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ZERO-SUM BALANCED BINARY SEQUENCES

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Abstract

For every positive integer $n \equiv 0 \bmod 4$, we construct a zero-sum $\{\pm 1\}$-sequence of length n which is balanced, i.e., whose associated Steinhaus triangle contains as many $+1$'s as -1's. This implies the existence of balanced binary sequences of every length $m \equiv 0$ or $3 \bmod 4$, thereby providing a new solution to a problem posed by Steinhaus in 1963.

1. Introduction

Let $X = x_1 x_2 \dots x_n$ be a binary sequence of length n, with $x_i = \pm 1$ for all i. We define its *derived sequence* ∂X by $\partial X = y_1 y_2 \dots y_{n-1}$, where y_i is the product of x_i and x_{i+1} for all i. This is a binary sequence again, of length $n - 1$. By convention, $\partial X = \emptyset$ if $n \leq 1$, where $\emptyset$ stands for the empty sequence of length 0. Iterating the derivation process, we denote by $\partial^k X$ the kth derived sequence of X, defined recursively as usual by $\partial^0 X = X$ and $\partial^k X = \partial(\partial^{k-1} X)$ for $k \geq 1$.

The Steinhaus *triangle* (or *derived triangle*) of X is the collection $\Delta X = \{X, \partial X, \dots, \partial^{n-1} X\}$

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of iterated derived sequences of X. For example, if $X = + + - +$, then

$$\Delta X = \begin{array}{ccccccc} + & & + & & - & & + \\ & + & & - & & - & \\ & & - & & + & & \\ & & & - & & & \end{array}.$$

Definition 1 *Let X be a finite binary sequence. We say that X*
- *is* zero-sum *if its entries sum to 0;*
- *is* balanced *if its Steinhaus triangle ΔX is zero-sum, i.e., if the entries of ΔX sum to 0.*

For example, the above binary sequence $X = + + - +$ is balanced, as its Steinhaus triangle contains exactly 5 $+$'s and 5 $-$'s. This concept was introduced by Steinhaus in [4] with the following problem: does there exist a balanced binary sequence of length m, for every $m \equiv 0$ or 3 mod 4? (Without this necessary condition, ΔX would contain an odd number of terms.) Steinhaus' problem was first solved positively by Harborth in 1972 [3]. New solutions with special properties, such as symmetry/antisymmetry for instance, recently appeared in [1] and [2].

The present paper is concerned with binary sequences X which are both zero-sum and balanced, or equivalently, such that both X and ∂X are balanced. We show that such sequences exist in all lengths $n = 4k$.

Theorem 2 *For every positive integer $n \equiv 0 \bmod 4$, there exists a binary sequence X of length n which is both zero-sum and balanced.*

This result provides one more solution of Steinhaus' original problem.

Corollary 3 *For every positive integer $m \equiv 0$ or $3 \bmod 4$, there exists a binary sequence X, of length m, which is balanced.*

Proof. If $m \equiv 0 \bmod 4$, we are done by Theorem 2. If $m \equiv 3 \bmod 4$, then by Theorem 2 again, there exists a binary sequence Y of length $n = m + 1$ which is both zero-sum and balanced. Set $X = \partial Y$. Note that the derived triangle ΔY is the concatenation of Y (as its first line) and of ΔX. Now, since Y and ΔY are both zero-sum, it follows that ΔX itself is zero-sum. This means that X is a balanced binary sequence of length m, as needed.

Theorem 2 answers a problem proposed by M. Kervaire and listed as open in [1]. Its proof is given in Section 2. The relevant sequences have been constructed by an algorithmic procedure explained in Section 3 and refined in Section 4. The last section describes one instance of unpredictable behavior in the construction procedure.

2. Explicit Solutions

Given a binary sequence $X = x_1 x_2 \dots x_n$ of length n and an integer $1 \leq i \leq n$, we denote by

$$X[i] = x_1 \dots x_i$$

the initial segment of length i of X, and by

$$X^\infty = x_1 x_2 \dots x_n \, x_1 x_2 \dots x_n \dots$$

the infinite periodic sequence with period X. If $Y = y_1 y_2 \dots$ is another binary sequence, finite or infinite, we denote by

$$X \, Y = x_1 x_2 \dots x_n \, y_1 y_2 \dots$$

the concatenation of X and Y. If $Y = y_1 \dots y_m$ is finite, we say that the sequence

$$X \, Y^\infty = x_1 x_2 \dots x_n \, y_1 \dots y_m \, y_1 \dots y_m \dots$$

is *eventually periodic*, with initial segment X and period Y.

