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Solution of Nonlinear Volterra-Hammerstein Integral Equations *Via* Rationalized Haar Functions

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Rationalized Haar functions are developed to approximate the solutions of the nonlinear Volterra-Hammerstein integral equations. Properties of Rationalized Haar functions are first presented, and the operational matrix of integration together with the product operational matrix are utilized to reduce the computation of integral equations to into some algebraic equations. The method is computationally attractive, and applications are demonstrated through illustrative examples.

Keywords: Rationalized Haar; Volterra-Hammerstein; Nonlinear; Integral equations

AMS Classifications: 45D05, 45Lxx

1. INTRODUCTION

Orthogonal functions have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem.

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The approach is based on converting the underlying differential equations into integral equations through integration, approximating various signals involved in the equation by truncated orthogonal series $\phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_{k-1}(t)]^T$ and using the operational matrix of integration P to eliminate the integral operations. The elements $\phi_0(t)$, $\phi_1(t), \ldots, \phi_{k-1}(t)$ are the basis functions, orthogonal on a certain interval, and the matrix P can be uniquely determined based on the particular orthogonal functions. Orthogonal functions have also been proposed to solve linear integral equations. Special attention has been given to applications of Walsh functions [1], block-pulse functions [2], Laguerre series [3], Legendre polynomials [4], Chebyshev polynomials [5] and Fourier series [6].

The orthogonal set of Haar functions is a group of square waves with magnitude of $+2^{(i/2)}$, $-2^{(i/2)}$ and 0, $i = 0, 1, 2, \ldots$ [7]. The use of the Haar functions comes from the rapid convergence feature of Haar series in expansion of function compared with that of Walsh series [8]. Lynch *et al.* [9] and [10] have rationalized the Haar transform by deleting the irrational numbers and introducing the integral powers of two. This modification results in what is called the rationalized Haar (RH) transform. The RH transform preserves all the properties of the original Haar transform and can be efficiently implemented using digital pipeline architecture [10]. The corresponding functions are known as RH functions. The RH functions are composed of only three amplitudes $+1$, -1 and 0.

In Refs. [11–13] the authors offered a numerical method to solve linear differential equations and its application to function evaluation. The method used was based on the stairstep approximation using Haar functions and on mathematical manipulation using quasi-binary numbers. However, there are some difficulties for practical use as in [11–13]. This is because the Haar functions have magnitudes of $+2^{(i/2)}$, $-2^{(i/2)}$, and 0, $i = 0, 1, 2, \ldots$, and an operating system dealing with quasi-binary numbers is required for speedy manipulation. To overcome these difficulties, RH functions were used in [14–16]. In these references the RH functions operational matrix of integration was applied to solve linear ordinary differential equations [14], first and second order linear partial differential equations [15] and variational problems [16].

In the present paper we are concerned with the application of RH functions to the numerical solution of nonlinear Volterra-Hammerstein integral equations of the form

$$y(t) = f(t) + \int_0^t \kappa(t,s)g(s,y(s))ds, \quad 0 \le t < 1 \tag{1}$$

where f, g and κ are given continuous functions, with $g(s,y)$ nonlinear in y. We assume that Eq. (1) has a unique solution y to be determined.

Several numerical methods for approximating the solution of Hammerstein integral equations are known. For Fredholm-Hammerstein integral equations, the classical method of successive approximations was introduced in [17]. A variation of the Nystrom method was presented in [18]. A collocation-type method was developed in [19]. In [20], Brunner applied a collocation-type method to Eq. (1) and integro-differential equations, and discussed its connection with the iterated collocation method. Guoqiang [21] introduced and discussed the asymptotic error expansion of a collocation-type method for Volterra-Hammerstein integral equations. The methods in [19] and [21] transform a given integral equation into a system of nonlinear equations, which has to be solved with some kind of iterative method. In [19], only in favorable cases the definite integrals involved in the solution may be evaluated analytically, while in [21] the integrals involved in the solution have to be evaluated at each time step of the iteration.

