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Survival Maximization for a Laguerre Population

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A population whose evolution is approximately described by a Laguerre diffusion process is considered. Let $Y(t)$ be the number of individuals alive at time t and $T(y, t_0)$ be the first time $Y(t)$ is equal to either 0 or $d(> 0)$, given that $Y(t_0) = y$ is in $(0, d)$. The aim is to minimize the expected value of a cost criterion in which the final cost is equal to 0 if $Y(T) = d$ and to ∞ if $Y(T) = 0$. The case when the final cost is 0 (respectively ∞) if T is greater than or equal to (resp. less than) a fixed constant s is also solved explicitly. In both cases, the risk sensitivity of the optimizer is taken into account.

1 BIOLOGICAL BACKGROUND AND INTRODUCTION

1.1 Biological Background [See Ref. 1, pp. 176–182]

Assume that a population is composed of A-type and a-type individuals and that the size of the population is a constant N. Assume also that there are currently i A-type individuals and that before maturity *mutation* converts an A-type (respectively a-type) individual to an a-type (resp. A-type) with probability p_{Aa} (resp. p_{aA}). Then the expected proportion of mature A-type individuals before reproduction is given by

$$p_i = \frac{i(1 - p_{Aa}) + (N - i)p_{aA}}{N}.$$

Next, let X be the number of A-type individuals in the next generation; the *Wright-Fisher model* postulates that

$$P[X = j] = C_j^N p_i^j (1 - p_i)^{N-j}.$$

That is, it assumes that the probability that any individual will produce an A-type individual is equal to p_i, independently for each of the N individuals in the population. Therefore, the

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population process evolves as a Markov chain. For N large, we can show that this process can be approximated by a diffusion process. In particular, if we set

$$p_{Aa} = \frac{\gamma_1}{N^d} \quad \text{and} \quad p_{aA} = \frac{\gamma_2}{N},$$

where $\gamma_i \in (0, \infty)$ for $i = 1, 2$ and $0 < d < 1$, then the limiting process (as N tends to ∞) is a diffusion process $\{X(t), t \geq 0\}$ with infinitesimal parameters

$$\mu_X(x) = \gamma_2 - \gamma_1 x \quad \text{and} \quad \sigma_X^2 = x$$

which is sometimes called a *Laguerre* diffusion process. This process is also used in branching theory. Notice that the orders of mutation rates are assumed to be different.

The model corresponds to the case when the number of A-type individuals is of the order N^d. The state space is the interval $[0, \infty)$ and is interpreted as follows: if $X(t)$ goes to infinity, then the number of A-types becomes larger than order N^d, whereas if $X(t)$ reaches zero, then this number becomes smaller than order N^d. Therefore, the survival of the A-type individuals in the population is endangered if d is small and $X(t)$ reaches zero.

1.2 Introduction

In this paper, we consider the problem of optimally controlling a diffusion process $\{X(t), t \geq 0\}$ with infinitesimal parameters

$$\mu_X(x) = \alpha + \beta x \quad \text{and} \quad \sigma_X^2 = \sigma^2 x,$$

where $\sigma > 0$ and where α and β can take any real values. Hence, the Laguerre process is the special case when $\sigma = 1$, $\alpha > 0$ and $\beta < 0$.

More precisely, we consider the controlled diffusion process $\{Y(t), t \geq 0\}$ defined by the stochastic differential equation

$$dY(t) = (\alpha + \beta Y(t))dt + b[Y(t)]u(t)dt + \sigma Y^{1/2}(t)dW(t), \tag{1}$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion and the controlled variable $u(t)$ must be chosen so as to minimize the cost criterion

$$C(\theta, y) := -\frac{1}{\theta} \ln E[e^{-\theta J(y)}] \tag{2}$$

in which

$$J(y) := \int_{t_0}^{T(y)} \frac{1}{2} u^2(t) \, dt + K[T, Y(T)], \tag{3}$$

where

$$T(y, t_0) := \inf\{t > t_0 : Y(t) = 0 \text{ or } d \ (> 0) | Y(t_0) = y \in (0, d)\} \tag{4}$$

and $K(\cdot, \cdot)$ is a general termination cost function. Furthermore, in (2) θ is a parameter that takes the risk sensitivity of the optimizer into account [see Ref. 2, p. 4]. If θ is positive

(respectively negative), the optimizer is said to be risk-seeking (resp. risk-averse), whereas we have:

$$\lim_{\theta \to 0} C(\theta, y) = E[J(y)],$$

which is the risk-neutral case.

