Explicit Solution of the Jump Problem for the Laplace Equation and Singularities at the Edges

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The boundary value problem for the Laplace equation outside several cuts in a plane is studied. The jump of the solution of the Laplace equation and the jump of its normal derivative are specified on the cuts. The problem is studied under different conditions at infinity, which lead to different uniqueness and existence theorems. The solution of this problem is constructed in the explicit form by means of single layer and angular potentials. The singularities at the ends of the cuts are investigated.

Keywords: Harmonic functions; Cracks; Edges

1. INTRODUCTION

In the jump problem for the Laplace equation outside cuts in a plane, we specify the jump of the solution and the jump of its normal derivative at the cuts. This problem is closely related to the Dirichlet and Neumann problems outside cuts in a plane, which are used to model cracks in solids or screens in fluids [1–10]. This problem is also closely related to a transmission problem for the Helmholtz equation, where the jump of unknown function and the jump of its normal derivative are given on the closed curves [11–14]. The attempt to formulate the jump problem outside an open arc for the 2-D Laplace

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equation with one boundary condition of jump type is contained in [10], but the problem in [10] was not uniquely solvable and so was not well-posed. In the present paper we give a well-posed formulation of boundary value problem outside cuts in a plane for the Laplace equation with two boundary conditions of jump type. Moreover, we construct an explicit solution of our problem in the form of a single layer potential and an angular potential [4, 10]. It should be stressed that our solution is explicit for cuts of an arbitrary shape. This is the basic advantage of the jump problem over Dirichlet and Neumann problems outside cuts in a plane, since the explicit solution can not be obtained in Dirichlet and Neumann problems for cuts of an arbitrary shape. In the present paper we also give explicit formulas for singularities of the solution gradient at the ends of cuts. It appears that these singularities are weaker than in the Dirichlet and Neumann problems outside cuts in a plane [4, 5]. It is found that singularities in the jump problem are logarithmic, while in the Dirichlet and Neumann problems they are generally of power 1/2. The jump problem for the Laplace equation presented in this paper can be effectively used to model different physical phenomena in cracked media. Our results can be also helpful in problems on crack determination [15 – 17, 20].

The jump problem for analytic complex functions has been studied in [21]. However, our problem cannot be reformulated in terms of analytic complex theory, because complex function related to real harmonic solution of our problem is not analytic outside cuts, it may have logarithmic branching point at infinity. So, the theory of analytic complex functions [21] does not apply. If one requires that the solution in our problem to be bounded at infinity, then our problem may be not solvable, more precisely, it is solvable under the special solvability condition. The related jump problem in analytic complex theory [21] is always solvable, even if it is required that its solution tends to zero at infinity.

2. FORMULATION OF THE PROBLEM

By a simple open curve we mean a non-closed smooth arc of finite length without self-intersections [5]. In the plane $x = (x_1, x_2) \in R^2$ we consider simple open curves $\Gamma_1, \ldots, \Gamma_N \in C^{2,\lambda}$, $\lambda \in (0, 1]$, so that they do not have common points. We put $\Gamma = \bigcup_{n=1}^N \Gamma_n$. We assume that

each curve Γ_n is parametrized by the arc length s:

$$\Gamma_n = \{x : x = x(s) = (x_1(s), x_2(s)), s \in [a_n, b_n]\}, \quad n = 1, \dots, N,$$

so that $a_1 < b_1 < \cdots < a_N < b_N$. Therefore points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence. Below the set of the intervals on the Os axis $\bigcup_{n=1}^{N}[a_n, b_n]$ will be denoted by Γ also.

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos \alpha(s), \sin \alpha(s))$, where $\cos \alpha(s) = x_1'(s)$, $\sin \alpha(s) = x_2'(s)$. Let $\mathbf{n}_x = (\sin \alpha(s), -\cos \alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated counterclockwise by an angle of $\pi/2$. We consider Γ as a set of cuts. The side of Γ which is on the left when the parameter s increases will be denoted by Γ^+ and the opposite side will be denoted by Γ^-.

