



 SYMMETRIES IN K -BONACCI ADIC SYSTEMS

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Abstract

In this paper we study the symmetries of the k -bonacci adic systems and their geometrical realizations known as Rauzy fractals. We consider the maximal invariant subset under the involution or “mirror image” in $\{0, 1\}^{\mathbb{N}}$ that exchanges 0 with 1. We explore the dynamical properties of the induced dynamical system on this set and their geometrical realizations as subsets of Rauzy fractals.

1. Introduction

In this paper we explore some symmetries of the k -bonacci adic systems, $(\mathcal{R}(k), \mathcal{T}_k)$, and their geometric realizations known as Rauzy fractals.

These adic systems are symbolic dynamical systems with symbols in the alphabet $\{0, 1\}$. The adic map \mathcal{T}_k is given by “addition by 1” in $\mathcal{R}(k)$; it will be defined in Section 2. We consider the maximal subsystems invariant under the involution or “mirror image” $\Psi : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$, which consists of mapping each 0 into 1 and vice versa. Let $\mathcal{P}(k)$ be the maximal subset of $\mathcal{R}(k)$ invariant under Ψ . We obtain this set using the concept of the product automaton; see Section 3. We show in Theorem 1 there is a continuous map \mathcal{S}_k on $\mathcal{P}(k)$ such that the dynamical system $(\mathcal{P}(k), \mathcal{S}_k)$ is semi-conjugated to $(\mathcal{R}(k-1), \mathcal{T}_{k-1})$, and the semi-conjugacy is 2 to 1. So we can interpret $\mathcal{P}(k)$ as a double cover of $\mathcal{R}(k-1)$, and the “successor map” in $\mathcal{R}(k-1)$, i.e., \mathcal{T}_{k-1} , is lifted continuously to a map \mathcal{S}_k on $\mathcal{P}(k)$. We explore how the symmetries in the symbolic systems are reflected in their geometrical representations; we also show that Theorem 1 does not have a geometric counterpart. However, the dynamical system $(\mathcal{P}(k), \mathcal{S}_k)$ can be realized in \mathbb{R}^{k-1} as a permutation in $k-1$ pieces; see Propositions 6 and 7.

The systems $(\mathcal{P}(k), \mathcal{S}_k)$ were first introduced by the author in [22], where their metrical properties were studied. Unlike there, in the present paper we use the language of automata theory and in particular the notion of the product automaton

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in order to introduce and study these systems. The properties of the geometrical realizations, and the symmetries involved, were not addressed before.

Here we do not consider the relation of the dynamical systems studied with substitution dynamical systems [17, 18].

2. Adic Systems

Let $\mathcal{A} = \{1, \dots, k\}$ be a finite alphabet and $W \subset \mathcal{A}^* = \cup_{i \geq 0} \mathcal{A}^i$ a set of finite words on \mathcal{A} . An *automaton* over \mathcal{A} , $\mathfrak{A} = (Q, W, E, I)$, is a direct graph labelled by elements of \mathcal{A} . We let Q be the set of *states*, $I \subset Q$ is the set of *initial states*, W is the set of *labels* and $E \subset Q \times W \times Q$ is the set of labelled *edges* or *transitions*. If $(p, w, q) \in E$ we say that w is a transition between p and q . Here we will consider W a finite set, $Q = \mathcal{A}$ and $I = \{1\}$. A *finite path in the automaton* \mathfrak{A} is a word in E ; that is, it is a sequence of transitions: $(p_n, a_n, q_n)(p_{n-1}, a_{n-1}, q_{n-1}) \cdots (p_0, a_0, q_0)$ such that $q_i = p_{i-1}$ for $1 \leq i \leq n$ and $p_n \in I$; however, we usually denote the paths using only the labels, i.e., $a_n \cdots a_0$. The automaton reads words from left to right. See [8] for more on automata theory.

The *transition matrix* of an automaton is the matrix M whose m_{ij} entry is defined as the number of different transitions from i to j . We shall assume that the automaton is *primitive*, i.e., for some integer $m > 0$ each entry of M^m is positive.