Our objective is to maximize the survival time of the population (of A-type individuals), so that we would like $Y(T)$ to take on the value d (and *not* 0). To do so, we will consider two special cases for the termination cost function K. In Section 2, we will take

$$K_1[T, Y(T)] = \begin{cases} 0 & \text{if } Y(T) = d, \\ \infty & \text{if } Y(T) = 0. \end{cases} \tag{5}$$

Actually, we could be less extreme and set

$$K_1[T, Y(T)] = \begin{cases} c_1 & \text{if } Y(T) = d, \\ c_2 & \text{if } Y(T) = 0, \end{cases}$$

with $c_2 \gg c_1$. However, here we prefer to treat the limiting case above.

Next, in Section 3, we will let d tend to ∞ and choose

$$K_2[T, Y(T)] = \begin{cases} 0 & \text{if } T \geq s, \\ \infty & \text{if } T < s, \end{cases} \tag{6}$$

where s is a fixed constant. That is, we will force the process $\{Y(t), t \geq 0\}$ to remain above 0 at least until time s. Here again we could replace 0 and ∞ by c_1 and c_2 respectively.

Finally, some concluding remarks will be made in Section 4.

2   FORCING THE POPULATION TO REACH *d* BEFORE 0

First we prove the following lemma.

LEMMA 2.1   *For the diffusion process $\{X(t), t \geq 0\}$ with infinitesimal parameters $\mu_X(x) = \alpha + \beta x$ and $\sigma_X^2(x) = \sigma^2 x$, the origin is*

$$\begin{cases} \text{an exit boundary} & \text{if } \alpha \leq 0, \\ \text{a regular boundary} & \text{if } 0 < \alpha < \dfrac{\sigma^2}{2}, \\ \text{an entrance boundary} & \text{if } \alpha \geq \dfrac{\sigma^2}{2}. \end{cases}$$

Proof   The results above follow from the formulae [see Ref. 3, p. 279]

$$\sigma_1 := \int_0^b \int_y^b \frac{2}{\sigma^2 x} \left(\frac{x}{y}\right)^{2\alpha/\sigma^2} \exp\left[\frac{2\beta(x-y)}{\sigma^2}\right] dx\, dy = \infty \Leftrightarrow \alpha \geq \frac{\sigma^2}{2}$$

and

$$\mu_1 := \int_0^b \int_y^b \frac{2}{\sigma^2 y} \left(\frac{y}{x}\right)^{2\alpha/\sigma^2} \exp\left[\frac{2\beta(y-x)}{\sigma^2}\right] dx\, dy = \infty \Leftrightarrow \alpha \leq 0.$$

Remarks

(1) A boundary $x = b$ is said to be *accessible* if the probability of reaching it in finite time is strictly positive. Both exit and regular boundaries are accessible, whereas entrance (and natural) boundaries are *inaccessible*. Furthermore, a boundary is accessible if and only if the quantity σ_1 in the proof above is finite. Then, the boundary is *regular* if and only if $\mu_1 < \infty$.

(2) Notice that the type of boundary at the origin does not depend on the constant β.

Now, if the constant α in the stochastic differential equation (1) is greater than or equal to $\sigma^2/2$, we can write that the optimal control is $u^* \equiv 0$ because the controlled process $\{Y(t), t \geq 0\}$ is then identical to the uncontrolled diffusion process $\{X(t), t \geq 0\}$ and, according to the previous lemma, it cannot reach the origin. Hence, we can write that $C(\theta, y) \equiv 0$ if $u(t) \equiv 0$. Since $C(\theta, y)$ cannot be negative, we have indeed $u^* \equiv 0$. Therefore, we will assume from now on that $\alpha < \sigma^2/2$.

