quad \forall \zeta \in \mathcal{G}, \tag{1}$$

$$\mathcal{G} := \{\zeta \in \mathbb{C}^n\colon \zeta \neq 0,\ \zeta^* S_i \zeta \leq 0,\ \forall i = 1, \dots, m\}, \tag{2}$$

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where Θ and S_i are given Hermitian matrices. It is trivial to verify that a sufficient condition for (1) is given by

$$\exists \tau_i > 0 \quad \text{such that} \quad \Theta < \sum_{i=1}^{m} \tau_i S_i. \tag{3}$$

The $\mathcal{S}$-procedure [1] is to replace the multiple inequality constraint in (1) by the single inequality in (3) with multipliers τ_i. While this procedure is concerned with the quadratic forms on $\mathbb{C}^n$, an extension is available [2] to the case of the quadratic forms on $\mathcal{L}_2$, the set of square integrable vector-valued functions.

In general, the $\mathcal{S}$-procedure on $\mathbb{C}^n$ is conservative, i.e. (3) is only sufficient for (1) and may not be necessary. Nevertheless, the condition (3) can be efficiently verified by searching for the parameters τ_i which is a finite dimensional convex feasibility problem. Indeed, the $\mathcal{S}$-procedure and the aforementioned extension have been shown to be useful for developing various methods for control systems analysis and synthesis [2–4].

When applying the $\mathcal{S}$-procedure, the main concern is whether or not the procedure is conservative for the particular condition at hand. This fact gives rise to the following fundamental question: When does the $\mathcal{S}$-procedure yield an exact (nonconservative) condition? This question has already been extensively studied by Yakubovich and others. It is shown (for the nonstrict inequality case) that the $\mathcal{S}$-procedure on $\mathbb{C}^n$ is exact if $m \leq 2$ and that for $m > 2$ there are Θ and S_i such that the $\mathcal{S}$-procedure is conservative [1,4,5]. Moreover, the $\mathcal{S}$-procedure on $\mathcal{L}_2$ is known to be exact regardless of the number of constraints m [2].

In this paper, we generalize the $\mathcal{S}$-procedure on $\mathbb{C}^n$ in the following manner: note that the set $\mathcal{G}$ in (2) can be characterized by

$$\mathcal{G} = \{\zeta \in \mathbb{C}^n : \zeta \neq 0, \ \zeta^* S \zeta \leq 0, \ \forall S \in \mathcal{S}\} \tag{4}$$

where

$$\mathcal{S} := \left\{ \sum_{i=1}^{m} \tau_i S_i : \ \tau_i > 0, \ \forall i = 1, \ldots, m \right\}.$$

Then the $\mathcal{S}$-procedure is to replace condition (1), defined together with (4), by the existence of $S \in \mathcal{S}$ such that $\Theta < S$. Now, if we consider a general class of matrices $\mathcal{S}$ instead of the one given above, the S-procedure is still valid, i.e. the latter condition is sufficient to guarantee (1). We call this the generalized S-procedure.

The first contribution of this paper is to show conditions on $\mathcal{S}$ under which the generalized S-procedure is exact, and give a specific set $\mathcal{S}$ that satisfies the conditions. The second contribution is to show that the celebrated Kalman–Yakubovich–Popov (KYP) lemma [6,7] and its extension to the finite frequency condition simply follow from the generalized S-procedure. The finite frequency KYP lemma thus obtained is useful for solving various control problems including the integrated design of dynamical systems [8] and the computation of the structured singular value (upper bound) [9].

2 THE GENERALIZED S-PROCEDURE

Let us first introduce the notion of *lossless sets*, which will turn out to be a class of $\mathcal{S}$ in (4) leading to an *exact* (nonconservative) generalized S-procedure.

DEFINITION 1 *A subset $\mathcal{S}$ of $n \times n$ Hermitian matrices is said to be* lossless *if it has the following properties:*

(a) *$\mathcal{S}$ is convex.*
(b) $S \in \mathcal{S} \Rightarrow \tau S \in \mathcal{S}\ \forall \tau > 0$.
(c) *For each nonzero matrix $H \in \mathbb{C}^{n\times n}$ such that*

$$H = H^* \geq 0, \quad \mathrm{tr}(SH) \leq 0\ \ \forall S \in \mathcal{S},$$

there exist vectors $\zeta_i \in \mathbb{C}^n$ $(i = 1, \ldots, r)$ such that

$$H = \sum_{i=1}^{r} \zeta_i \zeta_i^*, \quad \zeta_i^* S \zeta_i \leq 0\ \ \forall S \in \mathcal{S},$$

where r is the rank of H.