Let

$$\mathcal{R} = \{ \mathbf{a} = a_0 a_1 \dots \in W^{\mathbb{N}} : a_n \dots a_0 \text{ is a path in } \mathfrak{A} \text{ for all } n \in \mathbb{N} \},$$

where \mathfrak{A} is an automaton. We consider in \mathcal{R} the topology induced from the product topology of $W^{\mathbb{N}}$, i.e., the usual metric topology on 1-sided infinite sequences over a finite alphabet.

We can order lexicographically all the finite paths in \mathfrak{A} , so the addition by 1 of a given element in W^* is its consecutive element. Due to the definition of \mathcal{R} , addition by 1 is well-defined in \mathcal{R} , and we will denote this map by \mathcal{T} . There are only a finite number of maximal elements of \mathcal{R} . If \mathbf{a} is a maximal element of \mathcal{R} , then $\mathcal{T}(\mathbf{a})$ is the minimal element of \mathcal{R} . Here we suppose that there is only one minimal element in \mathcal{R} . Hence we have the dynamical system $(\mathcal{R}, \mathcal{T})$, which is called the *adic system*. If the transition matrix is primitive then the dynamical systems $(\mathcal{R}, \mathcal{T})$ is primitive, i.e., every orbit is dense. We note that an adic system is sometimes also called *stationary Markov compactum* or *stationary Bratteli-Vershik system*; see [30, 27, 7, 13, 17, 3] for further details. These dynamical systems are related to general numeration systems; for details see [20, 6, 10, 11].

Let $\mathcal{A} = \{1, \dots, k\}$, $W = \{0, 1\}$ where 0 denotes the empty word, $Q = \mathcal{A}$,

$$E = \{(1, 0, 1), (1, 1, 2), (j, 1, j + 1), (j, 0, 1), (k, 0, 1) : 2 \leq j \leq k - 1\}$$

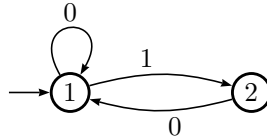


Figure 1: The Fibonacci automaton.

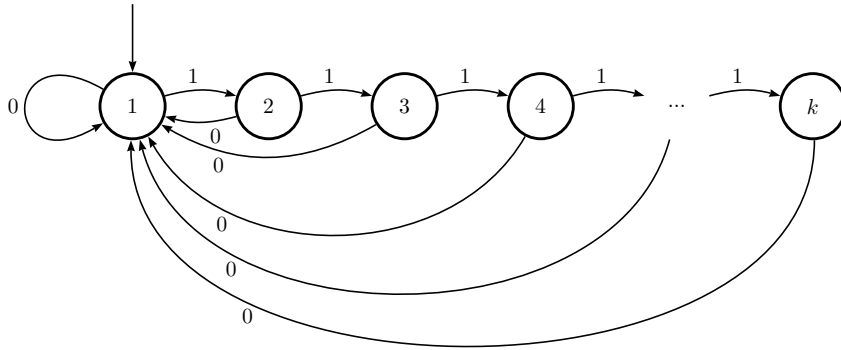


Figure 2: The k -bonacci automaton.

and $I = \{1\}$; see Figure 2. This is known as the k -bonacci automaton; we shall denote it by $\mathfrak{A}(k)$. For $k = 2$ it is called the *Fibonacci or golden mean automaton*; see Figure 1. The corresponding adic system is denoted by $(\mathcal{R}(k), \mathcal{T}_k)$, in order to emphasize the dependency on k .

It is easy to check that the automaton $\mathfrak{A}(k)$ is symmetrical in the sense that $a_n \dots a_0$ is a path in $\mathfrak{A}(k)$ if and only if $a_0 \dots a_n$ is a path in $\mathfrak{A}(k)$. Then

$$\begin{aligned} \mathcal{R}(k) &:= \{ \mathbf{a} = a_0 a_1 \dots \in \{0, 1\}^{\mathbb{N}} : a_n \dots a_0 \text{ is a path in } \mathfrak{A}(k) \text{ for all } n \in \mathbb{N} \} \\ &= \{ \mathbf{a} = a_0 a_1 \dots \in \{0, 1\}^{\mathbb{N}} : a_0 \dots a_n \text{ is a path in } \mathfrak{A}(k) \text{ for all } n \in \mathbb{N} \} \\ &= \{ \mathbf{a} = a_0 a_1 \dots \in \{0, 1\}^{\mathbb{N}} : a_i \cdot a_{i+1} \cdot \dots \cdot a_{i+k-1} \cdot a_{i+k} = 0, \forall i \geq 0 \}, \end{aligned}$$

where $a_i \cdot a_j$ denotes the usual product of natural numbers, i.e., the space $\mathcal{R}(k)$ consists of all one-sided infinite sequences of 0 and 1 such that there are no k consecutive 1s.

