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Guided Waves in a Fluid-Loaded Transversely Isotropic Plate

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Dispersion relations are obtained for the propagation of symmetric and antisymmetric modes in a free transversely isotropic plate. Dispersion curves are plotted for the first four symmetric modes for a magnesium plate immersed in water. The first mode is highly damped and switches over to the second mode when the normalized frequency exceeds 12.

Key words: Wave-guide; Fluid-loaded; Plate; Transversely isotropic; Dispersion relations; Attenuation spectrum

AMS Classification: *73D15, 73D25*

1 INTRODUCTION

Two types of independent guided waves can exist in a free elastic plate.

(1) S H or shear horizontal wave: whose polarization is parallel to the surface of the elastic plate.
(2) Lamb wave: Here the displacement possesses a longitudinal as well as a transverse component.

If the elastic plate is immersed in an ideal fluid, the SH wave is not affected since its displacement, being parallel to the interface, is not coupled with the fluid. However the Lamb wave has a normal component and, if its phase velocity exceeds that of the sound in the surrounding fluid, energy is radiated into the fluid. This fact justifies the term "leaky Lamb waves," which is used to describe them.

Lamb modes in an infinite elastic isotropic plate were first treated by Lord Rayleigh [1] and Lamb [2]. The Rayleigh–Lamb frequency equation giving the mode spectrum was solved only for special cases until Mindlin introduced the method of bounds [3, 4].

Wave propagation in an isotropic plate immersed in an ideal fluid was first investigated by Osborne and Hart [5]. Rokhlin *et al.* [6] have studied the complete wave spectrum in a fluid-loaded plate. They found that, when the ratio of the densities of the fluid and the elastic material, ρ_0/ρ_1, is small, the mode spectrum of the loaded plate is only slightly different from that of a free plate. However an increase in this ratio leads to an interaction between various modes until in the limiting case $\rho_0/\rho_1 \to \infty$, the spectrum transforms into that of a plate

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clamped on its surfaces with slip boundary conditions. Recent work of Freedman [7] extended the parameter range over which the results of [6] are valid.

In this paper we shall study the propagation of guided waves in a transversely isotropic (TI) plate loaded by an inviscid fluid. Nagy [8] and Ahmad [9] have recently studied this problem for cylindrical geometry. However, to the best of our knowledge, the problem of "leaky waves" in a TI plate is being treated here for the first time. In Section 2, we present the mathematical formulation of the problem and derive the generalized Rayleigh–Lamb equation. The dispersion relation governing a fluid-loaded TI plate is also obtained. In Section 3, we present the solutions of these equations, for a magnesium plate immersed in water, in the form of dispersion curves. We also find the attenuation spectrum for the symmetric modes.

2 MATHEMATICAL FORMULATION. FREE PLATE

We use the representation of Buchwald [10] in which potential functions ϕ, ψ and χ are defined in the following manner:

$$u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \chi}{\partial x_2}, \tag{2.1}$$

$$u_2 = \frac{\partial \phi}{\partial x_2} - \frac{\partial \chi}{\partial x_1}, \tag{2.2}$$

$$u_3 = \frac{\partial \psi}{\partial x_3}, \tag{2.3}$$

where u denotes the displacement vector and satisfies the equation of motion

$$c_{ijkl} \frac{\partial^2 u_i}{\partial x_j \partial x_k} = \rho_1 \frac{\partial^2 u_i}{\partial t^2}. \tag{2.4}$$

where ρ_1 denotes the density of the elastic material. When $u_i, i = 1, 2, 3, \ldots$ are substituted in (2.4), it is found that, for a transversely isotropic material, the potential functions must satisfy the following equations

$$c_{11}\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2}\right) + c_{44}\left(\frac{\partial^2 \phi}{\partial x_3^2}\right) + (c_{13} + c_{44})\frac{\partial^2 \psi}{\partial x_3^2} = \rho_1 \frac{\partial^2 \phi}{\partial t^2}, \tag{2.5}$$