The following is one of our main results and formally states that the generalized S-procedure is exact if the set $\mathcal{S}$ in (4) is lossless.

THEOREM 1 (The generalized S-procedure) *Let a Hermitian matrix Θ and a subset $\mathcal{S}$ of Hermitian matrices be given. Suppose $\mathcal{S}$ is lossless.*

Then the following statements are equivalent.

(i) $\zeta^*\Theta\zeta < 0 \ \forall \zeta \in \mathcal{G} := \{\zeta \in \mathbb{C}^n: \zeta \neq 0, \ \zeta^* S \zeta \leq 0 \ \forall S \in \mathcal{S}\}$.
(ii) *There exists* $S \in \mathcal{S}$ *such that* $\Theta < S$.

To prove this theorem, the following lemma is useful. The lemma is a version of the separating hyper-plane theorem [10] and has been derived in e.g. [11].

LEMMA 1 *Let* $\mathcal{X}$ *be a convex subset of* $\mathbb{C}^m$, *and* $F : \mathcal{X} \to \mathbb{C}^{n\times n}$ *be a Hermitian-valued affine function. The following statements are equivalent.*

(i) *The set* $\{x\text{: } x \in \mathcal{X}, \ F(x) < 0\}$ *is empty.*
(ii) $\exists$ *nonzero* $H = H^* \geq 0$ *s.t.* $\mathrm{tr}(F(x)H) \geq 0 \ \ \forall x \in \mathcal{X}$.

We now prove Theorem 1.

Proof (ii) $\Rightarrow$ (i) is trivial. To show the converse, suppose (ii) does not hold, i.e. there is no $S \in \mathcal{S}$ such that $\Theta < S$. Then, from Lemma 1, there exists a nonzero matrix H such that

$$H = H^* \geq 0, \quad \mathrm{tr}((\Theta - S)H) \geq 0 \ \ \forall S \in \mathcal{S}.$$

Since $\mathcal{S}$ is lossless, we have from property (b) of Definition 1 that

$$\mathrm{tr}(SH) \leq 0 \ \ \forall S \in \mathcal{S}, \quad \mathrm{tr}(\Theta H) \geq 0.$$

The first condition in turn implies the existence of the vectors ζ_i in property (c), and the second condition becomes

$$\mathrm{tr}(\Theta H) = \sum_{i=1}^{r} \zeta_i^* \Theta \zeta_i \geq 0.$$

Hence, there exists an index k such that $\zeta_k^* \Theta \zeta_k \geq 0$. Noting that $\zeta_k \in \mathcal{G}$, we conclude that (i) does not hold.

The significance of Theorem 1 can be explained as follows. Given a condition as in (1), Theorem 1 may be used to *equivalently* convert the condition to a numerically verifiable condition of the form given in statement (ii) of Theorem 1. To make sure that the conversion is exact, first we have to characterize the set $\mathcal{G}$ as in (4) for some set $\mathcal{S}$. Then we need to check if $\mathcal{S}$ is lossless. Of course these steps are usually non-trivial, but can be done for some class of $\mathcal{G}$ that is relevant to control systems analysis. We will do this next.

3 THE FINITE FREQUENCY KYP LEMMA

Consider the class of $\mathcal{G}$ described by

$$\mathcal{G} := \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \mathbb{C}^{2n} : f = j\omega g, \text{ for some } \omega \in \mathbb{R}, \ |\omega| \le \omega_0 \right\}, \qquad (5)$$

where $\omega_0 > 0$ is a given real scalar. Viewing $j\omega$ as the Laplace operator s, it is easily seen that this set is related to (input, output) signals (f, g) of an integrator. Thus it is not surprising that the set $\mathcal{G}$ plays a key role in the analysis of dynamical systems.

The following result identifies the set $\mathcal{S}$ that characterizes the set $\mathcal{G}$ in (5) through the definition in (4).