In this paper, we apply RH functions to solve the nonlinear Volterra-Hammerstein integral equations given in Eq. (1). The method is first applied to an equivalent integral equation $z = g(t, y(t))$, $t \in [-1, 1]$ where the solution z is approximated by a RH function with unknown coefficients. The operational matrices of integration and product together with Newton-Cotes nodes [22] are then used to evaluate the unknown coefficients and find approximate solutions for $y(t)$. It is known that spectral projection methods provide highly accurate approximations for the solutions of operator equations in functions spaces, provided that these solutions are sufficiently smooth [23]. Our approach is different from the methods initiated in [20] and [21]. The major difference between our analysis and those of [20] and [21] being the fact that for polynomial interpolation, uniform convergence of interpolants can not be guaranteed for every continuous function, regardless of the choice of the interpolation nodes [24]. Moreover, the uniform convergence under suitable conditions using the spectral methods is established in [24] and [25] for nonlinear Fredholm-Hammerstein [24] and nonlinear Volterra-Hammerstein [25] integral equations. The advantages of the proposed method are that:

(1) using RH functions properties the integrals involved in the solution are calculated once and a set of nonlinear algebraic equations are obtained.
(2) using Newton-Cotes nodes, these nonlinear equations are solved and as a result the solution to Eq. (1) is calculated. The Haar transform is much faster than the Fourier transform, and it is even faster than the Walsh transform [26].

Illustrative examples are included to demonstrate the validity and applicability of the technique.

2. PROPERTIES OF RATIONALIZED HAAR FUNCTIONS

2.1. Rationalized Haar Functions

The RH functions $RH(r,t)$, $r=1,2,3,\ldots$ are composed of three values $+1$, -1 and 0 and can be defined on the interval $[0,1)$ as [14]

$$RH(r,t)=\begin{cases} 1, & J_1 \le t < J_{(1/2)} \\ -1, & J_{(1/2)} \le t < J_0 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where

$$J_u=\frac{j-u}{2^i}, \quad u=0,\ \frac{1}{2},\ 1.$$

The value of r is defined by two parameters i and j as

$$r=2^i+j-1, \quad i=0,1,2,3,\ldots \quad j=1,2,3,\ldots,2^i.$$

$RH(0,t)$ is defined for $i=j=0$ and is given by

$$RH(0,t)=1, \quad 0\le t<1. \tag{3}$$

A set of the first eight RH functions is shown in Figures 1–8, where, $r=0,1,2,\ldots,7$. The first two functions are nonzero over the whole

FIGURE 1 $RH(0,t)$ obtained for $i=0$ and $j=0$.

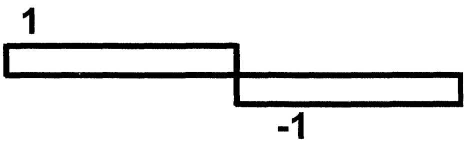

FIGURE 2 $RH(1,t)$ obtained for $i=0$ and $j=1$.

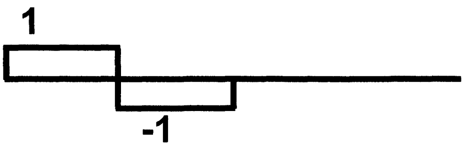

FIGURE 3 $RH(2,t)$ obtained for $i=1$ and $j=1$.

FIGURE 4 $RH(3,t)$ obtained for $i=1$ and $j=2$.

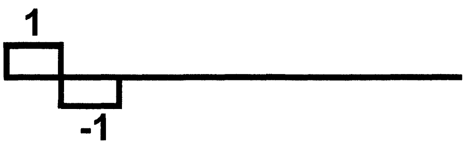

FIGURE 5 $RH(4,t)$ obtained for $i=2$ and $j=1$.

FIGURE 6 $RH(5,t)$ obtained for $i=2$ and $j=2$.

interval on $0 \leq t < 1$, the rest are nonzero only over a portion on $0 \leq t < 1$. Haar functions become increasingly localized as their number increases and can be generated recursively [27].