$$(c_{13} + c_{44})\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2}\right) + c_{44}\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2}\right) + c_{33}\frac{\partial^2 \psi}{\partial x_3^2} = \rho_1 \frac{\partial^2 \psi}{\partial t^2}, \tag{2.6}$$

$$\frac{1}{2}(c_{11} - c_{12})\left(\frac{\partial^2 \chi}{\partial x_1^2} + \frac{\partial^2 \chi}{\partial x_2^2}\right) + c_{44}\left(\frac{\partial^2 \chi}{\partial x_3^2}\right) = \rho_1 \frac{\partial^2 \chi}{\partial t^2}. \tag{2.7}$$

The c_{ij} are the components of the stiffness tensor, with the x_3-axis chosen along the axis of symmetry of the material. We assume that the axis of symmetry of the material, *i.e.* the x_3-axis is normal to the surface of the plate. The origin is chosen in the middle so that the planes $x_3 = \pm h$ form the boundaries. The x_1-axis is chosen in the direction of propagation of waves. The problem is then independent of the x_2 coordinate. The potential χ represents the

SH wave and Eq. (2.7) shows that it is uncoupled from the P and SV waves which are coupled in Eqs. (2.5) and (2.6). Since we are interested in the propagation of a Lamb wave, we take $\chi = 0$, and the problem, as in the case of an isotropic material, becomes that of a plane strain.

We assume solution of Eqs. (2.5) and (2.6) in the form

$$\phi(x_1, x_3, t) = A\cos(px_3)\exp i(kx_1 - \omega t), \tag{2.8}$$

$$\psi(x_1, x_3, t) = B\cos(px_3)\exp i(kx_1 - \omega t). \tag{2.9}$$

The above choice corresponds to the symmetric modes of the plate. The antisymmetric modes are obtained by replacing cosine with sine in the above equations. Substituting these expressions in Eqs. (2.5) and (2.6) we find that A and B satisfy the system of equations

$$[c_{44}p^2 - (\rho_1\omega^2 - c_{11}k^2)]A + (c_{13} + c_{44})p^2 B = 0, \tag{2.10}$$

$$(c_{13} + c_{44})k^2 A + [c_{33}p^2 - (\rho_1\omega^2 - c_{11}k^2)]B = 0. \tag{2.11}$$

For a nontrivial solution p^2 must be determined from the equation

$$c_{33}c_{44}p^4 - Ep^2 + F = 0, \tag{2.12}$$

where

$$E = [c_{44}(\rho_1\omega^2 - c_{44}k^2)] + c_{33}(\rho_1\omega^2 - c_{11}k^2) + (c_{13} + c_{44})^2 k^2,$$

and

$$F = (\rho_1\omega^2 - c_{11}k^2)(\rho_1\omega^2 - c_{44}k^2).$$

Define

$$\Delta = \sqrt{E^2 - 4c_{33}c_{44}F}.$$

Then the two roots of Eq. (2.12) can be written as

$$p_1^2 = \frac{E - \Delta}{2c_{33}c_{44}}, \qquad p_2^2 = \frac{E + \Delta}{2c_{33}c_{44}}. \tag{2.13}$$

Define the amplitude ratios, B/A (see Eqs. (2.10), (2.11)),

$$q_1 = \frac{(c_{13} + c_{44})k^2}{c_{33}p_1^2 - (\rho_1\omega^2 - c_{11}k^2)},$$

$$q_2 = \frac{(c_{13} + c_{44})k^2}{c_{33}p_2^2 - (\rho_1\omega^2 - c_{11}k^2)}.$$

The general expressions for the potential, defined in Eqs. (2.8) and (2.9) now become

$$\phi(x_1, x_3, t) = [A_1\cos(p_1x_3) + A_2\cos(p_2x_3)]\exp i(kx_1 - \omega t), \tag{2.14}$$

$$\psi(x_1, x_3, t) = [q_1A_1\cos(px_3) + q_2A_2\cos(p_2x_3)]\exp i(kx_1 - \omega t). \tag{2.15}$$