When we choose the termination cost function defined in (5), the optimal control problem is time-invariant, so that we can assume that $t_0 = 0$. Let $F(y)$ be the minimum expected cost incurred from the initial state $y(=y(0))$. That is,

$$F(y) := \inf_{u(t), 0 \leq t \leq T(y)} C(\theta, y).$$

Assuming that $F(y)$ exists and is twice differentiable, we can show that it satisfies the dynamic programming equation

$$\inf_u \left[\frac{1}{2} u^2 + (\alpha + \beta y + b(y)u)F'(y) - \theta \frac{\sigma^2}{2} y[F'(y)]^2 + \frac{\sigma^2}{2} y F''(y) \right] = 0,$$

where $u = u(0)$. This non-linear ordinary differential equation is valid for $0 < y < d$; the boundary condition is

$$F(y) = K_1(y) \quad \text{if } y = 0 \text{ or } d.$$

The minimizing u is given by

$$u^* = -b(y)F'(y), \tag{7}$$

so that

$$(\alpha + \beta y)F'(y) - \frac{1}{2} b^2(y)[F'(y)]^2 - \theta \frac{\sigma^2}{2} y[F'(y)]^2 + \frac{\sigma^2}{2} y F''(y) = 0. \tag{8}$$

PROPOSITION 2.1 *If $\alpha < \sigma^2/2$ and $b[Y(t)]$ is equal to $Y^{1/2}(t)$, then the optimal control u^* is given by*

$$u^* = \frac{y^{1/2}}{\gamma} \frac{y^{-2\alpha/\sigma^2} e^{-2\beta y/\sigma^2}}{\int_0^y z^{-2\alpha/\sigma^2} e^{-2\beta z/\sigma^2} \, \mathrm{d}z}$$

for $0 < y < d$, where

$$\gamma = \frac{1}{\sigma^2} + \theta$$

is assumed to be positive.

Proof Notice that we have:

$$\gamma\sigma^2 y = b^2(y) + \theta\sigma^2 y \tag{9}$$

when $b(y) = y^{1/2}$. Then the transformation [see Ref. 2, p. 223]

$$G(y) = e^{-\gamma F(y)} \tag{10}$$

linearizes the ordinary differential equation (8) to

$$(\alpha + \beta y)G'(y) + \frac{1}{2}\sigma^2 y G''(y) = 0 \tag{11}$$

for $0 < y < d$. The boundary condition is

$$G(y) = e^{-\gamma K_1(y)} = \begin{cases} 1 & \text{if } y = d, \\ 0 & \text{if } y = 0. \end{cases} \tag{12}$$

Remark The function $G(y)$ can be interpreted as

$$G(y) = E[e^{-\gamma K_1[X(T)]}|X(0) = y] = P[X(T) = d|X(0) = y],$$

where $\{X(t), t \geq 0\}$ is the *uncontrolled* process corresponding to $\{Y(t), t \geq 0\}$ [see Ref. 2, p. 224]. This probability does indeed satisfy the linear ordinary differential equation (11), subject to the boundary condition (12) [see Ref. 4, p. 231].

Solving equation (11) subject to (12) is an easy task. We find that

$$G(y) = \frac{\int_0^y z^{-2\alpha/\sigma^2} e^{-2\beta z/\sigma^2} \, dz}{\int_0^d z^{-2\alpha/\sigma^2} e^{-2\beta z/\sigma^2} \, dz} \tag{13}$$

for $0 \leq y \leq d$. The optimal solution u^* is then deduced from (10) and (7).

Remarks

(1) Suppose that the cost function $J(y)$ defined in (3) is generalized to

$$J(y) = \int_{t_0}^{T(y)} \frac{1}{2} q[Y(t)]u^2(t) \, dt + K[T, Y(T)],$$

where $q(\cdot)$ is positive. Then the transformation used to linearize the ordinary differential equation (8) will work if there exists a constant γ such that

$$\gamma\sigma^2 y = \frac{b^2(y)}{q(y)} + \theta\sigma^2 y.$$

For instance, the problem has the same optimal solution if $b[Y(t)] \equiv 1$ and $q[Y(t)] = 1/Y(t)$. These choices for the functions b and q imply that the smaller the value of $Y(t)$ is, the more expensive it is to control the process.

