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ON INTEGERS NONREPRESENTABLE BY A GENERALIZED ARITHMETIC PROGRESSION

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Abstract

We consider those positive integers that are not representable as linear combinations of terms of a generalized arithmetic progression with nonnegative integer coefficients. To do this, we make use of the numerical semigroup generated by a generalized arithmetic progression. The number of integers nonrepresentable by such a numerical semigroup is determined as well as that of its dual. In addition, we find the number and the sum of those integers representable by the dual of the semigroup that are not representable by the semigroup itself.

1. Introduction

Let $a_1, \ldots, a_\nu$ be relatively prime positive integers. It is natural to ask which integers are representable as linear combinations of $a_1, \ldots, a_\nu$ with nonnegative integer coefficients. A nonnegative integer x is said to be *representable* by $a_1, \ldots, a_\nu$ (or, if the context is clear, representable) if there exist nonnegative integers $x_1, \ldots, x_\nu$ such that $x = x_1a_1 + \ldots + x_\nu a_\nu$ and *nonrepresentable* otherwise. It is well known that all sufficiently large integers are representable. Hence, it is natural to ask what is the value of the largest nonrepresentable integer. This problem is known as the Frobenius problem as it is said to have been mentioned by Frobenius in his lectures. In general, the Frobenius problem is very difficult [2], [3]. Of course, there are other ways to learn about nonrepresentable integers. In [4], Brown and Shiue suggest studying the sum of the nonrepresentable integers. A variation of these ideas is considered by Tripathi in [15]. To introduce this, we must first set up some notation.

Let $\mathbb{N}_0$ denote the monoid of nonnegative integers under addition. A submonoid of $\mathbb{N}_0$ is called a numerical semigroup. Given $a_1, \ldots, a_\nu$ as above, let $S = \langle a_1, \ldots, a_\nu \rangle$ where

$$\langle a_1, \ldots, a_\nu \rangle := \left\{ \sum_{i=1}^{\nu} x_i a_i : x_i \in \mathbb{N}_0 \right\}.$$

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Then S is a submonoid of $\mathbb{N}_0$, called the numerical semigroup generated by $a_1, \ldots, a_\nu$. The integers $a_1, \ldots, a_\nu$ are called generators of S. Clearly, an integer x is representable by $a_1, \ldots, a_\nu$ if and only if $x \in S$. For this reason, we say that x is representable by S if $x \in S$ and nonrepresentable by S otherwise. Given a numerical semigroup S, there exist positive integers $a_1, \ldots, a_\nu$ such that $S = \langle a_1, \ldots, a_\nu \rangle$ and $\nu \leq a_1$. Let $n(S) := |\mathbb{N}_0 \setminus S|$ denote the number of integers nonrepresentable by S. The dual of a numerical semigroup S is $B(S) := \{x \in \mathbb{N}_0 : x + S \setminus \{0\} \subseteq S\}$. One can check that $B(S)$ is also a numerical semigroup. The type of S is $n^*(S) := |B(S) \setminus S|$. In [15], two new parameters are introduced: $g^*(S) := \min\{x : x \in B(S) \setminus S\}$ and $s^*(S) := \sum_{x \in B(S) \setminus S} x$. For a general reference on numerical semigroups, see [7], [6], or [1].

The number of integers nonrepresentable by a numerical semigroup S whose generators are in arithmetic progression has been determined [14] as well as the parameters $n^*(S)$, $g^*(S)$, and $s^*(S)$ [15]. Numerical semigroups of this form were first studied in [11]. In this work, we focus our attention on numerical semigroups S with generators that are in a generalized arithmetic progression. A generalized arithmetic progression is a sequence of the form $a, ha + d, ha + 2d, \ldots, ha + kd$ where a, d, h, k are positive integers. Numerical semigroups generated by generalized arithmetic progressions have been studied in [8], [12], [5], and [10]. In [9], the dual of such a semigroup is determined. Here, we use this to find the number of integers nonrepresentable by the dual of the semigroup, the type of S, $g^*(S)$, and $s^*(S)$. In addition, we modify the methods of [14] to find $n(S)$, the number of integers nonrepresentable by S.

Throughout this paper, S will denote the numerical semigroup

$$S = \langle a, ha + d, ha + 2d, \ldots, ha + kd \rangle ,$$

where a, d, h, k are positive integers such that a and d are relatively prime, $a \geq 2$, and $k \leq a - 2$. For our purposes, we may assume that $k \geq 2$. Otherwise, S is a numerical semigroup with two generators. For a numerical semigroup S with two generators, the number of nonrepresentable integers has been determined [13] and the Frobenius problem has been solved [2]. Brown and Shiue determined the sum of the integers nonrepresentable by such S [4]. Using the fact that $B(S) = S \cup \{g(S)\}$ for a numerical semigroup S with two generators, it is easy to see that $n^*(S) = 1$, $g^*(S) = g(S) = s^*(S)$ where $g(S)$ denotes the largest integer nonrepresentable by S.