Theorem 4 *Let $S_0 = I_0 \, P_0^\infty$ and $S_4 = I_4 \, P_4^\infty$ be the eventually periodic infinite binary sequences with respective initial segments*

$$I_0 {=} + - - + - + - +$$
$$I_4 {=} - - + + + + - -$$

of length 8, and periods

$$P_0 {=} - - - + - + + + - - + - + - + + + + - - - - + +$$
$$P_4 {=} - + - + + - - - - + + + - - + - + - + + - + + -$$

of length 24. Then, for every integer $m \geq 0$, the initial segments $S_0[8m]$ of length $8m$ of S_0, and $S_4[8m+4]$ of length $8m+4$ of S_4, are both zero-sum and balanced.

This is Theorem 2 again, in a more detailed version. The remainder of this Section is devoted to its proof. As such sequences are hard to dig out with the required properties, we shall explain in the next two sections how they were discovered.

Proof.

- **The case of $S_0[8m]$.** Let $T_{8m} = \Delta S_0[8m]$ denote the derived triangle of $S_0[8m]$. We shall show that T_{8m} is made of 10 bricks, all triangles and diamonds of sidelength 8, assembled in an eventually periodic structure. It will therefore be easy to compute the entry sum of T_{8m} and show, as required, that it equals zero.

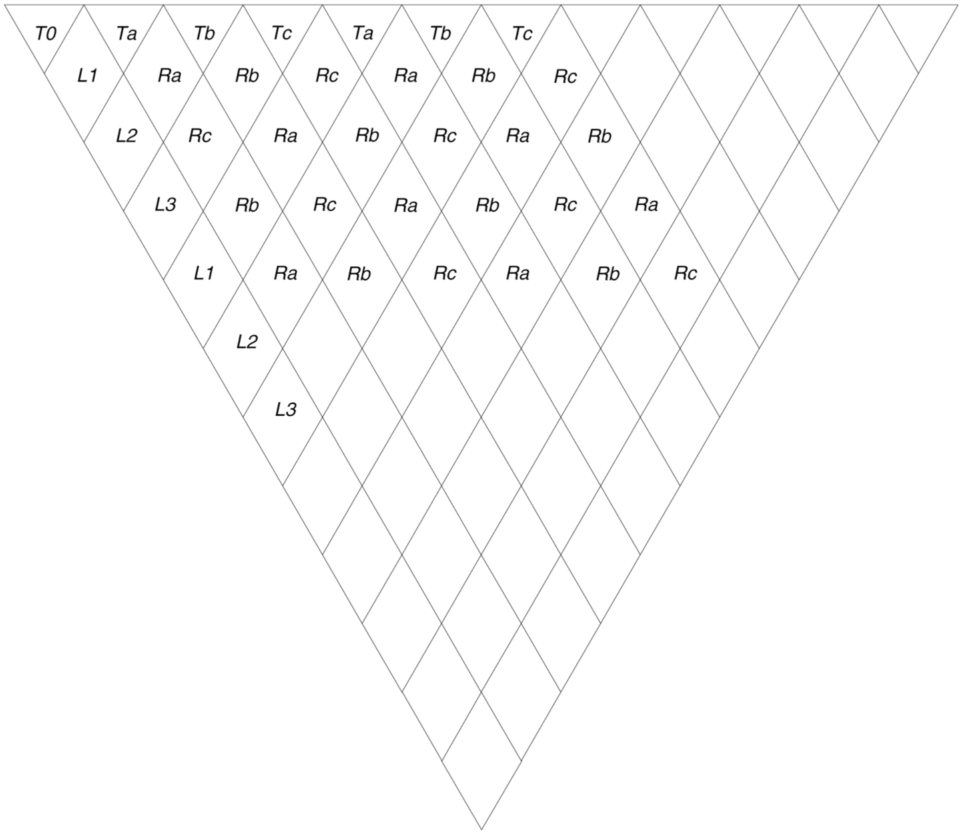

Figure 1: Structure of the Derived Triangle T_{8m} of $S_0[8m]$

Given any collection X of ±1's, we denote by $\sigma(X)$ the sum of its entries.

First, it is easily checked that $\sigma(S_0[8m]) = 0$, using the eventual periodicity of the sequence $S_0 = I_0\,P_0^{\infty}$. Indeed, we have $\sigma(I_0) = 0$, and P_0 is the concatenation of three sequences of length 8 each summing to 0.

We must further show that $\sigma(T_{8m}) = 0$ as well. Assume for the moment that T_{8m} is structured as in Figure 1, with two types of bricks: triangles named $T0, Ta, Tb, Tc$, and diamonds named $L1, L2, L3, Ra, Rb, Rc$. Here are these 10 building bricks.