We say that the function $u(x)$ belongs to the smoothness class $\mathbf{K}$ if the following conditions are satisfied:

(1) $u(x) \in C^0(\overline{R^2 \setminus \Gamma}) \cap C^2(R^2 \setminus \Gamma)$ and $u(x)$ is continuous at the ends of Γ;

(2) $\nabla u \in C^0(\overline{R^2 \setminus \Gamma} \setminus X)$, where X is a point set, consisting of the endpoints of Γ: $X = \bigcup_{n=1}^{N}(x(a_n) \cup x(b_n))$;

(3) in the neighbourhood of any point $x(d) \in X$, for some constants $C > 0$ and $\varepsilon > -1$, the inequality

$$|\nabla u| < C|x - x(d)|^{\varepsilon} \tag{1}$$

holds, where $x \to x(d)$ and $d = a_n$ or $d = b_n$ for $n = 1, \dots, N$.

Remark In the definition of the class $\mathbf{K}$ we consider Γ as a set of cuts in a plane. In particular, the notation $C^0(\overline{R^2 \setminus \Gamma})$ denotes a class of functions, which are continuously extended on Γ from the left and right, but their values on Γ from the left and right can be different, so that the functions may have a jump across Γ.

We introduce 3 classes of functions $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ with the different behaviour at infinity.

The function $u(x)$ belongs to $\mathbf{M}_1$, if there exist constants C_1, C_2, C_3, C_4 such that the estimates

$$|u(x) - C_1 \ln|x| - C_2| < C_3|x|^{-1}, \quad |\nabla u(x)| < C_4|x|^{-1}$$

hold as $|x| = \sqrt{x_1^2 + x_2^2} \to \infty$.

The function $u(x)$ belongs to $\mathbf{M}_2$, if for some constants $\mathcal{C}_1$, $\mathcal{C}_2$, the estimates

$$|u(x)| < \mathcal{C}_1, \quad |\nabla u(x)| < \mathcal{C}_2 |x|^{-2}$$

hold as $|x| = \sqrt{x_1^2 + x_2^2} \to \infty$.

The definition of the class $\mathbf{M}_3$ can be formulated in the same way as the definition of $\mathbf{M}_2$, but instead of the first inequality from $\mathbf{M}_2$ we require the following inequality

$$|u(x)| < \mathcal{C}_1 |x|^{-1}.$$

Let us formulate the jump problem for the harmonic functions in $R^2 \backslash \Gamma$.

Problem $(\mathbf{U}_j(j = 1, 2, 3))$ Find a function $u(x)$ of class $\mathbf{K}$, so that $u(x)$ satisfies the Laplace equation in $R^2 \backslash \Gamma$

$$\Delta u = 0, \quad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2, \tag{2}$$

satisfies the jump boundary conditions

$$u(x)|_{x(s) \in \Gamma^+} - u(x)|_{x(s) \in \Gamma^-} = f_1(s), \tag{3a}$$

$$\frac{\partial u}{\partial \mathbf{n}}\bigg|_{x(s) \in \Gamma^+} - \frac{\partial u}{\partial \mathbf{n}}\bigg|_{x(s) \in \Gamma^-} = f_2(s), \tag{3b}$$

and meets the following conditions at infinity

$$u(x) \in \mathbf{M}_j \quad (j = 1, 2, 3). \tag{4}$$

All conditions of the problem must be fulfiled in a classical sense.

Thus, we consider 3 problems $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$. They differ in conditions at infinity (4). In case of the problems $\mathbf{U}_2$, $\mathbf{U}_3$ the solution must be bounded at infinity. In case of $\mathbf{U}_1$ the solution may have a logarithmic singularity at infinity.

Conditions (1) at the ends of Γ in the formulation of the class $\mathbf{K}$ ensure the absense of point sources at the ends of Γ. If $f_1(s) = f_2(s) = 0$ on $\gamma \subset \Gamma$, then Eq. (2) holds on γ and $u(x)$ is analytic on γ.

THEOREM 1 (1) *The solution of the problem U_1 is defined up to an arbitrary additive constant.* (2) *The necessary condition for the solvability of the problems U_2, U_3 is*

$$\int_{\Gamma} f_2(s)ds = 0. \tag{5}$$

The solution of U_2 is defined up to an arbitrary additive constant. There is at most one solution of the problem U_3.

By $\int_{\Gamma} \cdots d\sigma$ we mean

$$\sum_{n=1}^{N} \int_{a_n}^{b_n} \cdots d\sigma.$$

Now we prove the theorem. The limit values of functions on Γ^+ and Γ^- will be denoted by the superscripts "$+$" and "$-$" respectively.