LEMMA 2 *Let a real scalar ω_0 and complex vectors f and g be given. The following statements are equivalent.*

(i) *There exists a real scalar ω such that $f = j\omega g$, $|\omega| \le \omega_0$.*

(ii) $\begin{bmatrix} f \\ g \end{bmatrix}^* \begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \le 0, \forall$ *complex matrices* $P = P^*$, $Q = Q^* > 0$.

Proof Suppose (i) holds. Then

$$\begin{bmatrix} f \\ g \end{bmatrix}^* \begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = (\omega^2 - \omega_0^2)(g^* Q g) \le 0$$

and hence (ii) holds. Conversely, if (ii) is satisfied,

$$\operatorname{tr}(ff^* - \omega_0^2 gg^*)Q + \operatorname{tr}(gf^* + fg^*)P \le 0$$

holds for all $P = P^*$ and $Q = Q^* > 0$. It can readily be verified that this implies

$$ff^* - \omega_0^2 gg^* \le 0, \quad gf^* + fg^* = 0.$$

It now follows from Lemma III.4 of [11] that (i) holds.

Let us now give a result that shows the losslessness of the set $\mathcal{S}$ related to $\mathcal{G}$ defined in (5). Its proof is rather technical and will be given later to keep the presentation streamlined.

LEMMA 3 *Let a scalar* $\omega_0 > 0$ *and a matrix* $F \in \mathbb{C}^{2n\times k}$ *be given. Define a subset of Hermitian matrices by*

$$\mathcal{S} := \left\{ F^* \begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} F \colon P = P^*,\ Q = Q^* > 0 \right\}.$$

Then the set $\mathcal{S}$ *is lossless.*

The following theorem is a generalization of the KYP lemma [6,7] where a frequency domain condition is required to hold only for a given low frequency band. The result is a simple consequence of the generalized $\mathcal{S}$-procedure.

THEOREM 2 *Let a scalar* $\omega_0 > 0$ *and matrices* $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$ *and a Hermitian matrix* $\Theta \in \mathbb{C}^{(n+m)\times(n+m)}$ *be given. Suppose* A *has no eigenvalues on the imaginary axis. Then the following statements are equivalent.*

(i) *The finite frequency condition*

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} < 0, \quad \forall |\omega| \le \omega_0$$

holds.

(ii) *There exist Hermitian matrices* $P, Q \in \mathbb{C}^{n\times n}$ *such that* $Q > 0$ *and*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P \\ P & \omega_0^2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta < 0.$$

If matrices A, B *and* Θ *are all real, the equivalence still holds when restricting* P *and* Q *to be real.*

Proof Note that (i) holds if and only if

$$\zeta^* \Theta \zeta < 0 \quad \forall \zeta \in \mathcal{G}$$

where

$$\mathcal{G} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{C}^{n+m} \colon w \neq 0,\ j\omega x = Ax + Bw \text{ for some } \omega \in \mathbb{R},\ |\omega| \leq \omega_0 \right\}.$$

Defining

$$\begin{bmatrix} f \\ g \end{bmatrix} := F \begin{bmatrix} x \\ w \end{bmatrix}, \quad F := \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$$

and applying Lemma 2, the set $\mathcal{G}$ can be characterized as

$$\mathcal{G} = \{ \zeta \neq 0 \colon \zeta^* S \zeta \leq 0\ \forall S \in \mathcal{S} \}$$

where

$$\mathcal{S} := \left\{ F^* \begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} F \colon P = P^*,\ Q = Q^* > 0 \right\}.$$

From Lemma 3, the set $\mathcal{S}$ is lossless and hence the $\mathcal{S}$-procedure in Theorem 1 yields (i) $\Leftrightarrow$ (ii).

Finally, to prove the real case result, assume that there exist (complex) Hermitian matrices P and Q satisfying the condition in statement (ii). Then, noting that

$$(M + jN) = (M + jN)^* > 0 \Leftrightarrow \begin{bmatrix} M & -N \\ N & M \end{bmatrix} = \begin{bmatrix} M & -N \\ N & M \end{bmatrix}' > 0 \qquad (6)$$

holds for any real square matrices M and N, one can show that the real parts of P and Q also satisfy the same condition.

A simple change of variables in Theorem 2 yields a characterization of another frequency domain condition where the inequality is required to hold in an arbitrarily given frequency interval.