+ − − + − + − +	− − − + − + + +	− − + − + − + +	+ + − − − − + +
− + − − − − −	+ + − − − + +	+ − − − − − +	+ − + + + − +
− − + + + +	+ − + + − +	− + + + + −	− − + + − −
+ − + + +	− − + − −	− + + + −	+ − + − +
− − + +	+ − − +	− + + −	− − − −
+ − +	− + −	− + −	+ + +
− −	− −	− −	+ +
+	+	+	+
$T0$	Ta	Tb	Tc

−	+	−
+ −	− +	+ −
+ − −	− − −	− − +
+ − + +	+ + + −	+ + − −
+ − − + +	+ + + − +	− + − + +
+ − + − + −	− + + − − +	+ − − − + −
− − − − − − +	− − + − + − +	+ − + + − − +
− + + + + + − +	− + − − − − − +	+ − − + − + − +
− + + + + − −	− − + + + + −	− + − − − − −
− + + + − +	+ − + + + −	− − + + + +
− + + − −	− − + + −	+ − + + +
− + − +	+ − + −	− − + +
− − −	− − −	+ − +
+ +	+ +	− −
+	+	+
*L*1	*L*2	*L*3

−	+	−
− −	+ +	− −
− + +	− + −	+ + −
+ − + −	+ − − −	+ + − +
+ − − − +	− − + + +	− + − − +
− − + + − −	+ + − + + +	− − − + − −
+ + − + − + +	− + − − + + +	− + + − − + +
+ + − − − − + +	− − − + − + + +	− − + − + − + +
+ − + + + − +	+ + − − − + +	+ − − − − − +
− − + + − −	+ − + + − +	− + + + + −
+ − + − +	− − + − −	− + + + −
− − − −	+ − − +	− + + −
+ + +	− + −	− + −
+ +	− −	− −
+	+	+
Ra	*Rb*	*Rc*

As easily checked, these bricks have the following entry sums:

$$\begin{array}{ccc} & \sigma(T0)=\ \ 0; & \\ \sigma(Ta)=-2, & \sigma(Tb)=-2, & \sigma(Tc)=\ \ 4; \\ \sigma(L1)=\ \ 2, & \sigma(L2)=\ \ 0, & \sigma(L3)=-2; \\ \sigma(Ra)=\ \ 2, & \sigma(Rb)=\ \ 4, & \sigma(Rc)=-6. \end{array}$$

With this structure, it is easy to get $\sigma(T_{8m}) = 0$, by induction on m. First note that the triangle T_{8m+8} is obtained by gluing a band of width 8 to the right side of the triangle T_{8m}. We call it the *band difference* from T_{8m} to T_{8m+8}. For the induction step, it suffices to check that these band differences all have entry sum 0.

For $m = 1$, we have $T_8 = \Delta I_0 = T0$, and thus $\sigma(T_8) = 0$. For $m = 2$, the band difference from T_8 to T_{16}, being made of the two bricks Ta and $L1$, has entry sum $\sigma(Ta) + \sigma(L1) = 0$.

For $m = 3$, the band difference from T_{16} to T_{24} is made of the bricks Tb, Ra and $L2$, and thus again has entry sum 0. Finally, for $m = 4$, the band difference from T_{24} to T_{32} is made of the bricks Tc, Rb, Rc and $L3$, and hence also has entry sum 0.

Assume now $m \geq 5$. By the induction hypothesis, we have $\sigma(T_{8k}) = 0$ for all $1 \leq k \leq m-1$. Now observe on Figure 1 that, by periodicity, the band difference from $T_{8(m-1)}$ to T_{8m} is made of the same bricks as the band difference from $T_{8(m-4)}$ to $T_{8(m-3)}$, plus the three supplementary bricks Ra, Rb and Rc. Since $\sigma(Ra) + \sigma(Rb) + \sigma(Rc) = 0$, it follows that this band difference has entry sum 0 and, consequently, $\sigma(T_{8m}) = 0$ as claimed.

It remains to prove that the structure of the triangle T_{8m} is indeed as depicted in Figure 1. Let A, B be any two bricks from the set

$$\{T0, Ta, Tb, Tc, L1, L2, L3, Ra, Rb, Rc\},$$

and assume that they are *adjacent*, in the sense that, somewhere in the triangle T_{8m}, the rightmost entry of some brick labelled A is on the same line as, and left-adjacent to, the leftmost entry of some brick labelled B. For instance, the bricks $T0, Ta$ are adjacent, and so are the bricks $L1, Ra$.

Clearly, by the defining property of derived triangles, two adjacent bricks A, B in T_{8m} determine a *unique diamond* located on the southeast of A and on the southwest of B, that we denote $A * B$. For instance, $T0 * Ta = L1$ and $Ta * Tb = Ra$.