For an automaton $\mathfrak{A} = (Q, W, E, I)$ whose set of labels W is $\{0, 1\}$ we define its *dual automaton* as the automaton $\widehat{\mathfrak{A}} = (Q, W, \widehat{E}, I)$ where $\widehat{E} = \{(p, \widehat{w}, q) : (p, w, q) \in E\}$ where

$$\widehat{w} = \begin{cases} 0 & \text{if } w = 1 \\ 1 & \text{if } w = 0. \end{cases}$$

For an adic system $(\mathcal{R}, \mathcal{T})$ whose set of labels is $W = \{0, 1\}$, we define its *dual adic system* $(\widehat{\mathcal{R}}, \widehat{\mathcal{T}})$ using the dual automaton. Since taking the dual reverses the ordering in W^* , we have the relation $\mathcal{T}(\widehat{\mathcal{T}}(\widehat{\mathbf{a}})) = \mathbf{a}$, or equivalently $\widehat{\mathcal{T}}(\mathcal{T}(\mathbf{a})) = \widehat{\mathbf{a}}$ for any $\mathbf{a} \in \mathcal{R}$. We remark that $\widehat{\mathcal{T}}$ is still “adding by 1,” but on $\widehat{\mathcal{R}}$.

3. Product Automaton

Let \mathfrak{A}_1 and \mathfrak{A}_2 be two automata on the alphabets \mathcal{A}_1 and \mathcal{A}_2 respectively. The *product automaton* $\mathfrak{A}_1 \times \mathfrak{A}_2$ is defined as the automaton satisfying:

- set of states: $\mathcal{A}_1 \times \mathcal{A}_2$.
- labels: w is a label of $\mathfrak{A}_1 \times \mathfrak{A}_2$ if and only if w is a label of \mathfrak{A}_1 and \mathfrak{A}_2 .
- transitions: w is a transition between (p_1, q_1) and (p_2, q_2) if and only if w is a transition between p_i and q_i , for $i = 1$ and 2 .
- initial states: the product of the initial states of \mathfrak{A}_1 and \mathfrak{A}_2 .

And finally the connected component containing the initial state is taken.

The paths in $\mathfrak{A}_1 \times \mathfrak{A}_2$ are paths in \mathfrak{A}_1 and \mathfrak{A}_2 , so it is said that the paths in the product automaton are the common paths in each of the factors. The product automaton was used in [25, 21] in the study of adic systems.

Let \mathfrak{A} be an automaton with labels in $\{0, 1\}$. The product automaton $\mathfrak{A} \times \widehat{\mathfrak{A}}$ is called the *canonical product* of \mathfrak{A} . Let

$$\mathcal{P} = \left\{ \mathbf{a} = a_0 a_1 \dots \in W^{\mathbb{N}} : a_n \dots a_0 \text{ is a path in } \mathfrak{A} \times \widehat{\mathfrak{A}} \text{ for all } n \in \mathbb{N} \right\}.$$

Let $\Psi : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ be the involution $\Psi(\mathbf{a}) = \widehat{\mathbf{a}} = \widehat{a_0} \widehat{a_1} \dots$. By definition, \mathcal{P} is the maximal subset of \mathcal{R} invariant under Ψ .