(2) If the risk parameter θ is equal to $-1/\sigma^2$, then the ordinary differential equation (8) becomes

$$(\alpha + \beta y)F'(y) + \frac{\sigma^2}{2} y F''(y) = 0,$$

which is the same differential equation as $G(y)$ satisfies. However, the boundary condition is

$$F(y) = \begin{cases} 0 & \text{if } y = d, \\ \infty & \text{if } y = 0. \end{cases} \tag{14}$$

Therefore, we can conclude that the function $F(y)$ does not exist when $\theta = -1/\sigma^2$. Actually, $F(y)$ does not exist if $\theta \leq -1/\sigma^2$. This can be deduced from the probabilistic interpretation

$$e^{-\gamma F(y)} = G(y) = P[X(T) = d | X(0) = y] \in (0, 1) \quad \text{for } 0 < y < d$$

given above. Since $F(y)$ is non-negative, the constant γ cannot be negative.

Going back to the interpretation of the risk parameter θ given in Section 1, we can state that the optimizer must not be too risk-averse or pessimistic. On the other hand, we can allow the optimizer to be as risk-seeking or optimistic as we want. If θ increases, then the optimal control u^* decreases. Hence, a very risk-seeking optimizer is willing to take the risk of not using much control. If the controlled process $\{Y(t), t \geq 0\}$ hits the boundary d first, then the total cost is small. In the case of a risk-averse optimizer, he/she is afraid of receiving an infinite penalty for finishing at $y = 0$. Therefore, he/she is ready to use as large a control as needed (u^* tends to ∞ if θ decreases to $-1/\sigma^2$) to avoid this penalty.

(3) If $\beta \neq 0$, the definite integral that appears in the function $G(y)$ and/or u^* can be expressed in terms of the incomplete gamma function.

Particular Cases

(1) If $\alpha = \beta = 0$, we have:

$$G(y) = \frac{y}{d} \quad \text{and} \quad u^* = \frac{1}{\gamma y^{1/2}} \quad \text{for } 0 < y < d.$$

That is, $u^*(t) = [\gamma Y^{1/2}(t)]^{-1}$ and it follows that the optimally controlled process $\{Y^*(t), t \geq 0\}$ obeys the stochastic differential equation

$$dY^*(t) = \gamma^{-1}dt + \sigma\{(Y^*)^{1/2}(t)\}dW(t).$$

According to Lemma 2.1, this process cannot hit the origin if

$$\frac{1}{\gamma} \geq \frac{\sigma^2}{2} \Leftrightarrow \gamma \leq \frac{2}{\sigma^2}.$$

Thus if $(-\sigma^2 <) \theta \leq \sigma^{-2}$, the optimal control assures the optimizer not to receive the infinite penalty incurred for finishing at $y = 0$. When the optimizer is more risk-seeking, he/she is willing to take the risk of receiving this infinite penalty. Using (13) with $\alpha = 1/\gamma$ and $\beta = 0$, we find that this risk is given by

$$1 - G(y) = 1 - \left(\frac{y}{d}\right)^{1-2/(\gamma\sigma^2)}$$

if $\theta > 1/\sigma^2$. Notice that, in this case, we have $E[J(y)] = \infty$; however, $C(\theta, y)$ is finite.

(2) If $\beta = 0$, but $\alpha \neq 0$, we find that

$$u^* = \frac{(1 - 2\alpha/\sigma^2)}{\gamma} \frac{1}{y^{1/2}}.$$

Thererore, this situation is a simple generalization of the previous case.

(3) If $\alpha = 0$, but $\beta \neq 0$, we find that

$$G(y) = \frac{1 - e^{-2\beta y/\sigma^2}}{1 - e^{-2\beta d/\sigma^2}} \Rightarrow u^* = \frac{2\beta}{(1 + \theta\sigma^2)} \frac{y^{1/2}}{(e^{2\beta y/\sigma^2} - 1)}$$

for $0 < y < d$. Notice that both here and in the previous particular case, we have:

$$\lim_{\theta \downarrow -\sigma^{-2}} u^* = \lim_{y \downarrow 0} u^* = \infty.$$

Proposition 2.1 gives the optimal control in a special case, namely when the relationship in (9) holds, so that the optimal control problem can be reduced to a purely probabilistic problem. Next, the value of u^* will be computed when the relationship in (9) does not necessarily hold.