The structure of T_{8m}, as depicted in Figure 1, now simply follows from the easily checked relations:

$$\begin{array}{lll} Ta * Tb = Ra, & Tb * Tc = Rb, & Tc * Ta = Rc \\ T0 * Ta = L1, & L1 * Ra = L2, & L2 * Rc = L3 \\ Ra * Rb = Rc, & Rb * Rc = Ra, & Rc * Ra = Rb \\ & L3 * Rb = L1. & \end{array}$$

The argument proceeds as follows. By definition of $S_0[8m]$ and of its derived triangle, the first line of bricks in T_{8m} is the ultimately periodic sequence

$$T0, Ta, Tb, Tc, Ta, Tb, Tc, \ldots.$$

Now, it follows from the above relations that the second brick line in T_{8m} is the ultimately periodic sequence

$$L1, Ra, Rb, Rc, Ra, Rb, Rc, \ldots.$$

Similarly, the third, fourth and fifth brick lines in T_{8m} are, respectively, the sequences

$$L2, Rc, Ra, Rb, Rc, Ra, Rb, \ldots,$$

$$L3, Rb, Rc, Ra, Rb, Rc, Ra, \ldots,$$

$$L1, Ra, Rb, Rc, Ra, Rb, Rc, \ldots.$$

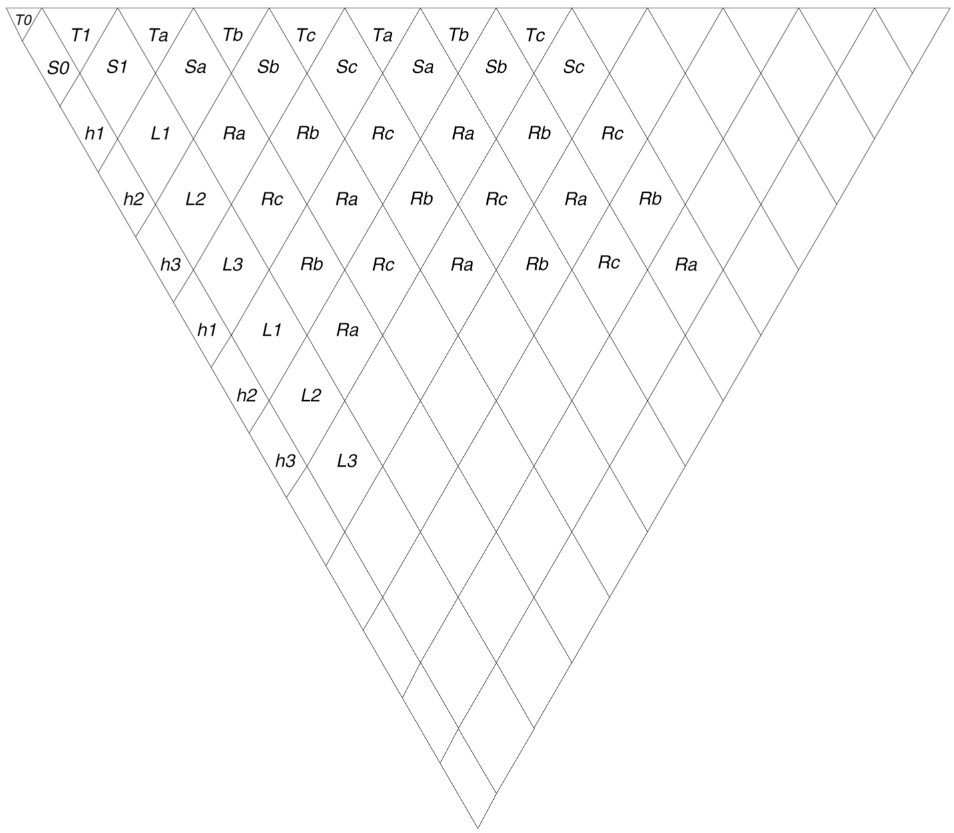

Figure 2: Structure of the Derived Triangle T_{8m+4} of $S_4[8m+4]$

Since the fifth brick line is equal to the second one, periodicity follows. This establishes the claimed structure of T_{8m}, and hence the equality $\sigma(T_{8m}) = 0$.

• **The case of** $S_4[8m+4]$**.** We denote by T_{8m+4} the derived triangle of $S_4[8m+4]$. This case is similar though slightly more complicated, since more bricks are needed to make up T_{8m+4}. Actually 19 bricks are needed: the triangle $T0$ of sidelength 4, the triangles $T1, Ta, Tb, Tc$ and the diamonds $S1, Sa, Sb, Sc, Ra, Rb, Rc, L1, L2, L3$ of sidelength 8, and finally the parallelograms $S0, h1, h2, h3$ of size 4×8. (See Figure 2.) Note that the common symbols between the two cases do not depict the same bricks. We shall only display $T0, T1, Ta, Tb, Tc$; the other bricks can easily be reconstructed by the defining property of derived triangles. We shall, however, give the entry sums of all the bricks.

$$\begin{array}{c} --++ \\ +-+ \\ -- \\ + \end{array}$$

$T0$

$++---+-+$	$+----+++$	$--+-+-++$	$-++--+-+$
$+-++---$	$-+++-++$	$+-----+$	$-+-+---$
$--+-++$	$-++--+$	$-++++-$	$----++$
$+---+$	$-+-+-$	$-+++-$	$+++-+$
$-++-$	$----$	$-++-$	$++--$
$-+-$	$+++$	$-+-$	$+-+$
$--$	$++$	$--$	$--$
$+$	$+$	$+$	$+$
$T1$	Ta	Tb	Tc

The 19 building bricks have the following entry sums. From this data and Figure 2, it is straightforward to check that $\sigma(T_{8m+4}) = 0$, as required.