COROLLARY 1 *Let real scalars* $\omega_1 \leq \omega_2$, *matrices* $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$ *and a Hermitian matrix* $\Theta \in \mathbb{C}^{(n+m)\times(n+m)}$ *be given. Suppose A has no eigenvalues on the imaginary axis. Then the following statements* *are equivalent.*

(i) *The finite frequency condition*

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} < 0, \quad \forall \omega_1 \le \omega \le \omega_2 \tag{7}$$

holds.

(ii) *There exist Hermitian matrices* $P, Q \in \mathbb{C}^{n \times n}$ *such that* $Q > 0$ *and*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta < 0, \tag{8}$$

where $\omega_c := (\omega_1 + \omega_2)/2$.

Proof Note that $\omega_1 \le \omega \le \omega_2$ is equivalent to $|\hat{\omega}| \le \hat{\omega}_{\max}$ where

$$\hat{\omega} = \omega - \omega_c, \quad \hat{\omega}_{\max} = (\omega_2 - \omega_1)/2.$$

Hence, the result follows by applying Theorem 2 to $(\hat{A}, B, \Theta)$ with $\hat{\omega}$ via the following transformation:

$$j\omega I - A = j\hat{\omega} I - \hat{A}, \quad \hat{A} := A - j\omega_c I.$$

When A, B and Θ are real matrices, one can show the following: If inequality (8) holds for

$$\begin{gathered} \omega_1 := \alpha, \quad \omega_2 := \beta, \\ P := P_R + jP_I, \quad Q := Q_R + jQ_I > 0 \end{gathered}$$

then the same inequality holds for

$$\begin{gathered} \omega_1 := -\beta, \quad \omega_2 := -\alpha, \\ P := P_R - jP_I, \quad Q = Q_R - jQ_I > 0. \end{gathered}$$

Thus the frequency domain condition (7) holds for $\omega_1 \le \omega \le \omega_2$, if and only if the same condition holds for $-\omega_2 \le \omega \le -\omega_1$.

When A and B are real, the finite frequency condition in Corollary 1 can be characterized by an LMI involving real matrices only. Such characterization is directly useful for numerical computation. The result follows from a straightforward application of the identity (6) and hence the proof is omitted.

COROLLARY 2 *Consider the finite frequency condition in Corollary* 1. *If A and B are real matrices, the condition is equivalent to the following:*

(iii) *There exist real symmetric matrices* $\mathcal{P}, \mathcal{Q} \in \mathbb{R}^{2n \times 2n}$ *of the form*

$$\mathcal{P} = \begin{bmatrix} P_R & -P_I \\ P_I & P_R \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} Q_R & -Q_I \\ Q_I & Q_R \end{bmatrix},$$

satisfying $\mathcal{Q} > 0$ *and*

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \end{bmatrix}' \begin{bmatrix} -\mathcal{Q} & \mathcal{P} + J\omega_c \mathcal{Q} \\ \mathcal{P} - J\omega_c \mathcal{Q} & -\omega_1 \omega_2 \mathcal{Q} \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \end{bmatrix} + \Phi < 0,$$

where

$$J := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad \mathcal{A} := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

and Φ *is defined in terms of the real and the imaginary parts of* Θ *as follows:*

$$\Theta = \begin{bmatrix} U_R & V_R \\ V_R' & W_R \end{bmatrix} + j \begin{bmatrix} U_I & V_I \\ -V_I' & W_I \end{bmatrix},$$

$$\Phi := \begin{bmatrix} U & V \\ V' & W \end{bmatrix}, \quad U := \begin{bmatrix} U_R & -U_I \\ U_I & U_R \end{bmatrix},$$

$$V := \begin{bmatrix} V_R & -V_I \\ V_I & V_R \end{bmatrix}, \quad W := \begin{bmatrix} W_R & -W_I \\ W_I & W_R \end{bmatrix}.$$

4 CONNECTION TO THE (D, G)-SCALING

The finite frequency KYP lemma (Theorem 2) shown in the previous section can also be derived through the losslessness theorem of the (D, G)-scaling upper bound of mixed μ [11]. In that case, we need some restrictions on matrix Θ to allow for an appropriate loop-shifting and its proof will no longer be self-contained, for the necessity proof relies on the losslessness of the (D, G)-scaling shown in [11]. Nevertheless, it would be of interest to outline the derivation of the finite frequency KYP lemma through the (D, G)-scaling.