PROPOSITION 2.2 *If $\alpha < \sigma^2/2$, the optimal value of the control variable is given by (7) with*

$$F'(y) = H(y) \left[c_0 - \int H(y) \left(\frac{b^2(y)}{\sigma^2(y)} + \theta \right) dy \right]^{-1},$$

where c_0 is a constant and

$$H(y) := y^{-2\alpha/\sigma^2} e^{-2\beta y/\sigma^2}.$$

Remark The constant c_0 must be determined by using the boundary conditions

$$F(d) = 0 \quad \text{and} \quad F(0) = \infty.$$

An example will be given after the proof of the proposition.

Proof Equation (8) can be rewritten as

$$(\alpha + \beta y)f(y) - \frac{1}{2}(b^2(y) + \theta\sigma^2 y)f^2(y) + \frac{1}{2}\sigma^2 y f'(y) = 0, \tag{15}$$

where $f(y) := F'(y)$. This non-linear first order ordinary differential equation is a Bernoulli equation which is linearized to

$$\frac{1}{2}\sigma^2 y \phi'(y) = (\alpha + \beta y)\phi(y) - \frac{1}{2}(b^2(y) + \theta\sigma^2 y)$$

by the transformation

$$\phi(y) = \frac{1}{f(y)}.$$

The expression for $F'(y)$ then follows at once from the solution of this last ordinary differential equation.

Remarks

(1) If we replace the cost function $J(y)$ by

$$J_1(y) := \int_{t_0}^{T(y)} \left[\frac{1}{2} u^2(t) + h(t) \right] dt + K[Y(T)],$$

where $h(t)$ is not identical to zero, then, as in the previous case, it is possible [see Ref. 5] to give a probabilistic interpretation to the function $\psi(y)$ defined by

$$\psi(y) = \exp\left[-\int \left(\frac{b^2(y)}{\sigma^2 y} + \theta \right) G(y)\, dy \right].$$

(2) If

$$\frac{b^2(y)}{\sigma^2 y} + \theta = 0,$$

then we have:

$$(\alpha + \beta y)f(y) + \frac{\sigma^2}{2} y f'(y) = 0.$$

We have seen above that the optimal control does not exist in this case. Actually, when the relationship in (9) does not hold, it is perhaps preferable to limit ourselves to the risk-neutral case $\theta = 0$.

Particular Case   To conclude this section, we will compute explicitly the optimal control u^* in a special case, namely that for which $\alpha = \beta = \theta = 0$ and $b[Y(t)] = Y(t)$. Notice that the relationship in (9) is not verified since there does not exist any constant γ such that

$$\gamma \sigma^2 y = y^2 + \theta \sigma^2 y$$

for all $0 < y < d$.
  We have:

$$H(y) = 1 \Rightarrow F'(y) = \left[c_0 - \frac{y^2}{2\sigma^2} \right]^{-1}.$$

As mentioned above, the constant c_0 must be chosen so as to satisfy the boundary condition (14). Assuming that $c_0 = 0$, we obtain that

$$F(y) = \frac{2\sigma^2}{y} + k_0.$$

We indeed have $F(0) = \infty$ and, by choosing the constant k_0 equal to $-2\sigma^2/d$, we satisfy the condition $F(d) = 0$. Hence, we deduce that c_0 is indeed equal to zero in this case. It follows that

$$u^* = \frac{2\sigma^2}{y}.$$

Therefore, the optimally controlled process $\{Y^*(t), t \geq 0\}$ satisfies the stochastic differential equation

$$dY^*(t) = 2\sigma^2 dt + \sigma\{(Y^*)^{1/2}(t)\}dW(t).$$

We know from Lemma 2.1 that the origin is an inaccessible boundary for this process. Notice that, as in the previous case, the optimally controlled process is of the same type as the corresponding uncontrolled process; to be more precise, it is a limiting case of the Laguerre diffusion process.

In the next section, to maximize the survival time of the population, instead of forcing the process $\{Y(t), t \geq 0\}$ to reach the level $d\ (>0)$ before 0, we will force $\{Y(t), t \geq 0\}$ to remain above 0 at least until a fixed time s.