$\sigma(T0) = 0$	$\sigma(T1) = -4$	$\sigma(Ta) = 4$	$\sigma(Tb) = -2$	$\sigma(Tc) = -2$
$\sigma(S0) = 4$	$\sigma(S1) = -4$	$\sigma(Sa) = -6$	$\sigma(Sb) = -10$	$\sigma(Sc) = -4$
$\sigma(h1) = 0$	$\sigma(L1) = 10$	$\sigma(Ra) = -4$	$\sigma(Rb) = 2$	$\sigma(Rc) = 2$
$\sigma(h2) = -2$	$\sigma(L2) = 18$	$\sigma(h3) = -2$	$\sigma(L3) = -4$	

The fact that T_{8m+4} does have the structure depicted in Figure 2 uses the same type of argument as in the preceding case; namely, that if A, B are two adjacent bricks, then they uniquely determine a third brick denoted $A * B$, lying southeast of A and southwest of B.

3. The Construction Method

The principal idea, as in [1], is to seek *strong solutions*, i.e., solutions s with the property that all initial segments of s of prescribed lengths are also solutions. On the one hand, this makes the problem easier to explore by computer, as strong solutions can be constructed by extending those already obtained in smaller lengths. On the other hand, this stronger requirement might force finitely many solutions only. A balance must be found, hopefully allowing strong solutions in all desired small lengths, yet sufficiently scarce so that large lengths can still be explored and *quasi-periodic* solutions, if any, can emerge.

In [1], a binary sequence X of length n is called *strongly balanced* if all its initial segments of length m, with $m \equiv n \bmod 4$, are also balanced. Now this condition turns out to be too strong here: if X has the property that all its initial segments of length $m \equiv n \bmod 4$ are both zero-sum and balanced, then $n = 0$, 4, or 8.

A weaker constraint consists in requiring only that initial segments of X of length $m \equiv n \bmod 8$ (instead of mod 4) be zero-sum and balanced. But then an opposite difficulty emerges: the number of strong solutions thus defined seems to explode with n, making it very hard to uncover easy-to-describe quasi-periodic solutions.

One way out consists in restricting the set of allowed extensions of length 8 of already constructed strong solutions. This idea does work and has allowed us to obtain Theorem 4. However, it requires some fine-tuning. Indeed, depending on the set of allowed extensions, the resulting construction algorithms exhibit completely different behaviors. In some instances, the process dies out after a few steps. In more favorable cases, after a vigorous initial growth, the number of strong solutions decreases and becomes periodic. Finally, in yet other cases, the number of strong solutions seems to explode. We shall present instances of all these phenomena, ending with a related easy-to-state but very challenging open problem.

To proceed with more details, we need the following notation.

Notation 5 *We denote by ZB_n the set of all zero-sum balanced binary sequences of length n, and by SZB_n the subset of ZB_n defined as*

$$SZB_n = \{X \in ZB_n : X[m] \in ZB_m \text{ for every } m \equiv n \bmod 8\}.$$

Here again, $X[m]$ denotes the initial segment of length m of X. The elements of SZB_n will be called *strongly zero-sum-balanced* sequences. Clearly, if $ZB_n \neq \emptyset$ then $n \equiv 0 \bmod 4$. Our purpose is to establish the converse. We shall in fact show that the subset SZB_n is nonempty whenever $n \equiv 0 \bmod 4$.

There is a simple algorithm to construct the set SZB_{n+8} assuming we already know SZB_n, based on the following.

Remark 6 *For each $X \in SZB_n$, and for each* zero-sum *binary sequence z of length 8, the extension Xz belongs to SZB_{n+8} if and only if Xz is balanced.*

Indeed, Xz is zero-sum as both X and z are, and since X is strongly zero-sum-balanced, it follows that Xz is strongly zero-sum-balanced whenever it is simply balanced.

The starting points are $n = 4$ and $n = 8$, where we have $SZB_n = ZB_n$ by definition. First, the number of zero-sum binary sequences of length 4 is $\binom{4}{2} = 6$ and, similarly, it is $\binom{8}{4} = 70$ in length 8. Among these zero-sum sequences, it is easy to select those which are balanced by constructing their Steinhaus triangles. We find:

$$\begin{aligned} SZB_4 &= \{--++,\ -+-+,\ +-+-,\ ++--\},\\ SZB_8 &= \{+--+-+-+,\ +-+--+-+,\ +-+-+--+,\\ &\quad -+-+-++-,\ -+-++-+-,\ -++-+-+-\}. \end{aligned}$$

Starting from SZB_8 and using Remark 6, it is algorithmically easy to successively build SZB_{16}, SZB_{24}, SZB_{32}, etc. We get the following cardinalities.

n	8	16	24	32	40	48	56
$\lvert SZB_n \rvert$	6	28	116	430	1386	3882	10094

Something similar occurs for $n \equiv 4 \bmod 8$. These results suggest that the number of strongly zero-sum-balanced sequences explodes with n, making them difficult to exploit. As indicated above, our way out is to restrict the allowed extensions of length 8 in the construction algorithm.

