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THE RALEIGH GAME

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Abstract

We present a game on 3 piles of tokens, which is neither a generalization of Nim, nor of Wythoff's game. Three winning strategies are given and validated. They are, respectively, recursive, algebraic and arithmetic in nature, and differ in their time and space requirements. The game is a birthday present for Ron Graham, but the margins of this abstract are too narrow to explain why.

–Dedicated to Ron Graham on his 70th birthday

1. Prologue

Let $\Delta_n := \lfloor\lfloor(n+1)\varphi\rfloor\varphi\rfloor - \lfloor\lfloor n\varphi\rfloor\varphi\rfloor$, where $\varphi = (1+\sqrt{5})/2$ is the golden section. Prove that for every $n \in \mathbb{Z}_{\geq 1}$,

(i) $\Delta_n \in \{2, 3\}$,

(ii) $\lfloor\lfloor(n+1)\varphi^2\rfloor\varphi\rfloor - \lfloor\lfloor n\varphi^2\rfloor\varphi\rfloor = 2\Delta_n - 1$,

(iii) $\lfloor n\varphi\rfloor + \lfloor n\varphi^2\rfloor = \lfloor\lfloor n\varphi^2\rfloor\varphi\rfloor$,

(iv) $\lfloor\lfloor n\varphi^2\rfloor\varphi\rfloor = \lfloor\lfloor n\varphi\rfloor\varphi^2\rfloor + 1$.

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2. Introduction

We consider 2-player take-away games on finitely many piles with finitely many tokens, without splitting piles into subpiles. **Throughout this paper, we adopt the convention that the player first unable to move loses; the opponent wins**. Nim is played on any finite number of piles; a move consists of selecting a pile, and removing from it any positive number of tokens [1]. The game ends when there are no more tokens. Wythoff's game is played on 2 piles of tokens and has 2 move rules: either make a Nim-move, or take the same (positive) number from both piles [13], [5], [6].

Most take-away games (without splitting piles) are variations of Nim. Very few are variations of Wythoff's game. In fact, if there are precisely 2 piles and a Nim-move is permitted, then the game does not have the Nim-strategy if and only if the move-rules permit taking the same positive number of tokens from both piles [2]. The strategy of games on more than 2 piles not possessing the Nim-strategy is rarely known.

The Raleigh game created here is played on 3 piles. Its winning strategy is neither that of Nim nor that of Wythoff's game. It's a *variation* of Wythoff's game, not a *generalization* thereof.

3. Game Description

As stated above, Wythoff's game is played on 2 piles of tokens and has 2 move rules. *Raleigh* is played on 3 piles of tokens and has 3 move rules. We denote positions of Raleigh by (a_1, a_2, a_3), with $0 \le a_1 \le a_2 \le a_3$.

Rules of move:

I. Any positive number of tokens from up to 2 piles can be removed.

II. From a nonzero position in which 2 piles have the same size, one can move to $(0, 0, 0)$.

III. If $0 < a_1 < a_2 < a_3$, one can remove the same positive number t from a_2 and a_3 and an arbitrary positive number from a_1, except that if $a_2 - t$ is the smallest component in the triple moved to, then $t \ne 3$.

Note that rule I implies that Raleigh's game is not a generalization of Wythoff's game.

What does Raleigh's game have to do with Ron Graham?

4. Recursive Characterization of the P-positions

Let $S_0 = (0,0,0)$, $S_1 = (1,2,3)$. If $S_m := (A_m, B_m, C_m)$ has already been defined for all $m < n$, then let

$$A_n = \text{mex}\{A_i, B_i, C_i : 0 \le i < n\} \quad (n \ge 0), \tag{1}$$

$$B_n = A_n + 1 \quad (n \ge 1),$$

$$C_n = \begin{cases} C_{n-1} + 3 & \text{if } \; A_n - A_{n-1} = 2 \\ C_{n-1} + 5 & \text{otherwise} \quad (n \ge 2). \end{cases} \tag{2}$$

These definitions clearly imply that each of A_n, B_n, C_n is a strictly increasing sequence. Let $S = \cup_{n=0}^{\infty} S_n$. A prefix of S of size 16 is shown in Table 1.

Table 1: P-positions of Raleigh.

n	A_n	B_n	C_n
0	0	0	0
1	1	2	3
2	4	5	8
3	6	7	11
4	9	10	16
5	12	13	21
6	14	15	24
7	17	18	29
8	19	20	32
9	22	23	37
10	25	26	42
11	27	28	45
12	30	31	50
13	33	34	55
14	35	36	58
15	38	39	63

The set of positions from which the second (Previous) player can force a win are the *P-positions* of a game.