FIGURE 7 $RH(6,t)$ obtained for $i=2$ and $j=3$.

FIGURE 8 $RH(7,t)$ obtained for $i=2$ and $j=4$.

The orthogonality property is given by

$$\int_0^1 RH(r,t)RH(v,t)dt = \begin{cases} 2^{-i} & \text{for } r = v \\ 0 & \text{for } r \neq v \end{cases}$$

where

$$v = 2^n + m - 1, \quad n = 0,1,2,\ldots \quad m = 1,2,\ldots,2^n.$$

2.2. Function Approximation

A function $f(t)$ defined over the interval $[0,1)$ may be expanded into RH functions as

$$f(t) = \sum_{r=0}^{+\infty} a_r RH(r,t), \tag{4}$$

where

$$a_r = 2^i \int_0^1 f(t)RH(r,t)dt, \quad r = 0,1,2,\ldots$$

with

$$r = 2^i + j - 1, \quad i = 0,1,2,3,\ldots \quad j = 1,2,3,\ldots,2^i \text{ and } r = 0 \text{ for } i = j = 0. \tag{5}$$

The series in Eq. (4) contains an infinite number of terms. If we let $i = 0,1,2,\ldots,\alpha$ then the infinite series in Eq. (4) is truncated up to its first k terms as

$$f(t) = \sum_{r=0}^{k-1} a_r RH(r, t) = A^T \phi(t), \tag{6}$$

where

$$k = 2^{\alpha+1}, \quad \alpha = 0, 1, 2, \ldots$$

The RH function coefficient vector A and RH function vector $\phi(t)$ are defined as

$$A = [a_0, a_1, \ldots, a_{k-1}]^T, \tag{7}$$

$$\phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_{k-1}(t)]^T, \tag{8}$$

where

$$\phi_r(t) = RH(r, t), \quad r = 0, 1, 2, \ldots, k-1.$$

Now let $\kappa(t, s)$ be a function of two independent variables defined for $t \in [0, 1)$ and $s \in [0, 1)$. Then κ can be expanded into RH functions as

$$\kappa(t, s) = \sum_{v=0}^{k-1} \sum_{r=0}^{k-1} h_{vr} \phi_r(t) \phi_v(s),$$

where

$$h_{vr} = 2^{i+n} \int_0^1 \int_0^1 \kappa(t, s) \phi_r(t) \phi_v(s) dt ds,$$

with v defined similarly to r in Eq. (5) and i, $n = 0, 1, 2, \ldots, \alpha$. Hence we have

$$\kappa(t, s) = \phi^T(t) H \phi(s), \tag{9}$$

where

$$H = (h_{vr})^T_{k \times k}. \tag{10}$$

If each waveform is divided into eight intervals, the magnitude of the waveform can be represented as

$$\hat{\Phi}_{8\times 8}=\begin{bmatrix}\phi_0\\ \phi_1\\ \vdots\\ \phi_7\end{bmatrix}=\begin{bmatrix}1&1&1&1&1&1&1&1\\ 1&1&1&1&-1&-1&-1&-1\\ 1&1&-1&-1&0&0&0&0\\ 0&0&0&0&1&1&-1&-1\\ 1&-1&0&0&0&0&0&0\\ 0&0&1&-1&0&0&0&0\\ 0&0&0&0&1&-1&0&0\\ 0&0&0&0&0&0&1&-1\end{bmatrix}. \tag{11}$$

In Eq. (11) the row denotes the order of the Haar function. The matrix $\hat{\Phi}_{k\times k}$ can be expressed as

$$\hat{\Phi}_{k\times k}=[\phi(1/2k),\phi(3/2k),\ldots,\phi((2k-1)/2k)]. \tag{12}$$