2. The number of nonrepresentable integers

In this section, we determine the number of integers nonrepresentable by a numerical semigroup generated by a generalized arithmetic progression.

Given an integer r, $0 \leq r \leq a - 1$, let

$$\mathcal{C}_r = \{x \in \mathbb{Z} : x \equiv r \bmod a\}$$

denote the equivalence class of r mod a and let

$$m_r = \min\{x \in S : x \in \mathcal{C}_r\}$$

denote the smallest element of the class $\mathcal{C}_r$ that is in S. To determine the number of integers nonrepresentable by S, we first modify [14, Lemma 1] and [14, Lemma 2] to obtain the following two results.

Lemma 2.1 The number of integers nonrepresentable by S is $n(S) = \frac{1}{a}\sum_{r=1}^{a-1} m_r - \frac{a-1}{2}$.

Proof. Clearly, each nonrepresentable integer is in one of the classes $\mathcal{C}_r$, $1 \le r \le a-1$, and the number of nonrepresentable integers in the class $\mathcal{C}_r$ is $\left\lfloor \frac{m_r}{a} \right\rfloor$. Hence,

$$n(S) = \sum_{r=1}^{a-1} \left\lfloor \frac{m_r}{a} \right\rfloor = \sum_{r=1}^{a-1} \frac{m_r - r}{a} = \frac{1}{a}\sum_{r=1}^{a-1} m_r - \frac{a-1}{2}.$$

Proposition 2.2 For each integer y, $1 \le y \le a-1$, the smallest element of S in the class $\mathcal{C}_{dy}$ is

$$m_{dy} = ah\left(\left\lfloor \frac{y-1}{k} \right\rfloor + 1\right) + dy.$$

Proof. Fix an integer y, $1 \le y \le a-1$. Write $y = kq + r$ where $0 \le r \le k-1$. Note that $ah\left(\left\lfloor \frac{y-1}{k} \right\rfloor + 1\right) + dy \in \{x \in \mathbb{N}_0 : x \equiv dy \bmod a, x \in S\}$ as

$$ah\left(\left\lfloor \frac{y-1}{k} \right\rfloor + 1\right) + dy = a\left(y_0 + y_1 h + y_2 h + \ldots + y_k h\right) + d\left(y_1 + 2y_2 + \ldots + k y_k\right)$$

where $y_k = q$, $y_r = 1$, $y_i = 0$ for all $i \ne r, k$ if $r \ne 0$, and $y_k = q$, $y_i = 0$ for all $i \ne k$ if $r = 0$. Hence $m_{dy} \le ah\left(\left\lfloor \frac{y-1}{k} \right\rfloor + 1\right) + dy$. Write $m_{dy} = a\left(x_0 + x_1 h + x_2 h + \ldots + x_k h\right) + d\left(x_1 + 2x_2 + \ldots + k x_k\right)$ for some $x_i \in \mathbb{N}_0$. This implies that $y \equiv x_1 + 2x_2 + \ldots + k x_k \bmod a$ and so $kq + r = y = x_1 + 2x_2 + \ldots + k x_k$. ¿From the definition of m_{dy} it follows that $x_i = y_i$ for all i, $0 \le i \le k$. Therefore, $m_{dy} = ah\left(\left\lfloor \frac{y-1}{k} \right\rfloor + 1\right) + dy$.

The next result is a generalization of [14, Theorem 1(ii)]. Let $c := \left\lfloor \frac{a-2}{k} \right\rfloor$ and $t := a - 2 - \left\lfloor \frac{a-2}{k} \right\rfloor$.

Theorem 2.3 The number of integers nonrepresentable by S is

$$n(S) = \frac{h\left(c+1\right)\left(a+t\right) + \left(d-1\right)\left(a-1\right)}{2}.$$

Proof. First note that the nonzero equivalence classes mod a can be represented by dy,