Theorem 1 *The collection S constitutes the set of P-positions of Raleigh.*

We begin by collecting a few properties of the set S.

Lemma 1 $A_{n+1} - A_n = B_{n+1} - B_n \in \{2, 3\}$ *for all* $n \in \mathbb{Z}_{\geq 1}$.

Proof. The equality follows from $B_n = A_n + 1$. Put $A_n = a$. Then $B_n = a + 1$. Now $a + 2$ was not assumed as A_m or B_m for $m < n$ since the sequences are increasing. If it also was not assumed as C_m, then $A_{n+1} = a + 2$ by (1). If $C_m = a + 2$, then $A_{n+1} = a + 3$ by (1).

Let $A = \cup_{i=1}^{\infty} A_i, \quad B = \cup_{i=1}^{\infty} B_i, \quad C = \cup_{i=1}^{\infty} C_i$.

Lemma 2 *The sets* A, B, C, *partition* $\mathbb{Z}_{\geq 1}$.

Proof. (1) implies that $A \cup B \cup C = \mathbb{Z}_{\geq 1}$. Suppose that $A_m = B_n$ for some $m, n \in \mathbb{Z}_{\geq 1}$. Then $m > n$ is impossible by (1). If $m < n$, then $B_n > A_n > A_m$ since A is a strictly increasing sequence, a contradiction.

Suppose that $A_m = C_n$ for some $m, n \in \mathbb{Z}_{\geq 1}$. Then $m > n$ is impossible by (1), as above. If $m < n$, then $A_m < B_m = A_m + 1 < A_n$, where the last inequality follows from (1). By comparing Lemma 1 with (2), we see that $A_n < C_n$, so $A_m < B_m < A_n < C_n$. Thus $A_m \neq C_n$ and $B_m \neq C_n$. It remains only to show that $B_m \neq C_n$ for $m > n$.

Table 1 shows that $B_m \neq C_1$ for all $m \geq 1$. Suppose that $B_m = C_n$ for some $m, n \in \mathbb{Z}_{\geq 2}$, $m > n$. Put $A_m = a$. Then $(A_m, B_m) = (a, a + 1 = C_n)$. We consider 2 cases.

(i) $A_m - A_{m-1} = 2$. Then $(A_{m-1}, B_{m-1}) = (a - 2, a - 1)$. Now $A_{m-1} \neq C_{n-1}$ as we have just seen. Thus either $C_n - C_{n-1} < a + 1 - (a - 2) = 3$ contradicting (2), or $C_n - C_{n-1} > a + 1 - (a - 2) = 3$, so $C_n - C_{n-1} = 5$ by (2). This implies $A_m - A_{m-1} = 3$, a contradiction.

By Lemma 1, the only other case is:

(ii) $A_m - A_{m-1} = 3$. Then $(A_{m-1}, B_{m-1}) = (a - 3, a - 2)$, so $C_{n-1} = a - 1$ by (1), since otherwise $a - 1$ would not be attained in S. Thus $C_n - C_{n-1} = 2$, contradicting (2).

Lemma 3 (i) $C_n - B_n$ *and* $C_n - A_n$ *are increasing functions of* n.

(ii) *Every positive integer appears in the multiset* $\{C_n - B_n, C_n - A_n : n \in \mathbb{Z}_{\geq 1}\}$.

Proof. By (2) and Lemma 1,

$$(C_{n+1} - C_n) - (A_{n+1} - A_n) = \begin{cases} 1 & \text{if } A_{n+1} - A_n = 2 \\ 2 & \text{if } A_{n+1} - A_n = 3. \end{cases}$$

Since $B_n = A_n + 1$, we have

$$(C_{n+1} - A_{n+1}) - (C_n - A_n) = (C_{n+1} - B_{n+1}) - (C_n - B_n) \in \{1, 2\}. \tag{3}$$

This already proves (i). Now if $C_n - B_n = t$ for some positive integer t, then $C_{n+1} - B_{n+1} = t+1$ or $t+2$. But $C_n - A_n = t+1$, so also in the latter case $t+1$ is assumed, establishing (ii).

Proof of Theorem 1. We show two things:

(A) Every move from any position in S results in a position outside S.