Using Eq. (6) we get

$$[f(1/2k), f(3/2k),\ldots, f((2k-1)/2k)]=A^T\hat{\Phi}_{k\times k}. \tag{13}$$

From Eqs. (10) and (13) we have

$$H=\left(\hat{\Phi}_{k\times k}^{-1}\right)^T\hat{H}\hat{\Phi}_{k\times k}^{-1}, \tag{14}$$

where

$$\hat{H}=(\hat{h}_{lp})_{k\times k},\quad \hat{h}_{lp}=\kappa((2l-1)/2k,(2p-1)/2k),\quad p,l=1,2,\ldots,k.$$

2.3. Operational Matrix of Integration

The integration of the $\phi(t)$ defined in Eq. (8) is given by

$$\int_0^t \phi(t')dt'=P\phi(t), \tag{15}$$

where $P=P_{k\times k}$ is the $k\times k$ operational matrix for integration and is given in [14] as

$$P_{k\times k}=\frac{1}{2k}\begin{bmatrix}2kP_{(k/2)\times(k/2)} & -\hat{\Phi}_{(k/2)\times(k/2)}\\ \hat{\Phi}_{(k/2)\times(k/2)}^{-1} & 0\end{bmatrix},$$

where $\hat{\Phi}_1 = [1]$, $P_1 = [1/2]$, $\hat{\Phi}_{k\times k}$ can be obtained similarly to $\hat{\Phi}_{8\times 8}$ in Eq. (11), and

$$\hat{\Phi}^{-1}_{k\times k} = \left(\frac{1}{k}\right)\hat{\Phi}^{T}_{k\times k}\text{diag.}$$
$$\left(1, 1, 2, 2, \underbrace{2^2, \ldots, 2^2}_{2^2}, \underbrace{2^3, \ldots, 2^3}_{2^3}, \ldots, \underbrace{\frac{k}{2}, \ldots, \frac{k}{2}}_{k/2}\right).$$

2.4. The Product Operational Matrix

Let the product of $\phi(t)\phi^T(t)$ be called the RH product matrix $\Psi_{k\times k}(t)$. That is,

$$\phi(t)\phi^T(t) = \Psi_{k\times k}(t).$$

To illustrate the calculation procedures we choose $k = 8$. Using Eqs. (2) and (3) we get

$$\phi_0(t)\phi_q(t) = \phi_0(t), \quad q = 0, 1, \ldots, 7$$

and for $p < q$, we have

$$\phi_p(t)\phi_q(t)$$
$$= \begin{cases} \phi_q(t), & \text{if } \phi_q(t) \text{ occurs during the first positive half wave of } \phi_p(t) \\ -\phi_q(t), & \text{if } \phi_q(t) \text{ occurs during the second negative half wave of } \phi_p(t) \\ 0, & \text{otherwise.} \end{cases}$$

Also, the square of any RH functions is a block pulse with magnitude of 1 during both the positive and negative half waves of RH functions. Thus we get

$$\Psi_{8\times 8}(t) =$$
$$\begin{bmatrix} \phi_0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 \\ \phi_1 & \phi_0 & \phi_2 & -\phi_3 & \phi_4 & \phi_5 & -\phi_6 & -\phi_7 \\ \phi_2 & \phi_2 & (\phi_0+\phi_1)/2 & 0 & \phi_4 & -\phi_5 & 0 & 0 \\ \phi_3 & -\phi_3 & 0 & (\phi_0-\phi_1)/2 & 0 & 0 & \phi_6 & -\phi_7 \\ \phi_4 & \phi_4 & \phi_4 & 0 & (\phi_0+\phi_1+2\phi_2)/4 & 0 & 0 & 0 \\ \phi_5 & \phi_5 & -\phi_5 & 0 & 0 & (\phi_0+\phi_1-2\phi_2)/4 & 0 & 0 \\ \phi_6 & -\phi_6 & 0 & \phi_6 & 0 & 0 & (\phi_0-\phi_1+2\phi_3)/4 & 0 \\ \phi_7 & -\phi_7 & 0 & -\phi_7 & 0 & 0 & 0 & (\phi_0-\phi_1-2\phi_3)/4 \end{bmatrix}$$