Let $u_j(x)$ be a solution of the problem $U_j(j=1,2,3)$. To apply energy equalities for harmonic functions, we envelope open curves by closed contours, tend contours to the curves and use the smoothness of the solution of the problem U_j. In this way we arrive at two identities

$$\int_{\Gamma} \left[\left(\frac{\partial u_j}{\partial \mathbf{n}_x} \right)^+ - \left(\frac{\partial u_j}{\partial \mathbf{n}_x} \right)^- \right] ds + \int_0^{2\pi} \frac{\partial u_j}{\partial r} r d\varphi = 0,$$

$$\|\nabla u_j\|_{L_2(C_r \backslash \Gamma)}^2 =$$

$$= \int_{\Gamma} u_j^+ \left(\frac{\partial u_j}{\partial \mathbf{n}_x} \right)^+ ds - \int_{\Gamma} u_j^- \left(\frac{\partial u_j}{\partial \mathbf{n}_x} \right)^- ds + \int_0^{2\pi} u_j \frac{\partial u_j}{\partial r} r d\varphi =$$

$$= \int_{\Gamma} \left\{ (u_j^+ - u_j^-) \left(\frac{\partial u_j}{\partial \mathbf{n}_x} \right)^+ + u_j^- \left[\left(\frac{\partial u_j}{\partial \mathbf{n}_x} \right)^+ - \left(\frac{\partial u_j}{\partial \mathbf{n}_x} \right)^- \right] \right\} ds +$$

$$+ \int_0^{2\pi} u_j \frac{\partial u_j}{\partial r} r d\varphi, \tag{6}$$

where C_r is the circle of the large radius r with the center in the origin, and φ is a polar angle. We suppose, that $\Gamma \subset C_r$. Putting boundary conditions (3b) in the first identity we obtain

$$\int_{\Gamma} f_2(s)ds = -\int_0^{2\pi} \frac{\partial u_j}{\partial r} r d\varphi. \tag{7}$$

If $j = 2, 3$, then the necessary condition (5) follows from (7) and conditions at infinity (4) as $r \to \infty$.

Now let $u_j^0(x)$ be a solution of the homogeneous problem $\mathbf{U}_j (j = 1, 2, 3)$. Substituting $u_j^0(x)$ in (6) and taking into account homogeneous boundary conditions (3) we get:

$$\|\nabla u_j^0\|_{L_2(C_r \setminus \Gamma)}^2 = \int_0^{2\pi} u_j^0 \frac{\partial u_j^0}{\partial r} r \, d\varphi. \tag{8}$$

Let $j = 1$. Putting $r \to \infty$ and using notations from the definition of the class $\mathbf{M}_1$, we have

$$\|\nabla u_1^0\|_{L_2(R^2 \setminus \Gamma)}^2 = \lim_{r \to \infty} \|\nabla u_1^0\|_{L_2(C_r \setminus \Gamma)}^2 =$$

$$= \lim_{r \to \infty} \left\{ (C_1 \ln r + C_2) \int_0^{2\pi} \frac{\partial u_1^0}{\partial r} r \, d\varphi + \right.$$

$$\left. + \int_0^{2\pi} (u_1^0 - C_1 \ln r - C_2) \frac{\partial u_1^0}{\partial r} r \, d\varphi \right\}.$$

It follows from (7) that the first integral is identically equal to zero, because $f_2(s) \equiv 0$ in case of the homogeneous problem. The second integral tends to zero as $r \to \infty$ thanks to the definition of the class $\mathbf{M}_1$.

Thus, $\|\nabla u_1^0\|_{L_2(R^2 \setminus \Gamma)}^2 = 0$ and therefore $u_1^0(x) \equiv \text{const}$.

Let $j = 2$ or $j = 3$. Putting $r \to \infty$ in (8) and using definitions of classes $\mathbf{M}_2$ and $\mathbf{M}_3$, we obtain $\|\nabla u_j^0\|_{L_2(R^2 \setminus \Gamma)}^2 = 0$. If $j = 2$, then $u_2^0(x) \equiv \text{const}$. If $j = 3$, then $u_3^0(x) \equiv 0$ according to the definition of the class $\mathbf{M}_3$. Now the statement of the theorem follows from the linearity of the problem $\mathbf{U}_j (j = 1, 2, 3)$.