Here we shall consider $\mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)}$; see Figure 3. Let $\mathfrak{B}(k)$ be the automaton of Figure 4. It can be easily checked that $\mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)}$ and $\mathfrak{B}(k)$ are equivalent, i.e., both automata recognize the same finite paths. Observe that $\mathfrak{B}(k)$ is also symmetrical, i.e., $a_n \dots a_0$ is a path in $\mathfrak{B}(k)$ if and only if $a_0 \dots a_n$ is also a path in $\mathfrak{B}(k)$. Let $\mathcal{P}(k)$ be the set of infinite paths on $\mathfrak{B}(k)$, for $k \geq 3$. The set $\mathcal{P}(2)$ consists of only two points, namely $(01)^\infty = 010101\dots$ and $(10)^\infty = 101010\dots$

Since $\mathcal{P}(k)$ is invariant under the involution Ψ ,

$$\begin{aligned} \mathcal{P}(k) &:= \left\{ \mathbf{a} = a_0 a_1 \dots \in \{0, 1\}^{\mathbb{N}} : a_n \dots a_0 \text{ is a path in } \mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)} \text{ for all } n \in \mathbb{N} \right\} \\ &= \left\{ \mathbf{a} = a_0 a_1 \dots \in \{0, 1\}^{\mathbb{N}} : a_0 \dots a_n \text{ is a path in } \mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)} \text{ for all } n \in \mathbb{N} \right\} \\ &= \left\{ \mathbf{a} = a_0 a_1 \dots \in \{0, 1\}^{\mathbb{N}} : 0 < \sum_{j=i}^{i+k} a_j < k, \forall i \geq 0 \right\}, \end{aligned}$$

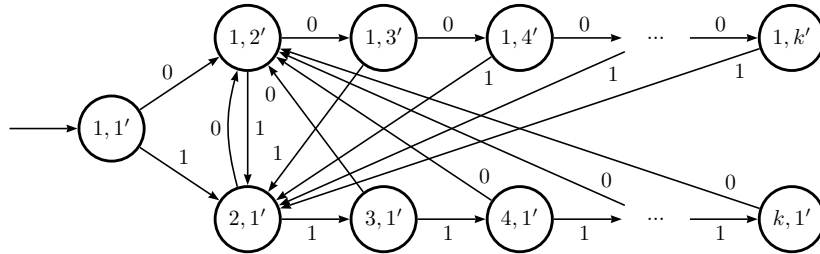


Figure 3: The canonical product of k -bonacci automaton, i.e., $\mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)}$.

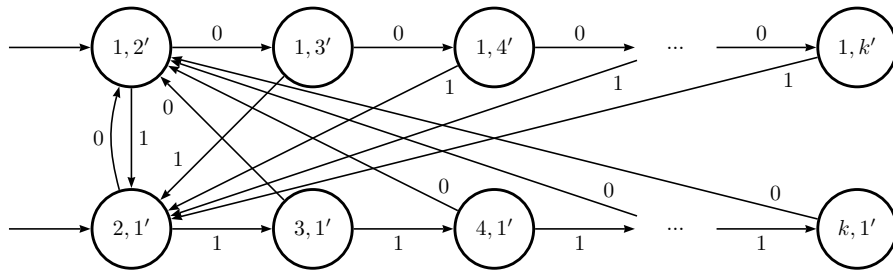


Figure 4: The automaton $\mathfrak{B}(k)$ is equivalent to $\mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)}$.

i.e., the space $\mathcal{P}(k)$ consists of all one-sided infinite sequences of 0 and 1 such that there are no k consecutive 1s nor k consecutive 0s.

Theorem 1. Let $(\mathcal{R}(k-1), \mathcal{T}_{k-1})$ be the $(k-1)$ -bonacci adic system and $\mathcal{P}(k)$ as before. For $k \geq 3$, there exist continuous maps $\phi_k : \mathcal{P}(k) \rightarrow \mathcal{R}(k-1)$ and $S_k : \mathcal{P}(k) \rightarrow \mathcal{P}(k)$ such that ϕ_k is 2 to 1 and semi-conjugates the dynamical system $(\mathcal{P}(k), S_k)$ with $(\mathcal{R}(k-1), \mathcal{T}_{k-1})$, i.e., the following diagram commutes

$$\begin{array}{ccc}
 \mathcal{P}(k) & \xrightarrow{S_k} & \mathcal{P}(k) \\
 \phi_k \downarrow & & \downarrow \phi_k \\
 \mathcal{R}(k-1) & \xrightarrow{\mathcal{T}_{k-1}} & \mathcal{R}(k-1).
 \end{array}$$

Moreover, let $\sigma(a_0a_1a_2\cdots) = a_1a_2\cdots$ be the shift map. Then the dynamical system $(\mathcal{P}(k), \sigma)$ is semi-conjugate to $(\mathcal{R}(k-1), \sigma)$, by the semi-conjugacy ϕ_k .