Let us first derive the finite frequency bounded-real lemma which is a special case of the finite frequency KYP lemma. Consider the $m \times p$ transfer function matrix

$$G(s) := C(sI - A)^{-1}B + D,$$

where matrices A, B, C and D are possibly complex. Suppose A has no eigenvalues on the imaginary axis. Then it can readily be verified [9] that the following identity holds for all real scalars ω and $\omega_0 > 0$:

$$G(j\omega) = \mathsf{C}(I - \delta\mathsf{A})^{-1}\delta\,\mathsf{B} + \mathsf{D} =: \mathsf{G}(\delta),$$

where

$$\delta := \omega/\omega_0,$$
$$\mathsf{M} := \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{bmatrix} := \begin{bmatrix} j\omega_0 A^{-1} & A^{-1}B \\ -j\omega_0 CA^{-1} & D - CA^{-1}B \end{bmatrix}.$$

From the standard μ-analysis, we have

$$\|G(j\omega)\| < 1, \quad \forall |\omega| \le \omega_0 \Leftrightarrow \|\mathsf{G}(\delta)\| < 1, \quad \forall |\delta| \le 1$$
$$\Leftrightarrow \det(I - \mathsf{M}\nabla) \ne 0, \quad \forall \nabla \in \boldsymbol{\nabla},$$

where

$$\boldsymbol{\nabla} := \{\mathrm{diag}(\delta I, \Delta) \colon \delta \in \mathbb{R},\ \Delta \in \mathbb{C}^{p \times m},\ |\delta| \le 1,\ \|\Delta\| \le 1\}.$$

Using the losslessness of the (D, G)-scaling with respect to the uncertainty ∇ consisting of one repeated real scalar δ and one full-block complex matrix Δ [11], the last condition is equivalent to the existence of complex matrices $\mathcal{D} = \mathcal{D}^* > 0$ and $\mathcal{G} = -\mathcal{G}^*$ such that

$$\begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \end{bmatrix}^* \begin{bmatrix} \mathcal{D} & \mathcal{G} \\ \mathcal{G}^* & -\mathcal{D} \end{bmatrix} \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} \mathsf{C} & \mathsf{D} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathsf{C} & \mathsf{D} \\ 0 & I \end{bmatrix} < 0.$$

Now, defining

$$P := j\mathcal{G}^*/\omega_0, \quad Q := \mathcal{D}/\omega_0^2$$

the congruent transformation by $\begin{bmatrix} A & B \\ 0 & -j\omega_0 I \end{bmatrix}$ yields

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P \\ P & \omega_0^2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0.$$

Clearly, $P = P^*$ and $Q = Q^* > 0$. Thus the existence of such P and Q is necessary and sufficient for the finite frequency bounded-real condition to hold.

We now consider the condition

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I_p \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1} B \\ I_p \end{bmatrix} < 0, \quad \forall |\omega| \le \omega_0. \tag{9}$$

Clearly, Θ must have at least p negative eigenvalues in order for this condition to hold. On the other hand, if all the eigenvalues are negative, the condition becomes trivial. Hence, it is reasonable to assume that Θ has both positive and negative eigenvalues, in which case, it can be written as

$$\Theta = \begin{bmatrix} C_1 & D_1 \\ C_2 & D_2 \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C_1 & D_1 \\ C_2 & D_2 \end{bmatrix}.$$

Let us also assume that D_2 is square ($p \times p$) and nonsingular. This is a restrictive assumption. Using the above expression for Θ, the condition in (9) can be described by

$$G_1(j\omega)^* G_1(j\omega) < G_2(j\omega)^* G_2(j\omega), \quad \forall |\omega| \le \omega_0,$$

where

$$G_i(s) := C_i(sI - A)^{-1} B + D_i \quad (i = 1, 2).$$

This condition is in turn equivalent to

$$\|G(j\omega)\| < 1, \ \forall |\omega| \le \omega_0, \quad G(s) := G_1(s) G_2(s)^{-1}.$$

It can be verified that a state space realization for $G(s)$ is given by

$$G(s) = \left(\begin{array}{c|c} A - BD_2^{-1}C_2 & BD_2^{-1} \\ \hline C_1 - D_1D_2^{-1}C_2 & D_1D_2^{-1} \end{array}\right).$$

Applying the finite frequency bounded-real condition to $G(s)$ and performing the congruent transformation with $\begin{bmatrix} I & 0 \\ C_2 & D_2 \end{bmatrix}$, it can be shown that the finite frequency KYP lemma (Theorem 2) holds.