(B) For every position outside S there is a move into a position in S.

(A) Clearly there is no legal move $S_1 \to S_0$. Suppose there are positions S_n and S_m with $m < n$, $n \geq 2$, such that there is a legal move $S_n \to S_m$. By Lemma 2, this move is necessarily of the form III. Since $B_i = A_i + 1$, there exists $t \in \mathbb{Z}_{\geq 1}$ such that either $(A_m, B_m, C_m) = (A_n - t, B_n - t, C_n - t)$, or $(A_m, B_m, C_m) = (B_n - t, A_n - t + 2, C_n - t)$, $t \neq 3$.

(i) $(A_n, B_n, C_n) = (A_m + t, B_m + t, C_m + t)$. Comparing the last components of the triples, we have $t \geq 3$ by (2). Comparing the first components, Lemma 1 implies $t \leq 3$. Hence $t = 3$, so $n = m + 1$. But a comparison of the first components then implies $t = 5$ by (2), a contradiction.

(ii) $(A_n, B_n, C_n) = (B_m + t - 2, A_m + t, C_m + t)$, $t \neq 3$. Comparing the last components and the proviso $t \neq 3$ imply $t \geq 5$. Comparing the middle components then shows that $n - m \geq 2$ (Lemma 1). Now the last equality implies $(C_n - B_n) - (C_m - B_m) = 1$. But (3) implies that $(C_n - B_n) - (C_m - B_m) > 1$ for $n - m \geq 2$, a contradiction. Thus (A) has been established.

(B) Let $(a_1, a_2, a_3) \notin S$, $0 \leq a_1 \leq a_2 \leq a_3$. If there is equality in any of these, a move of the form I or II leads to S_0. So we may assume that $0 < a_1 < a_2 < a_3$. By the complementarity of A, B, C, a_1 appears in precisely one component of precisely one S_n, $n \geq 1$. If $a_1 = C_n$, move $a_2 \to A_n$, $a_3 \to B_n$.

So suppose that $a_1 = B_n$. If $a_3 \geq C_n$, move $a_2 \to A_n$, $a_3 \to C_n$. So assume $a_3 < C_n$. Let $a_3 - a_2 = t$. By Lemma 3(ii), there exist $m \in \mathbb{Z}_{\geq 1}$ such that either (i) $C_m - B_m = t$, or (ii) $C_m - A_m = t$. In case (i) move $(a_1, a_2, a_3) \to (A_m, B_m, C_m)$. This is a legal move:

- $m < n$. Follows from Lemma 3(i) and $a_2 > a_1 = B_n$, since $C_m - B_m = a_3 - a_2 < a_3 - a_1 < C_n - B_n$.
- $a_1 = B_n > A_n > A_m$, so this move (as well as all others in the remainder of this proof) is of the form III.

In case (ii) move $(a_2, a_1, a_3) \to (A_m, B_m, C_m)$. This is also a legal move:

- $m < n$. We now have $C_m - A_m = a_3 - a_2 < a_3 - a_1 < C_n - B_n < C_n - A_n$.

- $a_1 = B_n > B_m$, since $n > m$.
- Suppose that $t = 3$. Then $C_m - B_m = 2$. But the first few entries of Table 1 and Lemma 3(i) show that $C_m - B_m$ never attains the value 2.

Now suppose that $a_1 = A_n$. If $a_3 > C_n$, move $a_2 \to B_n$, $a_3 \to C_n$. If $a_3 = C_n$, then $a_2 > B_n$, so move $a_2 \to B_n$. We may thus assume that $a_3 < C_n$. Let $a_3 - a_2 = t$. As above, there exist $m \in \mathbb{Z}_{\geq 1}$ such that either (i) $C_m - B_m = t$, or (ii) $C_m - A_m = t$. In case (i) move $(a_1, a_2, a_3) \to (A_m, B_m, C_m)$. It is a legal move:

- $m < n$. Follows from Lemma 3(i) and $a_2 > a_1 = A_n$, since $C_m - B_m = a_3 - a_2 < a_3 - a_1 \leq C_n - A_n - 1 = C_n - B_n$.
- $a_1 = A_n > A_m$, since $n > m$.

In case (ii) move $(a_2, a_1, a_3) \to (A_m, B_m, C_m)$. This is also a legal move:

- $m < n$. We now have $C_m - A_m = a_3 - a_2 < a_3 - a_1 < C_n - A_n$.
- By Lemma 1, $a_1 = A_n \geq A_{n-1} + 2 = B_{n-1} + 1 > B_m$.
- The above argument that $t \neq 3$ applies also here.