Proof. Let $\mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)}$ be the canonical product of the k -bonacci automaton; see Figure 3. If in this automaton we identify the state $(1, m')$ with $(m, 1')$ for $2 \leq m \leq k$, we get the automaton seen in Figure 5. If we erase the state $(1, 1')$ and define $(2, 1')$ as the initial state, we get the $(k-1)$ -bonacci automaton; see Figure 6. More precisely, let π be the map from the set of edges of the automaton $\mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)}$ to the set of edges of the automaton of Figure 6 defined as follows:

$$\begin{aligned} ((1, 2'), 1, (2, 1')) &\mapsto (1, 0, 1), & ((2, 1'), 0, (1, 2')) &\mapsto (1, 0, 1), \\ ((1, j'), 0, (1, j'+1)) &\mapsto (j-1, 1, j), & ((j, 1'), 1, (j+1, 1')) &\mapsto (j-1, 1, j), \\ ((1, j'), 1, (2, 1')) &\mapsto (j-1, 0, 1), & ((j, 1'), 0, (1, 2')) &\mapsto (j-1, 0, 1), \\ ((1, 1'), 0, (1, 2')) &\mapsto (0, 1, 1), & ((1, 1'), 1, (2, 1')) &\mapsto (0, 1, 1); \end{aligned}$$

where $2 \leq j \leq k$.

The map ϕ_k is the resulting map on $\mathcal{P}(k)$ after the identification described on the automaton $\mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)}$. Let $\mathbf{a} \in \mathcal{P}(k)$. Due to the symmetry of $\mathcal{P}(k)$ pointed out before, we can write it as:

$$\mathbf{a} = (p_0, a_0, p_1)(p_1, a_1, p_2)\cdots$$

where p_i are the states and a_i the labels in $\mathfrak{A}(k)$. Since $\pi(p_0, a_0, p_1) = (0, 1, 1)$, we define $\phi_k(\mathbf{a}) := \pi((p_1, a_1, p_2))\pi((p_2, a_2, p_3))\cdots$.

Since ϕ_k is continuous and bijective on each

$$\mathcal{P}(k)_i := \{\mathbf{a} = a_0a_1\dots \in \mathcal{P}(k) : a_0 = i\},$$

for $i = 0$ and 1 , the map \mathcal{T}_{k-1} lifts continuously to $\mathcal{S}_k^0 : \mathcal{P}(k)_0 \rightarrow \mathcal{P}(k)_0$ and $\mathcal{S}_k^1 : \mathcal{P}(k)_1 \rightarrow \mathcal{P}(k)_1$, where $\mathcal{S}_k^i = \phi_k^{-1} \circ \mathcal{T}_{k-1} \circ \phi_k$ for $i = 0, 1$, i.e., each \mathcal{S}_k^i is topologically conjugated to \mathcal{T}_{k-1} . We define

$$\mathcal{S}_k : \begin{array}{l} \mathcal{P}(k) \longrightarrow \mathcal{P}(k) \\ \mathbf{a} \longmapsto \mathcal{S}_k^i(\mathbf{a}); \end{array} \quad \text{if } \mathbf{a} \in \mathcal{P}(k)_i.$$

Since $\mathcal{P}(k)_0$ and $\mathcal{P}(k)_1$ are disjoint, the map \mathcal{S}_k is well-defined and the continuity follows from the continuity of \mathcal{S}_k^i . By construction we get

$$\phi_k(\mathcal{S}_k(\mathbf{a})) = \mathcal{T}_{k-1}(\phi_k(\mathbf{a})).$$

On the other hand, from the definition of ϕ_k , the spaces $\mathcal{P}(k)$ and $\mathcal{R}(k)$ are σ -invariant and it also follows that for $\mathbf{a} \in \mathcal{P}(k)$:

$$\phi_k(\sigma(\mathbf{a})) = \pi((p_2, a_2, a_3))\pi((p_3, a_3, a_4))\dots = \sigma(\phi_k(\mathbf{a})).$$