The boundary conditions for a free plate are that $\tau_{13}, \tau_{23}, \tau_{33}$ vanish when $x_3 = \pm h$ for all time and all x_1. Now

$$\tau_{13} = 2c_{44}\varepsilon_{13} = c_{44}\left(\frac{\partial u_t}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \tag{2.16}$$

$$\tau_{23} = 2c_{44}\varepsilon_{23} = c_{44}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right), \tag{2.17}$$

$$\tau_{33} = c_{13}\frac{\partial u_1}{\partial x_1} + c_{23}\frac{\partial u_2}{\partial x_2} + c_{33}\frac{\partial u_3}{\partial x_3}. \tag{2.18}$$

From Eq. (2.17) we see that the condition $\tau_{23} = 0$ is satisfied identically. Also vanishing of τ_{13} and τ_{33} at $x_3 = \pm h$ leads respectively to

$$A_1[p_1(1+q_1)\sin(p_1h)] + A_2[p_2(1+q_2)\sin(p_2h)] = 0, \tag{2.19}$$

and

$$A_1[c_{33}p_1^2q_1 + c_{13}k^2]\cos(p_1h) + A_2[c_{33}p_2^2q_2 + c_{13}k^2]\cos(p_2h) = 0. \tag{2.20}$$

For a nontrivial solution the determinant of the above system of equations must vanish.
This gives us

$$\frac{\tan(p_2h)}{\tan(p_1h)} = \frac{p_1(1+q_1)(c_{33}p_2^2q_2 + c_{13}k^2)}{p_2(1+q_2)(c_{33}p_1^2q_1 + c_{13}k^2)}. \tag{2.21}$$

The corresponding dispersion relation for the antisymmetric modes comes out as

$$\frac{\tan(p_1h)}{\tan(p_2h)} = \frac{p_1(1+q_1)(c_{33}p_2^2q_2 + c_{13}k^2)}{p_2(1+q_2)(c_{33}p_1^2q_1 + c_{13}k^2)}. \tag{2.22}$$

Eqs. (2.21) and (2.22) generalize the Rayleigh–Lamb equations respectively for the symmetric and the antisymmetric guided modes in a free plate [12, chap 6].

Fluid-Loaded Plate

The boundary conditions for a fluid loaded plate are

(1) Shear components of the stress, τ_{31} and τ_{32} must vanish on the interface.
(2) Normal component of stress must be continuous *i.e.*

$$\tau_{33} = -p, \tag{2.23}$$

where p is the pressure in the fluid.

(3) Normal component of the displacement must be continuous. The normal component of the displacement in the fluid is related to the pressure through the equation

$$-\frac{\partial p}{\partial x_3} = \rho_0\frac{\partial^2 u_3}{\partial t^2}, \tag{2.24}$$

where ρ_0 is the density of the fluid. We assume p to be of the form

$$p = p_0 f(x_3) \exp i\,(kx_1 - \omega t). \tag{2.25}$$

Since p satisfies the wave equation

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{2.26}$$

we must take

$$f(x_3) = e^{-ip_3|x_3|}, \tag{2.27}$$

where

$$p_3 = \sqrt{\frac{\omega^2}{c^2} - k^2}. \tag{2.28}$$

The choice (2.27) ensures that the wave is symmetrical in both positive and negative x_3-directions. Assume that the displacement for the symmetric modes inside the plate is obtained from the potential functions ϕ and ψ given by Eqs. (2.14) and (2.15) and the pressure p in the fluid by

$$p = p_0 \exp i\,(kx_1 - p_3|x_3| - \omega t), \tag{2.29}$$

The boundary condition, $\tau_{32} = 0$, is again identically satisfied. The condition $\tau_{31} = 0$ again leads to Eq. (2.19). Continuity of the normal components of the stress and the displacement gives respectively.