5. Algebraic Characterization of the P-positions

The recursive characterization enunciated in Theorem 1, provides an easy method to compute the P-positions.

How easy is it? If the initial position of the game is (a_1, a_2, a_3), the input size is $\log a_1 + \log a_2 + \log a_3$. The time needed to compute whether the position is a P-position or not, however, is proportional to $a_1 + a_2 + a_3$. So the algorithm isn't all that easy; in fact, it requires exponential space (and hence exponential time)!

Is there a polynomial-time strategy? In this and the next section we provide an answer to this question.

Theorem 2 *Let $\varphi = (1 + \sqrt{5})/2$ (golden section). For all $n \in \mathbb{Z}_{\geq 0}$,*

$$A_n = \lfloor \lfloor n\varphi \rfloor \varphi \rfloor, \quad B_n = \lfloor n\varphi^2 \rfloor, \quad C_n = \lfloor \lfloor n\varphi^2 \rfloor \varphi \rfloor.$$

For proving Theorem 2, put, for all $n \in \mathbb{Z}_{\geq 0}$,

$$A'_n = \lfloor \lfloor n\varphi \rfloor \varphi \rfloor, \quad B'_n = \lfloor n\varphi^2 \rfloor, \quad C'_n = \lfloor \lfloor n\varphi^2 \rfloor \varphi \rfloor,$$

$$A' = \cup_{n=1}^{\infty} A'_n, \quad B' = \cup_{n=1}^{\infty} B'_n, \quad C' = \cup_{n=1}^{\infty} C'_n.$$

Since $\varphi^2 = \varphi + 1 > \varphi > 1$, each of the sequences A'_n, B'_n, C'_n is strictly increasing. We begin by proving a few auxiliary results.

Lemma 4 *The sets A', B', C' partition $\mathbb{Z}_{\geq 1}$.*

Proof. Since $\varphi^{-1} + (\varphi^2)^{-1} = 1$, the sets $\cup_{n=1}^{\infty} \lfloor n\varphi \rfloor$ and B' split $\mathbb{Z}_{\geq 1}$ (see e.g., [6], §3). The result now follows, since then A' and C' split $\cup_{n=1}^{\infty} \lfloor n\varphi \rfloor$.

Lemma 5 *For all $n \in \mathbb{Z}_{\geq 1}$, $B'_n - A'_n = 1$.*

Proof. Clearly $B'_n - A'_n \geq \lfloor n\varphi^2 \rfloor - \lfloor n\varphi^2 \rfloor = 0$. But the sequences $\lfloor m\varphi \rfloor$, $\lfloor n\varphi^2 \rfloor$ are disjoint ($m, n \in \mathbb{Z}_{\geq 1}$) by Lemma 4. Hence the inequality is strict, so $B'_n - A'_n \geq 1$.

Conversely, we multiply the inequality $n\varphi < \lfloor n\varphi \rfloor + 1$ by φ, getting $n\varphi^2 < (\lfloor n\varphi \rfloor + 1)\varphi$. Therefore $\lfloor n\varphi^2 \rfloor \leq \lfloor (\lfloor n\varphi \rfloor + 1)\varphi \rfloor$. Again by complementarity, this inequality is strict. Hence $\lfloor n\varphi^2 \rfloor - \lfloor \lfloor n\varphi \rfloor \varphi \rfloor < \lfloor (\lfloor n\varphi \rfloor + 1)\varphi \rfloor - \lfloor \lfloor n\varphi \rfloor \varphi \rfloor \leq 2$, since $\varphi < 2$. Thus $B'_n - A'_n \leq 1$.

Lemma 6 *For all $n \in \mathbb{Z}_{\geq 1}$, $A'_n < B'_n < C'_n$, and $A'_n = \operatorname{mex}\{A'_i, B'_i, C'_i : 0 \leq i < n\}$.*

Proof. In the first paragraph of the proof of Lemma 5 we proved $A'_n < B'_n$. Clearly $B'_n \leq C'_n$. Since $B' \cap C' = \emptyset$ (Lemma 4), we actually have $B'_n < C'_n$, establishing the first part of the lemma.