In general we have

$$\Psi_{k\times k}(t)=\begin{pmatrix}\Psi_{(k/2)\times(k/2)}(t) & H^*_{(k/2)\times(k/2)}(t)\\ H^{*T}_{(k/2)\times(k/2)}(t) & D^*_{(k/2)\times(k/2)}(t)\end{pmatrix}, \tag{16}$$

where

$$\begin{aligned}\Psi_{1\times 1}(t)&=\phi_0,\\ H^*_{(k/2)\times(k/2)}(t)&=\hat{\Phi}_{(k/2)\times(k/2)}\ \mathrm{diag}[\phi_{(k/2)}(t),\phi_{(k/2)+1}(t),\ldots,\phi_{k-1}(t)],\\ D^*_{(k/2)\times(k/2)}(t)&=\mathrm{diag}[\hat{\Phi}^{-1}_{(k/2)\times(k/2)}[\phi_0(t),\phi_1(t),\ldots,\phi_{(k/2)-1}(t)]^T]^T.\end{aligned}$$

Furthermore, by multiplying the matrix $\Psi_{k\times k}(t)$ in Eq. (16) by the vector A in Eq. (7) we obtain

$$\Psi_{k\times k}(t)A=\tilde{A}_{k\times k}\phi(t), \tag{17}$$

where $\tilde{A}_{k\times k}$ is a $k\times k$ matrix given by

$$\tilde{A}_{k\times k}=\begin{pmatrix}\tilde{A}_{(k/2)\times(k/2)} & \tilde{H}_{(k/2)\times(k/2)}\\ \hat{H}_{(k/2)\times(k/2)} & \tilde{D}_{(k/2)\times(k/2)}\end{pmatrix},$$

with

$$\begin{aligned}\tilde{A}_{1\times 1}&=a_0,\\ \tilde{H}_{(k/2)\times(k/2)}&=\hat{\Phi}_{(k/2)\times(k/2)}\ \mathrm{diag}[a_{(k/2)},a_{(k/2)+1},\ldots,a_{k-1}],\\ \hat{H}_{(k/2)\times(k/2)}&=\mathrm{diag}[a_{(k/2)},a_{(k/2)+1},\ldots,a_{k-1}]\hat{\Phi}^{-1}_{(k/2)\times(k/2)},\end{aligned}$$

and

$$\tilde{D}_{(k/2)\times(k/2)}=\mathrm{diag}[[a_0,a_1,\ldots,a_{(k/2)-1}]\hat{\Phi}_{(k/2)\times(k/2)}].$$

For $k=4$ we have

$$\tilde{A}_{4\times 4}=\begin{bmatrix}a_0 & a_1 & a_2 & a_3\\ a_1 & a_0 & a_2 & -a_3\\ a_2/2 & a_2/2 & a_0+a_1 & 0\\ a_3/2 & -a_3/2 & 0 & a_0-a_1\end{bmatrix}.$$

3. NONLINEAR VOLTERRA-HAMMERSTEIN INTEGRAL EQUATIONS

Consider Volterra-Hammerstein integral equations given in Eq. (1). To solve for $y(t)$, we first approximate the solution not to the Eq. (1), but rather to an equivalent equation

$$z(t) = g(t, y(t)), \quad 0 \leq t < 1. \tag{18}$$

From Eq. (1) we get

$$z(t) = g(t,f(t) + \int_0^t \kappa(t,s)z(s)ds). \tag{19}$$

Suppose $z(t)$ can be expressed approximately as

$$z(t) = A^T\phi(t), \tag{20}$$

where A and $\phi(t)$ are given in Eqs. (7) and (8) respectively. Using Eqs. (9), (17) and (20) we have

$$\int_0^t \kappa(t,s)g(s,y(s))ds = \int_0^t \phi^T(t)H\tilde{A}\phi(s)ds.$$

From Eqs. (15) and (19) we get

$$z(t) = g(t,f(t) + \phi^T(t)H\tilde{A}P\phi(t)) \tag{21}$$

In order to construct the approximations for $z(t)$ we collocate Eq. (21) in k points. For a suitable collocation points we choose Newton-Cotes nodes as

$$t_p = \frac{2p-1}{2k}, \quad p = 1, 2, 3, \ldots, k. \tag{22}$$

By using Eqs. (12) and (22) we have

$$\phi(t_p) = \hat{\Phi}_{k\times k}e_p, \quad p = 1, 2, \ldots, k \tag{23}$$

where

$$e_p = \left[\underbrace{0, 0, \ldots, 0}_{p-1}, 1, \underbrace{0, \ldots, 0}_{k-p}\right]^T.$$