3 FORCING $\{Y(t), t \geq 0\}$ TO REMAIN ABOVE 0 UNTIL A FIXED TIME

In this section, we let

$$T(y, t_0) = \inf\{t > t_0: Y(t) = 0 | Y(t_0) = y > 0\}.$$

That is, we assume that the initial time is t_0 and we consider the case when $d \to \infty$ in (4). Moreover, we choose

$$J(y) = \int_{t_0}^{T(y)} \frac{1}{2} u^2(t)\, dt + K[T, Y(T)]$$

and we set the termination cost function $K[T, Y(T)]$ as in (6). Then, the optimal control problem is no longer time-invariant. The function F now depends on both y and the initial time t_0 and satisfies the non-linear partial differential equation

$$F_{t_0} + (\alpha + \beta y)F_y - \frac{1}{2}(b^2(y) + \theta\sigma^2 y)(F_y)^2 + \frac{1}{2}\sigma^2 y F_{yy} = 0. \tag{16}$$

PROPOSITION 3.1 *If $t_0 = 0$, $\sigma^2 = 2$, $\alpha < 1$, $\beta = 0$ and $b[Y(t)] = Y^{1/2}(t)$, then the optimal control u^* is given by*

$$u^* = \frac{y^{(1/2)-\alpha}s^{\alpha-1}e^{-y/s}}{\gamma P(1-\alpha, y/s)}$$

for $y > 0$, where we assume that $\gamma > 0$ and where $P(\cdot, \cdot)$ is the incomplete gamma function defined by

$$P(\lambda, x) = \int_0^x e^{-t}t^{\lambda-1}\, dt \quad \text{for } \lambda > 0.$$

Proof   Because the relationship in (9) does hold, we can state [see Ref. 6] that the optimal control is given by

$$u^* = \frac{y^{1/2}}{\gamma}\frac{R_y}{R},\tag{17}$$

where

$$R(y, t_0, s) = P[\tau(y, t_0) \geq s]$$

and

$$\tau(y, t_0) = \inf\{t > t_0 : X(t) = 0 | X(t_0) = y > 0\}.$$

That is, τ is the same as the random variable $T(y, t_0)$, but for the *uncontrolled* process $\{X(t), t \geq 0\}$ that corresponds to $\{Y(t), t \geq 0\}$.

Next, the author [see Ref. 7] has shown that if $\sigma^2 = 2$ and $\alpha < 1$, then the probability density function of $\tau(y, 0)$ is

$$f_\tau(t) = \frac{y^{1-\alpha}}{\Gamma(1-\alpha)} t^{\alpha-2} e^{-y/t}$$

for $t > 0$. It follows that

$$R(y, 0, s) = \int_s^\infty f_\tau(t)\,dt = \frac{1}{\Gamma(1-\alpha)} P\left(1 - \alpha, \frac{y}{s}\right).$$

The formula for u^* is then deduced at once.

Remarks

(1) We can of course evaluate explicitly the integral that defines the function $P(\cdot, \cdot)$ if $\alpha \in \{\ldots, -2, -1, 0\}$. For instance, if $\alpha = 0$ we obtain that

$$P\left(1, \frac{y}{s}\right) = 1 - e^{-y/s},$$

so that

$$u^* = \frac{y^{1/2}}{s\gamma(e^{y/s} - 1)}.$$

Notice that u^* decreases to 0 if y or γ tends to ∞ and/or s decreases to 0, whereas u^* tends to ∞ if y or γ decreases to 0 and/or s increases to ∞, which is logical.

(2) The conditions $t_0 = 0$ and $\sigma^2 = 2$ in the statement of Proposition 3.1 are not restrictive. The optimal solution could easily be generalized to the case when $\sigma^2 > 0$ and $t_0 > 0$. However, if $\beta \neq 0$ the computation of the function R above is much more difficult. Indeed, as is well known, the moment generating function

$$L(y; a) := E[e^{-a\tau(y,0)} | X(0) = y],$$

where we assume that $\text{Re}\{a\} > 0$, satisfies the ordinary differential equation

$$\frac{\sigma^2}{2} y L''(y; a) + (\alpha + \beta y)L'(y; a) = aL(y; a)\tag{18}$$

for $y > 0$, subject to the boundary condition

$$L(0; a) = 1. \tag{19}$$

When $\beta \neq 0$, setting $z = -2\beta y/\sigma^2$ we find that (18) is transformed into Kummer's equation:

$$zN''(z; a) + \left(\frac{2\alpha}{\sigma^2} - z\right)N'(z; a) = -\frac{a}{\beta}N(z; a),$$

where $N(z; a) = L(y; a)$. Its general solution can be expressed as

$$N(z; a) = c_1 M\left(-\frac{a}{\beta}, \frac{2\alpha}{\sigma^2}, z\right) + c_2 U\left(-\frac{a}{\beta}, \frac{2\alpha}{\sigma^2}, z\right),$$

where c_i is a constant for $i = 1, 2$, and $M(\cdot, \cdot, \cdot)$ and $U(\cdot, \cdot, \cdot)$ are confluent hypergeometric functions [see Ref. 8, p. 504]. Since

$$\lim_{y\to\infty} L(y; a) = 0,$$

we deduce from the formulae 13.1.4, 13.1.5 and 13.1.8 in Ref. [8] that if $\beta < 0$ we must set the constant c_1 equal to zero, so that

$$L(y; a) = c_2 U\left(-\frac{a}{\beta}, \frac{2\alpha}{\sigma^2}, -\frac{2\beta y}{\sigma^2}\right).$$

Finally, the boundary condition (19) implies [see Ref. 8, p. 508, formulae 13.5.10–13.5.12] that

$$L(y; a) = U\left(-\frac{a}{\beta}, \frac{2\alpha}{\sigma^2}, -\frac{2\beta y}{\sigma^2}\right)\frac{\Gamma(1 - \alpha/\beta - 2\alpha/\sigma^2)}{\Gamma(1 - 2\alpha/\sigma^2)}$$

for $y \geq 0$ (if $\beta < 0$).

Now, to obtain the probability density function of the random variable τ, and hence the function R, we must invert the Laplace transform L, which is not an easy task.

(3) If $b[Y(t)] \neq$ constant $Y^{1/2}(t)$, then the relationship in (9) is not satisfied and we cannot write that u^* is given by (17). In this case, we have to solve the non-linear partial differential equation (16), subject to the appropriate boundary condition, namely

$$F(0, t_0) = \begin{cases} \infty & \text{if } t_0 < s, \\ 0 & \text{if } t_0 \geq s. \end{cases}$$

We also have:

$$\lim_{y\to\infty} F(y, t_0) = 0,$$

because $P[T \geq s]$ increases to 1 if y tends to ∞ and $s \in \mathbb{R}^+$, so that $u^* \equiv 0$.

4 CONCLUSION

We have considered the problem of forcing a certain one-dimensional diffusion process to take on the value $d > 0$ before 0, and that of forcing this diffusion process to remain above 0 for at least a fixed time s. This diffusion process is a generalization of the Laguerre process which is used in genetics to model the evolution of a certain population. In both problems, we can state that the aim was to maximize the survival time of the population.

Note that the same diffusion process is used in financial mathematics as well. Indeed, the process $\{X(t), t \geq 0\}$ defined in Section 1, with parameters $\mu_X(x) = \alpha + \beta x$ and $\sigma_X^2(x) = \sigma^2 x$, is known as the Cox–Ingersoll–Ross (CIR) model in this field.

As mentioned in Section 1, in both cases treated in the present paper, we could have given a finite penalty for not reaching the objective, instead of infinite penalties as in (5) and (6). However, the formulae for the optimal control u^* would have been more complicated. Nevertheless, it would be a straightforward generalization to do so.

The problem considered in Section 2 was theoretically solved for any function $b[Y(t)]$ in the stochastic differential equation (1). The integrals that must be evaluated to obtain an explicit expression for u^* can, however, be rather difficult.

In the case of the problem studied in Section 3, we limited ourselves to the instance when $b[Y(t)] = Y^{1/2}(t)$, so that the relationship in (9) was satisfied. It would be interesting to find u^* when $b[Y(t)] \neq$ constant $Y^{1/2}(t)$. As we have seen, this entails solving a non-linear partial differential equation (see (16)), subject to the appropriate conditions. This can surely be accomplished in some special circumstances.

Finally, extensions of the problems treated in the present paper could be considered in two (or more) dimensions. Here again it should be possible to solve explicitly some particular problems, although the optimal solutions are likely to be very complicated.

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