4. Restricting Extensions

We need some more notation in order to explain our refinement of the above method.

Notation 7 *Let Z_8 denote the set of zero-sum binary sequences of length 8, ordered lexicographically.*

Again, the set Z_8 has $\binom{8}{4} = 70$ elements. Its first three elements are

$$\begin{aligned} z_1 &= - - - - + + + + \\ z_2 &= - - - + - + + + \\ z_3 &= - - + - - + + +, \end{aligned}$$

and its last three elements are

$$\begin{aligned} z_{68} &= + + - + + - - - \\ z_{69} &= + + + - + - - - \\ z_{70} &= + + + + - - - - \,. \end{aligned}$$

Given any subset $A \subset \{1, 2, \ldots, 70\}$, we shall denote by

$$Z_8[A] = \{z_i \,:\, i \in A\}$$

the subset of elements of Z_8 whose index belongs to A. Thus, for example,

$$Z_8[\{2,3\}] = \{z_2, z_3\} = \{- - - + - + + +, - - + - - + + +\},$$

and SZB_8, given in the preceding section, can be described as

$$SZB_8 = Z_8[\{22, 24, 30, 41, 47, 49\}].$$

We now introduce subsets $SZB_n(A)$ of SZB_n, parametrized by subsets $A \subset \{1, 2, \ldots, 70\}$, hoping to get a more tractable size growth.

Notation 8 *Let* $A \subset \{1, 2, \ldots, 70\}$*. We denote by* $SZB_n(A)$ *the subset of* SZB_n *defined recursively as follows. For* $n = 4$ *or* 8*, set* $SZB_n(A) = SZB_n$*. Assume now* $n > 8$*. Let* X *be a binary sequence of length* n*, and write* $X = X[n-8]z$ *where* z *is the tail of length 8 of* X*. Then, by definition,*

$$X \in SZB_n(A) \iff X[n-8] \in SZB_{n-8}(A) \text{ and } z \in Z_8[A].$$

In other words, a sequence X *belongs to* $SZB_n(A)$ *if it is built from an initial segment in* SZB_4 *or* SZB_8 *by successive extensions* $z_{i_1}, \ldots, z_{i_k}$ *of length 8 all belonging to the subset* $Z_8[A]$ *of* Z_8*.*

Experimenting with various subsets A, we obtain the following:

• For $n \equiv 0 \bmod 8$ and $A = \{1, 2, \ldots, 14\}$, the construction process vanishes after a few steps. In fact, we find that $|SZB_{56}(A)| = 1$ but $|SZB_n(A)| = 0$ for all $64 \le n = 8k$.

• Still for $n \equiv 0 \bmod 8$, the case $A = \{1, 2, \ldots, 15\}$ is the first one where the construction process does not vanish. We find that $|SZB_{96}(A)| = 2$ and, thereafter, $|SZB_n(A)| = 1$ for all $104 \le n = 8k$. This is where our sequence $I_0\, P_0^\infty$ of Theorem 4 comes from! Explicitly, we have

$$I_0 = z_{22} = + - - + - + -+$$

where $z_{22} \in SZB_8$ as required, and

$$P_0 = z_2\, z_7\, z_{15} = - - - + - + + + - - + - + - + + + + - - - - - ++,$$

so that

$$I_0\, P_0^\infty = z_{22}\; z_2 z_7 z_{15}\; z_2 z_7 z_{15}\; \ldots$$

Summarizing, we have

$$SZB_n(\{1, 2, \ldots, 15\}) = SZB_n(\{2, 7, 15\}) = \{I_0\, P_0^\infty[n]\}$$

for all $104 \le n = 8k$.

• For $n \equiv 4 \bmod 8$, starting from SZB_4 and using just the first 24 elements of Z_8 as allowed extensions, the construction process eventually dies away. However, with $A = \{1, 2, \ldots, 25\}$, we do get a nonvanishing process. It turns out that

$$|SZB_n(\{1, 2, \ldots, 25\})| = 1 \text{ for all } 140 \le n = 8k + 4.$$

Again, this is where our sequence $I_4\, P_4^\infty$ of Theorem 4 comes from. We have

$$I_4\, P_4^\infty = - - + + \;\; z_{25}\; z_5 z_7 z_{23}\; z_5 z_7 z_{23}\; z_5 z_7 z_{23}\; \ldots$$

In particular, we have $SZB_n(\{1, 2, \ldots, 25\}) = SZB_n(\{5, 7, 23, 25\})$ for all sufficiently large $n \equiv 4 \bmod 8$.