3. THE SOLUTION OF THE PROBLEM

To construct a solution of the problem $\mathbf{U}_j (j = 1, 2, 3)$ suppose that

$$f_1(s) \in C^{1,\lambda}(\Gamma), \quad f_2(s) \in C^{0,\lambda}(\Gamma), \quad \lambda \in (0, 1]; \tag{9a}$$

$$f_1(a_n) = f_1(b_n) = 0, \quad n = 1, \dots, N. \tag{9b}$$

At first we obtain a solution of the problem $\mathbf{U}_1$. The explicit solution of this problem can be constructed in the form of a sum of a single

layer potential and an angular potential [4, 10] for the Laplace Eq. (2). Consider a function

$$u(x) = v[f_1'](x) + w[f_2](x) + c, \qquad (10)$$

where c is an arbitrary constant,

$$w[f_2](x) = -\frac{1}{2\pi} \int_\Gamma f_2(\sigma) \ln |x - y(\sigma)| d\sigma$$

is a single layer potential for the Eq. (2), and $v[f_1'](x)$ is the angular potential [4, 10] for the Eq. (2)

$$v[f_1'](x) = -\frac{1}{2\pi} \int_\Gamma f_1'(\sigma) V(x, \sigma) d\sigma, \quad f_1'(\sigma) = \frac{d}{d\sigma} f_1(\sigma).$$

The kernel $V(x, \sigma)$ is defined (up to indeterminancy $2\pi m$, $m = \pm 1$, $\pm 2, \dots$) by the formulae

$$\cos V(x, \sigma) = \frac{x_1 - y_1(\sigma)}{|x - y(\sigma)|}, \quad \sin V(x, \sigma) = \frac{x_2 - y_2(\sigma)}{|x - y(\sigma)|},$$

where $y(\sigma) = (y_1(\sigma), y_2(\sigma)) \in \Gamma$,

$$|x - y(\sigma)| = \sqrt{(x_1 - y_1(\sigma))^2 + (x_2 - y_2(\sigma))^2}.$$

One can see, that $V(x, \sigma)$ is the angle between the vector $\overrightarrow{y(\sigma)x}$ and the direction of the Ox_1 axis. More precisely, $V(x, \sigma)$ is a many-valued harmonic function of x connected with $\ln |x - y(\sigma)|$ by the Cauchy–Riemann relations. Below by $V(x, \sigma)$ we denote an arbitrary fixed branch of this function, which varies continuously with σ along each curve Γ_n $(n = 1, \dots, N)$ for given fixed $x \notin \Gamma$. Under this definition of $V(x, \sigma)$, the potential $v[f_1'](x)$ is a many-valued function. In order that the potential $v[f_1'](x)$ be single-valued, the following additional conditions [10] must hold

$$\int_{a_n}^{b_n} f_1'(\sigma) d\sigma = f_1(b_n) - f(a_n) = 0, \quad n = 1, \dots, N.$$

Clearly, these conditions are satisfied due to our assumptions (9b). Integrating $v[f_1'](x)$ by parts and using (9b) we express the angular

potential in terms of a double layer potential

$$v[f_1'](x) = \frac{1}{2\pi} \int_\Gamma f_1(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln |x - y(\sigma)| d\sigma.$$

Consequently, the angular potential $v[f_1'](x)$ satisfies Eq. (2) outside Γ and belongs to the class $\mathbf{M}_3$. The single layer potential $w[f_2](x)$ belongs to the class $\mathbf{M}_1$.

So, it follows from properties of single layer and angular potentials [4, 14, 18] that the function (7) belongs to the class $\mathbf{K}$, satisfies Eq. (2) and meets conditions of class $\mathbf{M}_1$. It can be checked directly that the function (10) satisfies the boundary conditions (3) of the problem $\mathbf{U}_1$. Indeed, according to [4, 10], normal derivative of the angular potential $v[f_1'](x)$ is continuous across Γ. The single layer potential $w[f_2](x)$ is continuous across Γ in our assumptions. On the basis of the jump relations on Γ for the angular potential and for the normal derivative of the single layer potential, we obtain [4, 10]

$$u(x)|_{x(s) \in \Gamma_n^+} - u(x)|_{x(s) \in \Gamma_n^-} = v[f_1'](x)|_{x(s) \in \Gamma_n^+} - v[f_1'](x)|_{x(s) \in \Gamma_n^-} =$$

$$= \int_{a_n}^s \left(\frac{d}{d\sigma} f_1(\sigma) \right) d\sigma = f_1(s), \quad n = 1, \ldots, N,$$