$$A_1[c_{33}p_1^2 q_1 + c_{13}k^2]\cos(p_1 h) + A_2[c_{33}p_2^2 q_2 + c_{13}k^2]\cos(p_2 h) + p_0 \exp(-ip_3 h) = 0, \tag{2.30}$$

and

$$p_1 q_1 \sin(p_1 h) A_1 + p_2 q_2 \sin(p_2 h) A_2 - \frac{ip_3}{\omega^2 \rho_0}\exp(ip_3 h) p_0 = 0. \tag{2.31}$$

The dispersion relation for the fluid-loaded plate is obtained by setting the determinant of the system of Eqs. (2.19), (2.30) and (2.31) to zero. We get

$$\begin{vmatrix} p_1(1+q_1)\sin(p_1 h) & p_2(1+q_2)\sin(p_2 h) & 0 \\ (c_{33}p_1^2 q_1 + c_{13}k^2)\cos(p_1 h) & (c_{33}p_2^2 q_2 + c_{13}k^2)\cos(p_2 h) & 1 \\ p_1 q_1 \sin(p_1 h) & p_2 q_2 \sin(p_2 h) & \dfrac{-ip_3}{\omega^2 \rho_0} \end{vmatrix} = 0 \tag{2.32}$$

Define the dimensionless parameters

$$p_1 h = s_1, \quad p_2 h = s_2, \quad p_3 h = s_3, \quad kh = s, \quad \frac{c_{33}}{c_{13}} = a.$$

We can re-write Eq. (2.32) in the form

$$D_1 - \frac{i(h\omega)^2 \rho_0}{c_{13} s_3} D_2 = 0, \tag{2.33}$$

where

$$D_1 = s_1(1 + q_1)\,(a s_2^2 q_2 + s^2) \sin s_1 \cos s_2 - s_2(1 + q_2)(a s_1^2 q_1 + s^2) \sin s_2 \cos s_1, \tag{2.34a}$$

$$D_2 = s_1 s_2 (q_2 - q_1) \sin s_1 \sin s_2. \tag{2.34b}$$

Eq. (2.32) is the dispersion relation for the fluid-loaded plate. This generalizes the result of [6] for the isotropic plate. Note that the equation $D_1 = 0$ gives the modes for a free plate and the second term is a correction introduced by the fluid loading. In the limit $\rho_0 \to \infty$ the modes for a loaded plate will be determined by $D_2 = 0$.

3 RESULTS AND DISCUSSION

The wave numbers of any guided mode in a free or fluid-loaded plate can be obtained, at any frequency, by scanning respectively Eqs. (2.21) and (2.32) for their zeros. Dispersion relations are obtained in terms of the phase velocity $\omega/\mathrm{Re}(k)$ as a function of the frequency. These curves, for the first four modes, are plotted in Figure 1 where the solid curves are for a free magnesium plate and the dashed are for the same plate loaded by water.The material constants for magnesium are taken from [13] and are given in Table I.

The density of water is taken as $1000\,\mathrm{kg/m^3}$ and the velocity of sound in water is taken as $1475\,\mathrm{m/sec.}$ [14]. For an isotropic free plate, it is well known that, for high frequencies, the phase velocity of the lowest mode approaches the Rayleigh velocity. However velocity of the