5 PROOF OF THE LOSSLESSNESS OF THE SET $\mathcal{S}$

In this section, we prove Lemma 3. The following two lemmas are instrumental for the proof. Below, $(\cdot)^\dagger$ denotes the Moore–Penrose inverse of a matrix.

LEMMA 4 *Let complex matrices R and S be given. Suppose*

$$\|[R \quad S]\| \le 1, \quad R + R^* = 0. \tag{10}$$

Then there exists a matrix Q such that

$$\left\| \begin{bmatrix} R & S \\ -S^* & Q \end{bmatrix} \right\| \le 1, \quad Q + Q^* = 0. \tag{11}$$

Moreover, one such Q is given by

$$Q = -S^* R (I + R^2)^\dagger S.$$

Proof From the supposition, we have $\|R\| \le 1$ and hence $I - RR^* \ge 0$. Let $\Omega := (I - RR^*)^{1/2}$. From (10),

$$RR^* + SS^* \le I \Rightarrow SS^* \le \Omega^2.$$

This implies (e.g. [12]) that there exists a matrix C such that

$$S = \Omega C, \quad \|C\| \le 1.$$

Let

$$Q := -S^*\Omega^\dagger R\Omega^\dagger S = -S^*R(I-RR^*)^\dagger S.$$

Clearly, Q is skew Hermitian. Note that

$$\begin{aligned}\left\|\begin{bmatrix} R & S \\ -S^* & Q \end{bmatrix}\right\| &= \left\|\begin{bmatrix} R & \Omega C \\ -C^*\Omega & C^*\hat{Q}C \end{bmatrix}\right\| \\ &\leq \left\|\begin{bmatrix} I & 0 \\ 0 & -C^* \end{bmatrix}\right\| \left\|\begin{bmatrix} R & \Omega \\ -\Omega & \hat{Q} \end{bmatrix}\right\| \left\|\begin{bmatrix} I & 0 \\ 0 & -C \end{bmatrix}\right\| \\ &= \left\|\begin{bmatrix} R & \Omega \\ -\Omega & \hat{Q} \end{bmatrix}\right\|,\end{aligned}$$

where $\hat{Q} := -\Omega\Omega^\dagger R\Omega^\dagger\Omega$ and the last inequality holds due to $\|C\| \leq 1$. It can be verified that

$$R\Omega + \Omega R^* = 0.$$

Repeated use of this identity, after some manipulations, yields

$$\begin{bmatrix} R & \Omega \\ -\Omega & \hat{Q} \end{bmatrix}\begin{bmatrix} R & \Omega \\ -\Omega & \hat{Q} \end{bmatrix}^* = \begin{bmatrix} I & 0 \\ 0 & I - R(I-\Omega\Omega^\dagger)R^* \end{bmatrix} \leq I,$$

where the last inequality is due to the following fact:

$$I - \Omega\Omega^\dagger \geq 0 \Rightarrow 0 \leq I - R(I-\Omega\Omega^\dagger)R^* \leq I.$$

Hence we conclude that the norm condition in (11) holds.

LEMMA 5 *Let complex matrices Z and W of the same dimensions be given. The following statements are equivalent.*

(i) $WW^* \leq ZZ^*$ *and* $ZW^* + WZ^* = 0$.
(ii) *There exists a complex matrix* Δ *such that*

$$W = Z\Delta, \quad \|\Delta\| \leq 1, \ \Delta + \Delta^* = 0.$$

Proof (ii) $\Rightarrow$ (i) is trivial. To show the converse, suppose (i) holds. Then there exists ∇ such that

$$W = Z\nabla, \quad \|\nabla\| \leq 1.$$

This ∇ satisfies

$$Z(\nabla + \nabla^*)Z^* = 0.$$

If $Z^*Z > 0$, then $\nabla + \nabla^* = 0$ and we are done. So consider the case $Z^*Z \not> 0$. Let V be a Unitary matrix such that