$1 \leq y \leq a-1$. Then, according to Lemma 2.1 and Proposition 2.2,

$$\begin{aligned} n(S) &= \tfrac{1}{a}\sum_{y=1}^{a-1}\left(ah\left(\left\lfloor\tfrac{y-1}{k}\right\rfloor+1\right)+dy\right)-\tfrac{a-1}{2} \\ &= h\sum_{y=1}^{a-1}\left(\left\lfloor\tfrac{y-1}{k}\right\rfloor+1\right)+\tfrac{(d-1)(a-1)}{2} \\ &= h\sum_{y=0}^{a-2}\left(\left\lfloor\tfrac{y}{k}\right\rfloor+1\right)+\tfrac{(d-1)(a-1)}{2} \\ &= h\sum_{y=0}^{ck-1}\left(\left\lfloor\tfrac{y}{k}\right\rfloor+1\right)+h\sum_{y=ck}^{a-2}\left(\left\lfloor\tfrac{y}{k}\right\rfloor+1\right)+\tfrac{(d-1)(a-1)}{2} \\ &= hk\left(\sum_{i=1}^{c} i\right)+h\,(t+1)\,(c+1)+\tfrac{(d-1)(a-1)}{2} \\ &= h\,(c+1)\left(\tfrac{1}{2}kc+t+1\right)+\tfrac{(d-1)(a-1)}{2} \\ &= \tfrac{h}{2}\,(c+1)\,(a+t)+\tfrac{(d-1)(a-1)}{2}. \end{aligned}$$

3. Integers representable by the dual

In this section, we consider those integers that are representable by the dual of a semigroup generated by a generalized arithmetic progression but are not representable by the semigroup itself. We will use the following two results concerning S and its dual $B(S)$.

Lemma 3.1 [9, Proposition 2.5] The numerical semigroup S is

$$S = \left\{ la + jd : 0 \leq l, 0 \leq j \leq \left\lfloor \frac{l}{h} \right\rfloor k \right\}.$$

Lemma 3.2 [9, Lemma 2.7] The dual of S is

$$B(S) = S \cup \{(ch+h-1)\,a + jd : ck < j \leq a-1\}.$$

This allows us to obtain a generalization of the main result of [15].

Theorem 3.3 The set of integers nonrepresentable by S that are representable by the dual of S is

$$B(S) \setminus S = \{(ch+h-1)\,a + jd : ck < j \leq a-1\}.$$

Proof. By Lemma 3.2, we only need to show that $(ch+h-1)\,a + jd \notin S$ for all j, $ck < j \leq a-1$. Let $\alpha := (ch+h-1)\,a + jd$ for some j, $ck < j \leq a-1$. Suppose $\alpha \in S$. By Lemma 3.1, $\alpha = l'a + j'd$ for some $0 \leq l'$ and $0 \leq j' \leq \left\lfloor \frac{l'}{h} \right\rfloor k$. Then $(j-j')\,d = (l' - (ch+h-1))\,a$ which implies that $a \mid j - j'$ as $(a,d) = 1$. Since $l'a + j'd \leq (ch+h-1)\,a + (a-1)\,d$, either $j' \leq a-1$ or $l' \leq ch+h-1$. If $j' \leq j$, then $0 \leq j - j' \leq j \leq a-1$ implies $j = j'$ as $a \mid j - j'$. Hence, $l' = ch+h-1$. According to Lemma 3.1, this is a contradiction as $j' = j > ck \geq \left\lfloor \frac{ch+h-1}{h} \right\rfloor k = \left\lfloor \frac{l'}{h} \right\rfloor k$. Thus, it must be the case that $l' \leq ch+h-1$ and $j' > j$. Then $0 \leq j' - j \leq j' \leq \left\lfloor \frac{ch+h-1}{h} \right\rfloor k \leq ck \leq a-2$ implies that $j = j'$ as $a \mid j' - j$. As before, this gives a contradiction. Therefore, $\alpha \notin S$.

As a corollary, we find the number of integers nonrepresentable by the dual and also generalize of the final result of [15].

Corollary 3.4 The number of integers nonrepresentable by the dual of S is

$$n(B(S)) = \frac{h\,(c+1)\,(a+t) + (d-1)\,(a-1)}{2} - (a-1-ck)\,.$$

Moreover,

$$\begin{array}{rcl} g^*(S) & = & (ch+h-1)\,a + (ck+1)\,d, \\ s^*(S) & = & (a-ck-1)\,(ch+h-1)\,a + \frac{d}{2}\,(a\,(a-1) - ck\,(ck+1))\,, \text{ and} \\ n^*(S) & = & a-ck-1. \end{array}$$

Proof. By Theorem 2.3 and Theorem 3.3, we have that

$$n(B(S)) = n(S) - \mid B(S) \setminus S \mid = \frac{h\,(c+1)\,(a+t) + (d-1)\,(a-1)}{2} - (a-1-ck)\,.$$

¿From Theorem 3.3, it follows that

$$g^*(S) = \min\{(ch+h-1)\,a + jd : ck < j \leq a-1\} = (ch+h-1)\,a + (ck+1)\,d,$$

$$n^*(S) = \mid \{j \in \mathbb{N} : ck+1 \leq j \leq a-1\} \mid = a-ck-1,$$

and

$$\begin{array}{rcl} s^*(S) & = & \sum_{j=ck+1}^{a-1} (ch+h-1)\,a + jd \\ & = & (a-ck-1)\,(ch+h-1)\,a + \frac{d}{2}\,(a\,(a-1) - ck\,(ck+1))\,. \end{array}$$

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