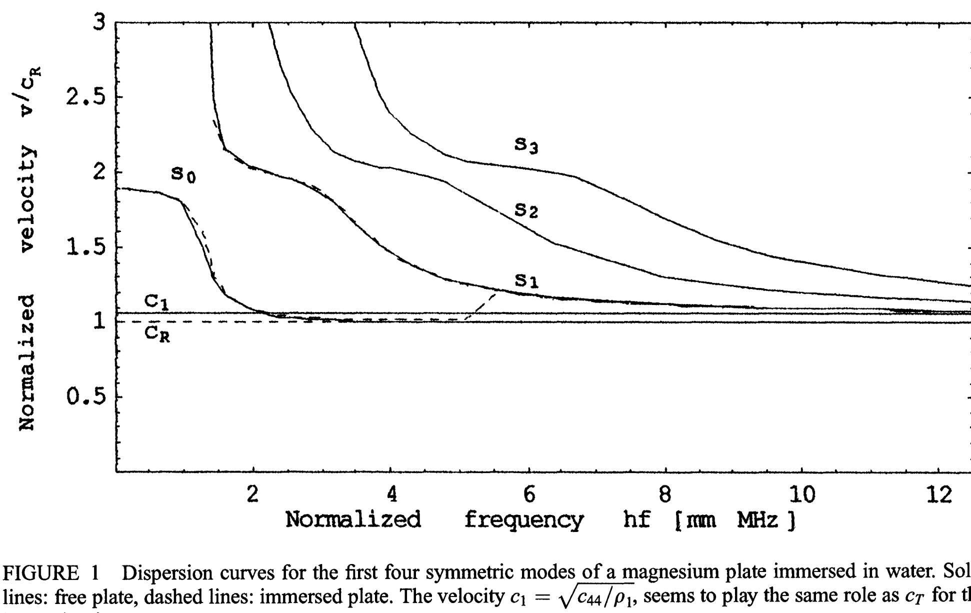

FIGURE 1 Dispersion curves for the first four symmetric modes of a magnesium plate immersed in water. Solid lines: free plate, dashed lines: immersed plate. The velocity $c_1 = \sqrt{c_{44}/\rho_1}$, seems to play the same role as c_T for the isotropic plate.

TABLE I Material Properties for Magnesium [13].

Stiffness $= 10^{10}\, N/m^2$					
c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	*Density* kg/m^3
5.97	2.62	2.17	6.17	1.64	1740

higher modes approaches c_T at high frequencies. Near the cut off frequency the dispersion relation, for the lowest mode, is of the form.

$$\omega = 2\sqrt{\frac{(\kappa^2 - 1)}{\kappa^2}} c_T k. \tag{3.1}$$

In the present case, both p_1 and p_2 appearing in Eq. (2.21) are complex for the lowest mode.In the limit of high frequency the left hand side of Eq. (2.21) approaches unity and the phase velocity is obtained from the following equation

$$\frac{p_1(1+q_1)}{p_2(1+q_2)} = \frac{c_{33}p_1^2 q_1 + c_{13}k^2}{c_{33}p_2^2 q_2 + c_{13}k^2}. \tag{3.2}$$

Eq. (3.2) is the generalization of the Rayleigh equation for a TI material. Its solution, for magnesium, gives $c_R = 2895.5$ m/sec., and the curves in Figure 1 are normalized with respect to this velocity.

When $k \ll 1$, Eq. (2.21) becomes

$$\frac{p_1^2(1+q_1)}{p_2^2(1+q_2)} = \frac{c_{33}p_1^2 q_1 + c_{13}k^2}{c_{33}p_2^2 q_2 + c_{13}k^2}. \tag{3.3}$$

The above equation can be easily solved numerically for any TI material. For magnesium, we find

$$\omega = 1.889 c_R k. \tag{3.4}$$

In Figure 1 are plotted the first four symmetric modes for a magnesium plate. The solid curve represents the mode for a free plate whereas the dashed curve represents the mode when the plate is immersed in water. The phase velocity is normalized with respect to the velocity of the Rayleigh wave on the surface of a half space of magnesium. The guided wave is only slightly perturbed due to fluid loading. However the zeroth symmetric mode "switches over" to the first mode when $af \cong 4.5$ mm MHz. This happens due to coincidence of the first free mode with the loaded zeroth mode of the plate. This type of mode switching has already been observed in plates [6] and rods [8, 9].

Figures 2 and 3 show the attenuation spectrum of the zeroth mode. The attenuation sharply increases as the normalized frequency exceeds unity until it merges into the first mode. It has been remarked by Nagy [9] that a normalized attenuation of 1 dB or higher renders a mode non propagating for all practical purposes. If we apply this criterion, we see that the zeroth mode ceases to be a propagating one when $af > 1.11$.