Equation (21) can be expressed as

$$z(t_p) = g\big(t_p, f(t_p) + e_p^T \hat{\Phi}_{k\times k}^T H\tilde{A}P\hat{\Phi}_{k\times k}e_p\big), \quad p = 1, 2, 3, \ldots, k. \quad (24)$$

Equation (24) can be solved for the unknowns a_r, $r = 0, 1, 2, \ldots, k-1$. The required approximations to the solution $y(t)$ in Eq. (1) are obtained and $y(t)$ is given by

$$y(t) = f(t) + \int_0^t \kappa(t,s)z(s)ds, \quad 0 \le t < 1.$$

Using Eqs. (9), (15) and (17) we get

$$y(t) = f(t) + \phi^T(t)H\tilde{A}P\phi(t) \quad (25)$$

where the matrices H and $\tilde{A}$ can be calculated from Eqs. (14) and (24).

4. ILLUSTRATIVE EXAMPLES

4.1. Example 1

Consider the nonlinear Volterra-Hammerstein integral equation

$$y(t) = f(t) + \int_0^t \kappa(t,s)y^2(s)ds, \quad 0 \le t < 1 \quad (26)$$

where $\kappa(t,s) = ts + 1$, and

$$f(t) = -\frac{1}{4}t^5 - \frac{2}{3}t^4 - \frac{5}{6}t^3 - t^2 + 1.$$

By using the method in Section 3, Eq. (26) is solved. The computational results for $k = 8$ and $k = 16$ together with the exact solution $y(t) = t + 1$ are given in Table I.

4.2. Example 2

Consider

$$y(t) = 1 + \sin^2(t) - \int_0^t 3\sin(t-s)y^2(s)ds, \quad 0 \le t < 1 \quad (27)$$

TABLE I Approximate and exact solutions for Example 1

t	*Approximate* $k=8$	*Approximate* $k=16$	*Exact*
0.0	1.034	1.005	1.0
0.1	1.089	1.112	1.1
0.2	1.191	1.204	1.2
0.3	1.325	1.301	1.3
0.4	1.411	1.400	1.4
0.5	1.521	1.502	1.5
0.6	1.594	1.599	1.6
0.7	1.710	1.700	1.7
0.8	1.849	1.801	1.8
0.9	1.941	1.912	1.9
1	1.987	1.999	2.0

TABLE II Approximates and exact solutions for Example 2

t	*Approximate* $k=16$	*Exact*
0.0	1	1
0.1	.9952	.9950
0.2	0.9800	0.9801
0.3	0.9554	0.9553
0.4	0.9210	0.9211
0.5	0.8775	0.8776
0.6	0.8255	0.8253
0.7	0.7648	0.7648
0.8	0.6969	0.6967
0.9	0.6217	0.6216
1	0.5405	0.5403

We solve Eq. (27) using the method in Section 3. The computational results for $k=16$ together with the exact solution $y(t)=\cos t$ are given in Table II.

5. CONCLUSION

The double rationalized Haar functions and the associated matrices of integration and product are applied to solve nonlinear Volterra-Hammerstein integral equations. The matrices $\hat{\Phi}_{k\times k}$ and $\hat{\Phi}_{k\times k}^{-1}$ introduced in Eqs. (12) and (14) contain many zeros, and these zeros make the Haar transform faster than other square functions such as Walsh and block-pulse functions, hence making rationalized Haar functions computationally very attractive. Examples with satisfactory results are used to demonstrate the application of this method.

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