5. A Critical Case

Here we consider $n \equiv 0 \bmod 8$ only. We have seen that for $A = \{1, 2, \ldots, 15\}$, we have $|SZB_n(A)| = 1$ for all sufficiently large n. On the other hand, for the full set SZB_n, its cardinality $|SZB_n|$ seems to explode with n. Is there an intermediate behavior, i.e., a suitable subset $A \subset \{1, 2, \ldots, 70\}$ for which the evolution of $|SZB_n(A)|$ looks unpredictable? After considerable experimentation, we may have found such a critical set.

To start with, if $A = \{1, 2, \ldots, 46\}$, nothing too surprising occurs. After reaching a height of 6437 at $n = 8 * 16 = 128$, the numbers $|SZB_n(A)|$ slowly go down and end up cycling as 15, 19, 19, 16, 17, 18 at $n = 8 * 81 = 648$.

However, with one more element, i.e., using $A = \{1, 2, \ldots, 47\}$, it becomes much harder to predict the behavior of $|SZB_n(A)|$. After a fast initial growth up to 9022 reached at $n = 128$ again, followed by a decay down to 25 at $n = 8*48 = 384$, these numbers meander for dozens of iterations (precisely, between the 34th and 99th ones) below 100. It is only at the 100th iteration, i.e., at $n = 800$, that the barrier of 100 is crossed again, with $|SZB_{800}(A)| = 126$. Erratical behavior goes on, yet with an overall slow growth, perhaps ultimately unbounded.

A closer examination reveals that few indices in $\{1, 2, \ldots, 47\}$ are actually used in these extensions for large n. Trying to pin down the essential elements, we have found the following critical subset. Let $A = \{15, 22, 34, 35, 47\}$. Then the behavior of $|SZB_n(A)|$ is very close to the one just described. In particular, we cannot answer the following question.

Problem 1 *Is the numerical sequence $|SZB_{8m}(\{15, 22, 34, 35, 47\})|$ bounded or unbounded?*

Our guess is that it is unbounded, but we cannot prove it. We can ask a still more specific question. Define

$$X = z_{22}z_{35}z_{47} = + - - + - + - + + + + - - - - + - + - + + - + -$$
$$Y = z_{15}z_{34}z_{47} = + + - - - - + + + + - + - - - + - + - + + - + - \,.$$

The sequence X is strongly zero-sum-balanced of length 24, i.e., it belongs to SZB_{24}. There are many elements of $SZB_{24m}(\{15, 22, 34, 35, 47\})$ which are concatenations of X and Y. Here are a few instances thereof:

$$X^{4k-1}Y,\ X^{4k}Y,\ X(X^3Y)^k,\ (X^4Y)^k,\ X^{11}(XY)^k,\ (X^{20}Y)^k,\ X^4Y(X^3YX^5Y)^k, \ldots$$

for all $k \geq 1$. We do know a few more such one-parameter families of words in X, Y giving rise to strongly zero-sum-balanced sequences. Finding infinitely many such families would settle Problem 1. But we are unable to do so. For instance, while $(X^4Y)^k$ and $(X^{20}Y)^k$ are words of the desired kind, we do not know any other words of the shape $(X^mY)^k$ having this property. We thus end up with a very challenging open problem.

Problem 2 *Describe all words in $X = z_{22}z_{35}z_{47}$ and $Y = z_{15}z_{34}z_{47}$ giving rise to strongly zero-sum-balanced binary sequences. Less ambitiously, are there infinitely many one-parameter families of such words?*