$$\left. \frac{\partial u(x)}{\partial \mathbf{n}_x} \right|_{x(s) \in \Gamma^+} - \left. \frac{\partial u(x)}{\partial \mathbf{n}_x} \right|_{x(s) \in \Gamma^-} = \left. \frac{\partial}{\partial \mathbf{n}_x} w[f_2](x) \right|_{x(s) \in \Gamma^+} -$$

$$- \left. \frac{\partial}{\partial \mathbf{n}_x} w[f_2](x) \right|_{x(s) \in \Gamma^-} = f_2(s),$$

where conditions (9b) for $f_1(s)$ have been employed. Thus, the function (10) is a solution of the problem $\mathbf{U}_1$. Note that (10) is an explicit solution of the problem $\mathbf{U}_1$ for curves $\Gamma_1, \ldots, \Gamma_N$ of an arbitrary shape. It can be verified by direct calculations that the condition (1) for $|\nabla u|$ is fulfied for any $\varepsilon \in (0, 1)$, i.e., for any small positive ε. In other words, $\nabla u(x)$ does not have power singularity at the ends of Γ. It will be shown in next section that ∇u has logarithmic singularity or, in certain cases, does not have singularity at all. Explicit formulas for singularities of ∇u at the ends of Γ will be presented and discussed in the next section.

THEOREM 2   *If conditions (9) hold, then the solution of the problem $\mathbf{U}_1$ exists and is given by the explicit formula (10).*

Let us consider the problems U_2, U_3. Suppose that functions $f_1(s)$, $f_2(s)$ from (3) meet conditions (9) and satisfy the necessary condition (5) for the solvability of these problems. Since U_2, U_3 differ from U_1 by more hard conditions at infinity (4), the solution of U_1 satisfies U_2 (or U_3), if corresponding conditions at infinity hold. The function (10) belongs to M_2 and so satisfies the problem U_2, if

$$\int_\Gamma f_2(s)\,ds = 0,$$

but this condition holds, because it coincides with the necessary condition (5), which is assumed to be valid. If, in addition, the constant c in (10) is equal to zero, then the function (10) belongs to M_3 and so satisfies U_3. We arrive at the assertion.

THEOREM 3 *If the conditions (9) and (5) hold, then the solution of the problem U_2 exists and is given by the explicit formula (10). If, in addition, $c = 0$ in (10), then this solution satisfies U_3.*

As stated in the Theorem 1, the solution of the problems U_1, U_2 is defined up to an arbitrary additive constant, while the solution of the problem U_3 is unique.

4. SINGULARITIES OF A GRADIENT OF A SOLUTION AT THE ENDS OF Γ

In this section by $u_j(x)$ we denote the solution of the problem $U_j (j = 1, 2, 3)$ ensured by the Theorems 2, 3. According to (1), ∇u_j may be unbounded at the ends of Γ. The explicit expressions for singularities of ∇u_j can be obtained from the formulas for singularities of derivatives of single layer and angular potentials near edges [4, 5]. Let $x(d)$ be one of the end-points of Γ. In the neighbourhood of $x(d)$ we introduce the system of polar coordinates $x_1 = |x - x(d)| \cos\varphi$, $x_2 = |x - x(d)| \sin\varphi$. We will assume that $\varphi \in (\alpha(d), \alpha(d) + 2\pi)$ if $d = a_n$ and $\varphi \in (\alpha(d) - \pi, \alpha(d) + \pi)$ if $d = b_n$ $(n = 1, \ldots, N)$. Recall that $\alpha(s)$ is the angle between the tangent vector τ_x to Γ at the point $x(s)$ and the direction of the Ox_1 axis. Hence, $\alpha(d) = \alpha(a_n + 0)$ if $d = a_n$ and $\alpha(d) = \alpha(b_n - 0)$ if $d = b_n$. Consequently the angle φ varies continuously in the neighbourhood of the point $x(d)$, cut along the contour Γ.

Recall that X is a set of end-points of Γ. Computing singularities of ∇u_j in the same way as in [4, 5] we arrive at the following assertion.