$$ZV = [Z_1 \quad 0],$$

where Z_1 is full column rank. Define R and S by

$$\begin{bmatrix} R & S \\ * & * \end{bmatrix} := V^* \nabla V,$$

where R is square with its dimension equal to the rank of Z and $*$ denotes irrelevant entries. Then

$$\begin{aligned} &\|[R \quad S]\| \leq 1 \Leftarrow \|\nabla\| \leq 1 \\ &R + R^* = 0 \Leftarrow Z(\nabla + \nabla^*)Z^* = Z_1(R + R^*)Z_1^* = 0. \end{aligned}$$

From Lemma 4, there exists Q such that

$$\Delta := V\begin{bmatrix} R & S \\ -S^* & Q \end{bmatrix} V^*, \quad \|\Delta\| \leq 1, \ \Delta + \Delta^* = 0.$$

For this Δ, we have

$$Z\Delta = [Z_1 \quad 0]\begin{bmatrix} R & S \\ -S^* & Q \end{bmatrix} V^* = Z\nabla = W.$$

Hence we conclude that (i) $\Rightarrow$ (ii).

We are now ready to prove Lemma 3.

Proof Properties (a) and (b) in Definition 1 are easily verified. To show property (c), let H be a nonzero matrix such that

$$H = H^* \geq 0, \quad \mathrm{tr}(HS) \leq 0 \ \forall S \in \mathcal{S}. \tag{12}$$

Since H is positive semi-definite, it admits a full rank factor $H = GG^*$, $G \in \mathbb{C}^{k \times r}$ where r is the rank of H. Defining

$$\begin{bmatrix} W \\ Z \end{bmatrix} := FG, \quad W, Z \in \mathbb{C}^{n \times r},$$

the latter condition in (12) can be written

$$\begin{aligned}&\operatorname{tr}(WW^* - \omega_0^2 ZZ^*)Q + \operatorname{tr}(WZ^* + ZW^*)P \leq 0\\&\forall P = P^*,\ Q = Q^* > 0.\end{aligned}$$

It can readily be verified that this condition is equivalent to

$$WW^* \leq \omega_0^2 ZZ^*, \quad WZ^* + ZW^* = 0.$$

From Lemma 5, there exists a matrix $\Delta \in \mathbb{C}^{r \times r}$ such that

$$W = \omega_0 Z \Delta, \quad \|\Delta\| \leq 1,\ \Delta + \Delta^* = 0.$$

Since Δ is skew-Hermitian with norm less than or equal to one, its spectral decomposition yields

$$\Delta = \sum_{i=1}^{r} \lambda_i u_i u_i^*, \quad |\lambda_i| \leq 1,\ \lambda + \bar{\lambda}_i = 0,\ \sum_{i=1}^{r} u_i u_i^* = I.$$

For $i = 1, \ldots, r$, define

$$\zeta_i := G u_i, \quad \begin{bmatrix} w_i \\ z_i \end{bmatrix} := \begin{bmatrix} W \\ Z \end{bmatrix} u_i = F \zeta_i, \quad w_i, z_i \in \mathbb{C}^n.$$

Then $H = \sum_{i=1}^{r} \zeta_i \zeta_i^*$ and

$$W u_i = \omega_0 Z \Delta u_i \Rightarrow w_i = \lambda_i \omega_0 z_i.$$

Hence we have

$$\begin{aligned}w_i w_i^* &= \omega_0^2 |\lambda_i|^2 z_i z_i^* \leq \omega_0^2 z_i z_i^*,\\ w_i z_i^* + z_i w_i^* &= \omega_0 (\lambda_i + \bar{\lambda}_i) z_i z_i^* = 0.\end{aligned}$$

These conditions imply

$$\operatorname{tr}(\zeta_i \zeta_i^* S) = \zeta_i^* S \zeta_i \leq 0 \quad \forall S \in \mathcal{S}$$

and we conclude that $\mathcal{S}$ satisfies property (c) of Definition 1.

6 CONCLUSION

We have given a generalization of the S-procedure, a powerful tool in control and optimization theories. As an application of the generalized S-procedure, the finite frequency KYP lemma is derived. These results are expected to be useful for control systems analysis and synthesis.

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