For $n \in \mathbb{Z}_{\geq 1}$, put $E_n := \operatorname{mex}\{A'_i, B'_i, C'_i : 0 \leq i < n\}$. Suppose that we have already shown that $A'_n = E_n$ for all $n < m$. Then also $A'_m = E_m$, because $A'_m < E_m$ would imply that either the sequence A'_i is not strictly increasing, or that $A' \cap (B' \cup C') \neq \emptyset$, contradicting Lemma 4. Also $A'_m > E_m$ would imply that the value E_m is never assumed in $A' \cup B' \cup C'$ because the sequences A'_n, B'_n, C'_n are strictly increasing, contradicting Lemma 4.

Lemma 7 *Let $n \in \mathbb{Z}_{\geq 1}$. Then $A'_{n+1} - A'_n = B'_{n+1} - B'_n \in \{2, 3\}$ and*

$$C'_{n+1} - C'_n = 2(A'_{n+1} - A'_n) - 1. \tag{4}$$

Proof. By Lemma 5, $A'_{n+1} - A'_n = B'_{n+1} - B'_n$. A direct computation shows that $B'_{n+1} - B'_n \in \{2, 3\}$.

By a simple computation and the first part of the present lemma, $C'_{n+1} - C'_n < (B'_{n+1} - B'_n)\varphi + 1 \leq 3\varphi + 1$. Thus $C'_{n+1} - C'_n \leq \lfloor 3\varphi + 1 \rfloor = 5$. Similarly, $C'_{n+1} - C'_n > (B'_{n+1} - B'_n)\varphi - 1 \geq 2\varphi - 1$, so $C'_{n+1} - C'_n \geq \lceil 2\varphi - 1 \rceil = 3$. We now show that $C'_{n+1} - C'_n = 4$ for no $n \in \mathbb{Z}_{\geq 1}$. Note that $C'_n + 1$ is necessarily in the sequence A': it cannot be in C' by the bounds we have just established for $C'_{n+1} - C'_n$, and it cannot be in B' because $B'_n = A'_n + 1$. Therefore $C'_n + 2 \in B'$, so $C'_n + 3 \in A' \cup C'$. If $C'_n + 3 \in A'$, then $C'_n + 4 \in B'$; and if $C'_n + 3 \in C'$, then $C'_n + 4 \in A'$. In any case $C'_n + 4 \notin C'$. Thus $C'_{n+1} - C'_n \in \{3, 5\}$.

Suppose now that $A'_{n+1} - A'_n = 2$ for some $n \in \mathbb{Z}_{\geq 1}$. Then $C'_{n+1} - C'_n < (B'_{n+1} - B'_n)\varphi + 1 = 2\varphi + 1$. Therefore $C'_{n+1} - C'_n \leq \lfloor 2\varphi + 1 \rfloor = 4$. Since $C'_{n+1} - C'_n \neq 4$, we see that necessarily $C'_{n+1} - C'_n = 3$. Similarly, if $A'_{n+1} - A'_n = 3$ for some $n \in \mathbb{Z}_{\geq 1}$, then necessarily $C'_{n+1} - C'_n = 5$. We have shown:

$$C'_{n+1} = \begin{cases} C'_n + 3 & \text{if } A'_{n+1} - A'_n = 2 \\ C'_n + 5 & \text{if } A'_{n+1} - A'_n = 3. \end{cases}$$

This can be encapsulated neatly in the form (4).

Proof of Theorem 2. We see that $(A'_0, B'_0, C'_0) = (0, 0, 0) = (A_0, B_0, C_0)$, $(A'_1, B'_1, C'_1) = (1, 2, 3) = (A_1, B_1, C_1)$. Also both A, B, C and A', B', C' partition $\mathbb{Z}_{\geq 1}$. Moreover, the recursive definition of A', B', C' is identical to that of A, B, C (Lemmas 5, 6, 7). Hence $A'_n = A_n$, $B'_n = B_n$, $C'_n = C_n$ for all $n \in \mathbb{Z}_{\geq 1}$.

It is easy to derive a constructive polynomial-time (hence polynomial-space) strategy from Theorem 2. The number φ has to be computed only to $O(\log a_1)$ bits. We leave the details to the reader. See also [6], §3.

6. Arithmetic Characterization of the P-positions

The *Fibonacci numbers* are given by $F_0 = 1$, $F_1 = 2$, and $F_{n+1} = F_n + F_{n-1}$ for all $n \in \mathbb{Z}_{\geq 1}$. The *Fibonacci numeration system* is a binary numeration system in which every positive integer N has a unique representation of the form $N = \sum_{i \geq 0} d_i F_i$, such that $d_i \in \{0, 1\}$, $d_i = 1 \Longrightarrow d_{i-1} = 0$, $i \geq 1$ [7].