THEOREM 4   *Let $x \to x(d) \in X$. Then in the neigbourhood of the point $x(d)$ the derivatives of the solution of the problem $U_j (j = 1, 2, 3)$ have the following behaviour*

$$\frac{\partial}{\partial x_1} u_j(x) = -(-1)^m \frac{f_1'(d)}{2\pi} [-\sin \alpha(d) \ln |x - x(d)| + \varphi \cos \alpha(d)] -$$

$$- (-1)^m \frac{f_2(d)}{2\pi} [\cos \alpha(d) \ln |x - x(d)| +$$

$$+ \varphi \sin \alpha(d)] + O(1),$$

$$\frac{\partial}{\partial x_2} u_j(x) = -(-1)^m \frac{f_1'(d)}{2\pi} [\cos \alpha(d) \ln |x - x(d)| + \varphi \sin \alpha(d)] +$$

$$+ (-1)^m \frac{f_2(d)}{2\pi} [-\sin \alpha(d) \ln |x - x(d)| +$$

$$+ \varphi \cos \alpha(d)] + O(1),$$

where $m = 0$ if $d = a_n$ and $m = 1$ if $d = b_n$ $(n = 1, \ldots, N)$.

Remark   By $O(1)$ we denote functions which are continuous at the point $x(d)$. Furthermore, the functions denoted by $O(1)$ are continuous in the neighbourhood of the point $x(d)$, cut along the contour Γ.

According to the Theorem 4, ∇u_j has logarithmic singularities at the ends of cuts Γ in general. However, if $f_1'(d) = f_2(d) = 0$ at the end $x(d) \in X$, then there is no any singularity of ∇u_j at the end $x(d)$. Moreover, ∇u_j is continuous at this end. If $f_1'(d) \neq 0$ or $f_2(d) \neq 0$, then ∇u_j has a logarithmic singularity at $x(d) \in X$.

Let us compare our results with singularities of a solution gradient in the Dirichlet and Neumann problems at the exterior of cuts in a plane. In these problems either Dirichlet or Neumann boundary condition has been specified on the cuts instead of (3). It was shown in [4, 5] that the solution gradient in the Dirichlet and Neumann problems in general tends at infinity as $O(|x - x(d)|^{-1/2})$ when $x \to x(d) \in X$. According to Theorem 4, the edge singularities of ∇u_j in the jump problem are generally logarithmic. Thus, the jump problem and Dirichlet/Neumann problem have as a rule different orders of singularities at the ends of cuts, so that the singularities in the

jump problem are weaker. We can conclude that the behaviour of the solution in the jump problem is essentially different from behaviour of the solution in the Dirichlet/Neumann problem. The discussed properties of singularities may be effectively used to select adequate model describing an appropriate physical model in cracked media, since cuts model cracks in solids.

In conclusion we stress that in the present paper we obtained an explicit solution of the jump problem. The explicit solutions were not obtained either for Dirichlet or for Neumann problems outside cuts in a plane if cuts have an arbitrary shape.

5. APPLICATIONS

Many models in different fields of physics are based on Laplace equation $\Delta u = 0$. For example, this equation is used to describe electrostatics, ideal fluid, stationary heat distribution, stationary electric current in semiconductors and so on. In these models u is a potential which is defined up to an arbitrary constant. It can be potential of an electric field in electrostatics or pressure in an ideal fluid. So, the real physical meaning has difference of potentials, and this difference is measured in experiments, for instance, voltage in the theory of electricity. The 1-st boundary condition in our jump problem is just a difference of potentials (jump) on sides of cuts. Wings, screens, cracks, electrodes in semiconductors are modeled by cuts in a 2-D case [22, 23]. The second boundary condition in our jump problem is the difference of $\partial u/\partial \mathbf{n}$ on sides of the cuts. In the dynamics of an ideal fluid this condition means the jump of normal velocities on wings. In the model of an electric current from electrodes in a semiconductor film this condition is the jump of normal density of electric current on electrodes [23] (in case of absence of a magnetic field).

Conditions at infinity discussed in our jump problem also have natural physical sense. Consider a model on stationary electric current from electrodes in a semiconductor [23]. If function u has a logarithmic growth at infinity (Problem U_1), then the model admits electric sources at infinity. Electric current may move from electrodes in the form of cuts to these sources. If function u is bounded at infinity (Problems

U_2, U_3), then the system of electrodes is closed, there are no electric sourses at infinity, and the electric current exists only between electrodes, since it may not move to infinity. The necessary solvability condition (5) reflects the conservation law of total current from all electrodes, because the system is closed. In the problem U_3 the semiconductor is earthed at infinity, *i.e.*, potential u tends to zero there.

Problems, where potential u has logarithmic growth at infinity are used in fluid dynamics also [24–26], though they are not so well-known as classical problems with potential bounded at infinity. For example, the problem on flow of an ideal fluid over a wing has been numerically studied in engineering research [24] in assumption that u is logarithmically singular at infinity.

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