□

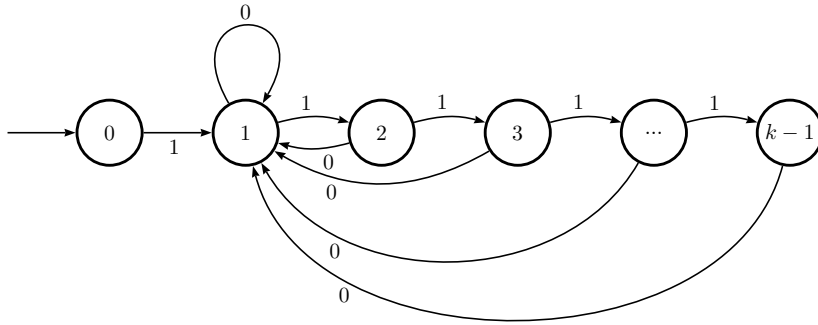


Figure 5: The automaton $\pi(\mathfrak{A}(k) \times \widehat{\mathfrak{A}(k)})$.

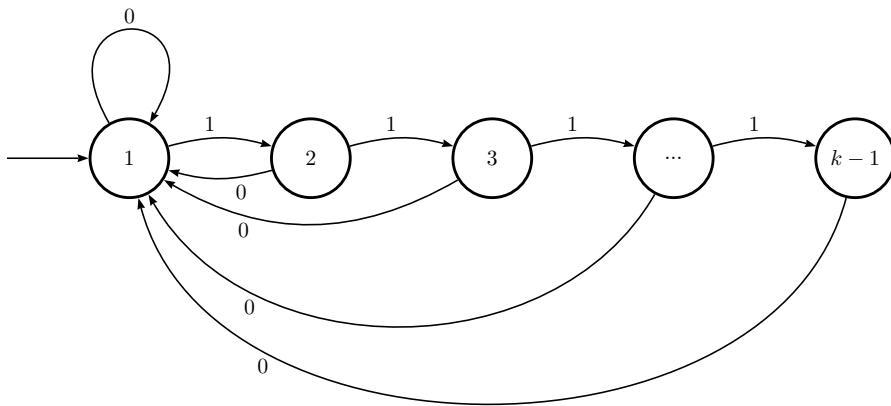


Figure 6: The automaton obtained from the automaton of Figure 5 after removing the state 0. This automaton is equivalent to the $(k - 1)$ -bonacci automaton.

Corollary 2. *Let Ψ be the involution defined before. Let $\mathcal{S}_k/\Psi : \mathcal{P}(k)/\Psi \rightarrow \mathcal{P}(k)/\Psi$ be the map corresponding to \mathcal{S}_k under the projection which maps $\mathcal{P}(k)$ into $\mathcal{P}(k)/\Psi$. Then the dynamical systems $(\mathcal{P}(k)/\Psi, \mathcal{S}_k/\Psi)$ and $(\mathcal{R}(k-1), \mathcal{T}_{k-1})$ are topologically conjugated.*

Proof. By the definition of the map ϕ_k , we get $\phi_k(\Psi(\mathbf{a})) = \phi_k(\mathbf{a})$. Since Ψ is a homeomorphism between $\mathcal{P}(k)_0$ and $\mathcal{P}(k)_1$, we have $\mathcal{P}(k)/\Psi$ is homeomorphic to $\mathcal{P}(k)_0$ and $\mathcal{P}(k)_1$. According to Theorem 1, $\phi_k/\Psi : \mathcal{P}(k)_j \rightarrow \mathcal{R}(k-1)$ is continuous and bijective for $j = 0, 1$. Since both spaces are compact, this map is a homeomorphism. Therefore $(\mathcal{P}(k)/\Psi, \mathcal{S}_k/\Psi)$ and $(\mathcal{R}(k-1), \mathcal{T}_{k-1})$ are topologically conjugated. \square

This Corollary shows that we can interpret $\mathcal{P}(k)$ as the double cover of $\mathcal{R}(k-1)$ with the “successor map” on $\mathcal{R}(k-1)$ and the corresponding lifted map on $\mathcal{P}(k)$. We point out that the map \mathcal{S}_k , i.e., lifted map on $\mathcal{P}(k)$, is not the adic map on this space.