For any $a \in \mathbb{Z}_{\geq 1}$ denote by $R(a)$ the representation of a in the Fibonacci numeration system. Thus $R(a) = (d_m, \ldots, d_0)$, if $a = \sum_{i \geq 0} d_i F_i$. Then the representation whose digits are $(d_m, \ldots, d_0, 0)$ is the *left shift* of $R(a)$.

Theorem 3 *$R(A)$ is the set of all numbers that end with a 1-bit in the Fibonacci numeration system, $R(B)$ is the set of all numbers that end with an odd number of 0-bits in the Fibonacci* *numeration system, and $R(C)$ is the set of all numbers that end in a nonzero even number of* 0*-bits in that system. Moreover, for every $n \in \mathbb{Z}_{\geq 1}$, $R(C_n)$ is the left shift of $R(B_n)$ in the Fibonacci numeration system.*

See Table 2 for an example.

Table 2: The P-positions and the Fibonacci numeration system.

n	A_n	B_n	C_n	13	8	5	3	2	1	21	13	8	5	3	2	1	n
1	1	2	3						1		1	0	0	1	0	0	16
2	4	5	8					1	0		1	0	0	1	0	1	17
3	6	7	11				1	0	0		1	0	1	0	0	0	18
4	9	10	16				1	0	1		1	0	1	0	0	1	19
5	12	13	21			1	0	0	0		1	0	1	0	1	0	20
6	14	15	24			1	0	0	1	1	0	0	0	0	0	0	21
7	17	18	29			1	0	1	0	1	0	0	0	0	0	1	22
8	19	20	32		1	0	0	0	0	1	0	0	0	0	1	0	23
9	22	23	37		1	0	0	0	1	1	0	0	0	1	0	0	24
10	25	26	42		1	0	0	1	0	1	0	0	0	1	0	1	25
11	27	28	45		1	0	1	0	0	1	0	0	1	0	0	0	26
12	30	31	50		1	0	1	0	1	1	0	0	1	0	0	1	27
13	33	34	55	1	0	0	0	0	0	1	0	0	1	0	1	0	28
14	35	36	58	1	0	0	0	0	1	1	0	1	0	0	0	0	29
15	38	39	63	1	0	0	0	1	0	1	0	1	0	0	0	1	30

Proof. Every term in the sequences $R(A_n)$, $R(C_n)$ must end in an even number of 1-bits in the Fibonacci numeration system since all of them have the form $\lfloor m\varphi \rfloor$, $(m \in \mathbb{Z}_{\geq 1})$, see [6], §4. In fact, every representation $N \in R(A)$ must end in 1, since $N+1 \in R(B)$ must end in an odd number of 1-bits. Suppose that there is a representation $N \in R(C_n)$ that ends in 1. Then $N+1$ can be in neither $R(A_n)$ nor in $R(C_n)$, so it is in $R(B_n)$. But then $N = (N+1) - 1$ is in $R(A_n)$, since $B_n - A_n = 1$, a contradiction.

Now $R(B_n)$ is the set of all representations ending in an odd number of 1-bits, so $R(C_n)$ is the set of all left shifts of the set of all representations $R(B_n)$. For any fixed $n \in \mathbb{Z}_{\geq 1}$, if $R(C_n)$ would not be the left shift of $R(B_n)$, then it would be assumed later on (by complementarity of A, B, C), contradicting the fact that the sequence C_n is increasing.

It is straightforward to derive another constructive polynomial-time strategy from the arithmetic characterization of the P-positions.

7. Epilogue

Although Raleigh's game is not a generalization of Wythoff's game, both games share the common underlying φ for the polynomial strategies, though in different manifestations. Also, for both games 3 different strategies were given, one recursive and exponential; and two polynomial ones.