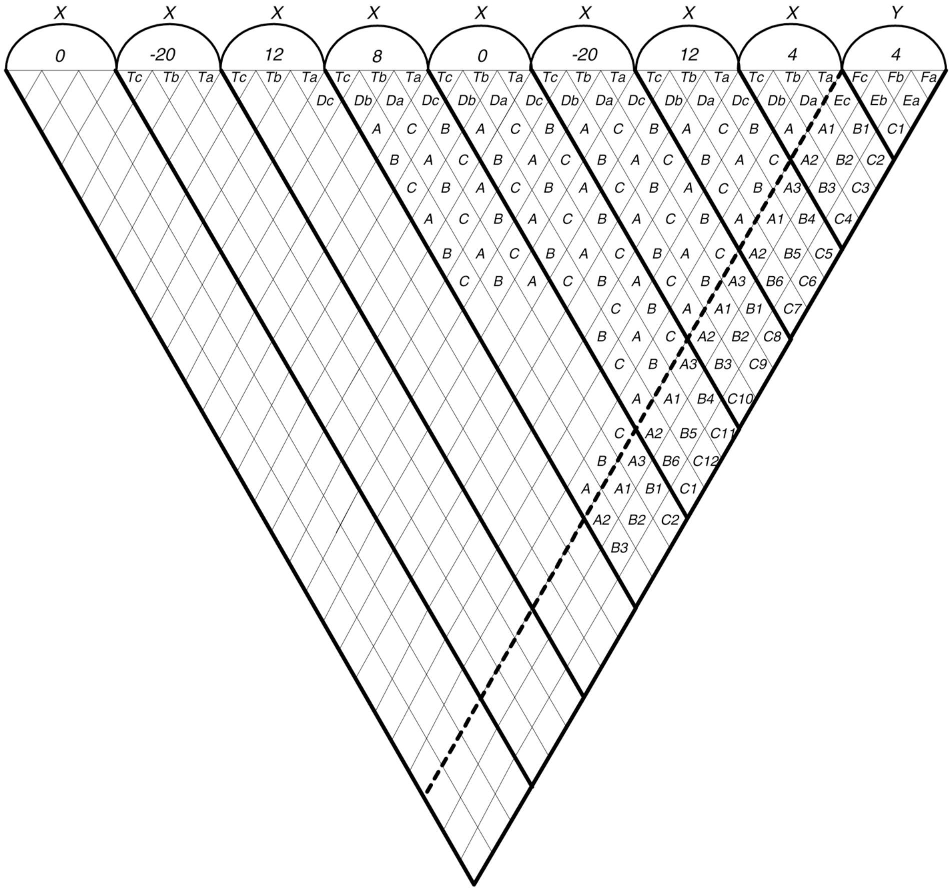

Figure 3: Structure of the Derived Triangle of X^8Y

The proof that the above words in X, Y give rise to strongly zero-sum-balanced sequences follows the same method as in Section 2, by exhibiting periodic structures in their derived triangles. We shall illustrate it for the words $X^{4k-1}Y$ and $X^{4k}Y$, with a picture and a few comments. (See Figure 3.) The corresponding derived triangles are, again, an essentially periodic assembly of a few bricks. The entry sums of these bricks are separately specified below. In Figure 3, the entry sum of each of the nine diagonal bands, corresponding to each successive letter of X^8Y, is given inside a bubble. Thus, a quick glance shows that the entry sums of the derived triangles of the words Y, XY, X^2Y, X^3Y, X^4Y, X^5Y, X^6Y, X^7Y, X^8Y are given by 4, 8, 20, 0, 0, 8, 20, 0, 0, respectively. More generally, for $t \geq 1$, the derived triangle $\Delta(X^tY)$ has entry sum 0 if $t \equiv 0$ or 3 mod 4, entry sum 8 if $t \equiv 1$ mod 4, and entry sum 20 if $t \equiv 2$ mod 4. In particular, the words $X^{4k-1}Y$ and $X^{4k}Y$ give rise to strongly zero-sum-balanced binary sequences, as announced.

$\sigma(T_a)$	$\sigma(T_b)$	$\sigma(T_c)$	$\sigma(F_a)$	$\sigma(F_b)$	$\sigma(F_c)$
0	4	0	-6	4	-2

$\sigma(A)$	$\sigma(B)$	$\sigma(C)$	$\sigma(D_a)$	$\sigma(D_b)$	$\sigma(D_c)$	$\sigma(E_a)$	$\sigma(E_b)$	$\sigma(E_c)$
2	2	-4	-2	-4	2	-6	4	-2

$\sigma(A_1)$	$\sigma(A_2)$	$\sigma(A_3)$	$\sigma(B_1)$	$\sigma(B_2)$	$\sigma(B_3)$	$\sigma(B_4)$	$\sigma(B_5)$	$\sigma(B_6)$
12	-14	2	-6	-2	0	-6	14	0

$\sigma(C_1)$	$\sigma(C_2)$	$\sigma(C_3)$	$\sigma(C_4)$	$\sigma(C_5)$	$\sigma(C_6)$	$\sigma(C_7)$	$\sigma(C_8)$	$\sigma(C_9)$	$\sigma(C_{10})$	$\sigma(C_{11})$	$\sigma(C_{12})$
6	0	2	14	12	-10	2	4	-10	-6	-8	-6

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References

[1] S. Eliahou and D. Hachez, On a problem of Steinhaus concerning binary sequences, Experimental Mathematics 13, No.2 (2004) 215-229.

[2] S. Eliahou and D. Hachez, On symmetric and antisymmetric balanced binary sequences, INTEGERS: Electronic Journal of Combinatorial Number Theory 5 (2005) #A06.

[3] H. Harborth, Solution of Steinhaus's Problem with Plus and Minus Signs, J. Comb. Th. (A) 12 (1972) 253-259.

[4] H. Steinhaus, One Hundred Problems in Elementary Mathematics. Elinsford, NY: Pergamon, 1963. New York: Dover, 1979 (reprint).