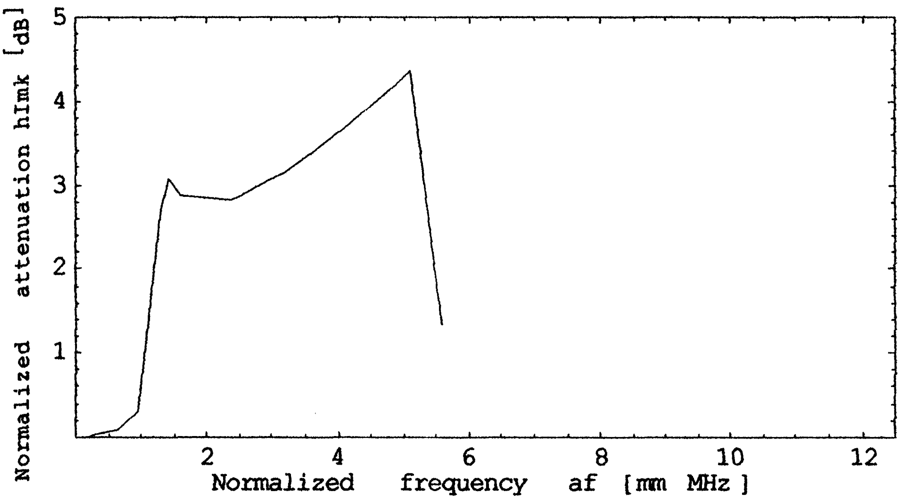

FIGURE 2 Leaky attenuation spectrum in the immersed plate for the first symmetric mode s_0.

Figure 3 shows the attention for the first and the second symmetric modes *i.e.* the s_1 and s_2 in the fluid-loaded plate. The solid curve represents the s_1 mode while the dashed curve represents the s_2 mode. Both modes are highly damped to start with, however the s_1 mode is undamped near $af \cong 12$ while the s_2 mode is undamped near $af \cong 25$. Beyond these frequencies the curves rise towards a maximum and then they slowly flatten.

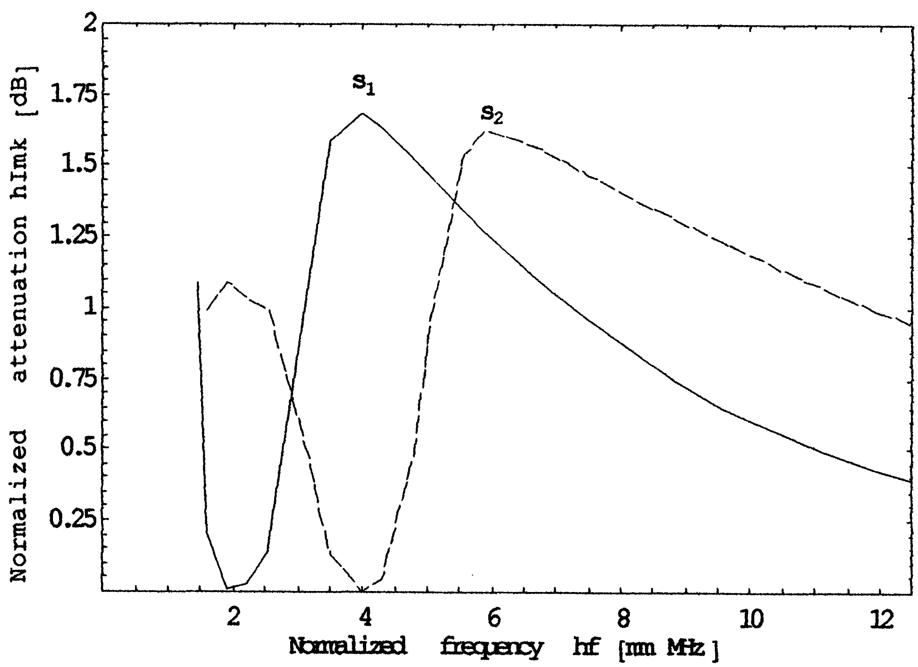

FIGURE 3 Leaky attenuation spectra in the immersed plate for the second mode, s_1, (solid line) and the third mode, s_2, (dashed line).

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