At the end of §3 we enquired about a connection between Ron Graham and Raleigh's game. We can now observe at least 4 independent connections:

- Historical. Sir Walter Raleigh (1552 - 1618) was, for some time, Governor of Jersey. Before moving to San Diego — which, he says, is a very apt place for retirees and their parents — Ron Graham had lived and worked in New Jersey, where he governed the Math Department at Bell Labs for many years.
- Geographic. RALEIGH is just south-west of GRAHAM, NC.
- Etymological. **R**on**A**ld **LE**w**I**s **G**ra**H**am.
- Mathematical. The main connection — to the game, not just to its name — is via the algebraic characterization of the P-positions given in §5, which leans heavily on the floor function. It enabled us to replace the recursive exponential strategy given in §4 by a polynomial one. Ron's fascination with the floor, ceiling and fractional part functions is evidenced in many of his papers. The entire ch. 3 of [10] is devoted to these functions, and I suspect that Ron is to blame for most of that beautiful chapter. The following is but a small sample of his works in this area: [8], [9], [10], [11], [12].

The identities (i) and (ii) of the Prologue have already been proved in the preceding sections. We now show how to prove (iii) and (iv), based on the identities established above.

Lemma 8 *Let $n \in \mathbb{Z}_{\geq 0}$. Then $\lfloor n\varphi \rfloor + \lfloor n\varphi^2 \rfloor = \lfloor \lfloor n\varphi^2 \rfloor \varphi \rfloor$.*

Proof. Put $D_n = \lfloor n\varphi \rfloor$. For $n \in \mathbb{Z}_{\geq 1}$, the following identity holds:

$$C_{n+1} - C_n = \begin{cases} 3 & \text{if } \; D_{n+1} - D_n = 1 \; \text{ if } A_{n+1} - A_n = B_{n+1} - B_n = 2 \\ 5 & \text{if } \; D_{n+1} - D_n = 2 \; \text{ if } A_{n+1} - A_n = B_{n+1} - B_n = 3. \end{cases}$$

This follows from (2), Lemma 1 and from $B_{n+1} - B_n = \lfloor (n+1)\varphi^2 \rfloor - \lfloor n\varphi^2 \rfloor = D_{n+1} - D_n + 1$ (since $\varphi^2 = \varphi + 1$).

We now proceed by induction on n. The statement holds trivially for $n = 0$. Suppose we proved it for all $n \leq m \;\; (m \geq 0)$. We have to show $C_{m+1} - D_{m+1} = B_{m+1}$. Now either $(C_{m+1} - C_m) - (D_{m+1} - D_m) = 2$, or $(C_{m+1} - C_m) - (D_{m+1} - D_m) = 3$. In the former case, $(C_{m+1} - D_{m+1}) - (C_m - D_m) = 2$. By the induction hypothesis, $C_m - D_m = B_m$. Hence $C_{m+1} - D_{m+1} = B_m + 2 = B_{m+1}$. The latter case is established similarly.

Lemma 9 *Let $n \in \mathbb{Z}_{\geq 1}$. Then* $\lfloor \lfloor n\varphi^2 \rfloor \varphi \rfloor = \lfloor \lfloor n\varphi \rfloor \varphi^2 \rfloor + 1$.

Proof. Put $G_n = \lfloor \lfloor n\varphi \rfloor \varphi^2 \rfloor$. Now

$$\begin{aligned} C_n &= \lfloor \lfloor n\varphi + n \rfloor \varphi \rfloor = \lfloor \lfloor n\varphi \rfloor \varphi + n\varphi \rfloor, \\ G_n &= \lfloor \lfloor n\varphi \rfloor (\varphi + 1) \rfloor = \lfloor \lfloor n\varphi \rfloor \varphi + \lfloor n\varphi \rfloor \rfloor. \end{aligned}$$

Thus, $C_n - G_n \geq 0$. But $C_m \cap G_n = \emptyset$ for all $m, n \in \mathbb{Z}_{\geq 1}$. Hence $C_n - G_n \geq 1$. Conversely, $C_n - G_n \leq \lfloor n\varphi^2 \rfloor \varphi - \lfloor n\varphi \rfloor \varphi^2 + 1 = \lfloor n\varphi \rfloor (\varphi - \varphi^2) + n\varphi + 1 = n\varphi - \lfloor n\varphi \rfloor + 1 < 2$. Thus, $C_n - G_n \leq 1$, so $C_n - G_n = 1$.

We note, incidentally, that the Graham family is also connected to the other polynomial strategy, the one based on a numeration system (§6). Fan Chung (= Ron Graham's wife) and Ron used an exotic ternary numeration system to investigate irregularities of distribution of sequences [3], [4] (a generalization of this numeration system is given in [7], §4). Therefore it is natural to devote this game to Ron. May he and his wife Fan Chung play it for an exponentially long time to come, always winning against their opponents in polynomial time!

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