4. Rauzy Fractals

The eigenvalues of the transition matrix of the k -bonacci automaton (i.e., the roots of the polynomial $x^k - x^{k-1} - \dots - x - 1$) are of the form $\gamma_1, \overline{\gamma_1}, \dots, \gamma_s, \overline{\gamma_s}, \gamma_{s+1}, \rho$ where $|\gamma_j| < 1$, $\gamma_1, \dots, \gamma_s$ are non-real complex, γ_{s+1} is real, this eigenvalue only exists if k is even; and ρ is real and bigger than 1; see [4]. Let

$$\mathfrak{R}(k) := \left\{ \left(\sum_{i \geq 0} a_i \gamma_1^i, \dots, \sum_{i \geq 0} a_i \gamma_s^i, \sum_{i \geq 0} a_i \gamma_{s+1}^i \right) \in \mathbb{R}^{k-1} : \mathbf{a} = a_0 a_1 \dots \in \mathcal{R} \right\}$$

if $k = 2s + 2$, and

$$\mathfrak{R}(k) := \left\{ \left(\sum_{i \geq 0} a_i \gamma_1^i, \dots, \sum_{i \geq 0} a_i \gamma_s^i \right) \in \mathbb{R}^{k-1} : \mathbf{a} = a_0 a_1 \dots \in \mathcal{R} \right\}$$

if $k = 2s + 1$.

Here we identify \mathbb{C}^s with \mathbb{R}^{2s} . This set is known as the *Rauzy fractal* of the k -bonacci automaton, first introduced by G. Rauzy for $k = 3$ in 1982 [19]; see Figure 7. In [19] it was proved, for $k = 3$, that it is compact, convex and simply connected. The geometrical and dynamical properties of this set have been studied extensively; see for instance [1, 2, 5, 12, 16, 17, 29, 24, 26].

The set $\mathfrak{R}(k)$ admits a natural partition $\{\mathfrak{R}(k)_1, \dots, \mathfrak{R}(k)_k\}$, which is also called the *natural decomposition*, where

$$\mathfrak{R}(k)_j := \left\{ \xi_k(\mathbf{a}) : \mathbf{a} = \underbrace{1 \cdots 1}_{j-1} 0 \cdots \right\}. \tag{1}$$

In Figure 7, $\mathfrak{R}(3)$ and its natural decomposition can be seen.

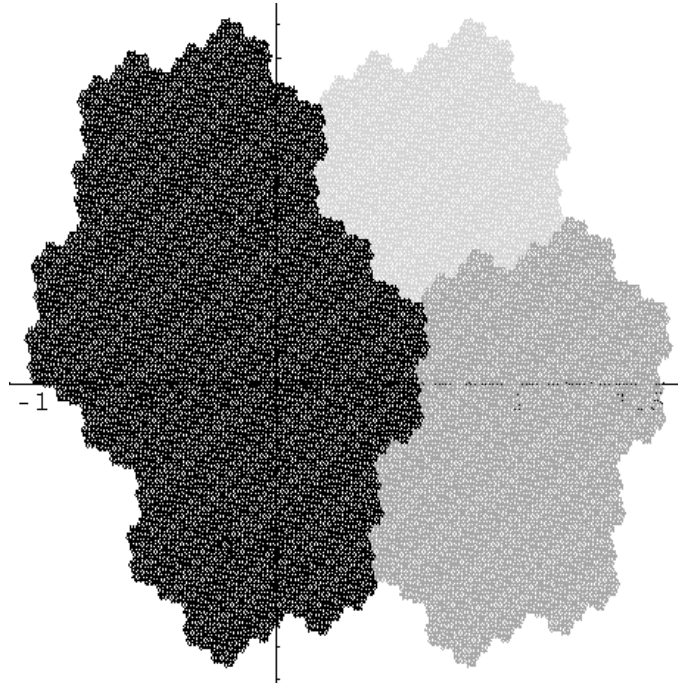


Figure 7: The Rauzy fractal $\mathfrak{R}(3)$ and its natural decomposition.

We define the map

$$\xi_k : \mathcal{R}(k) \rightarrow \mathbb{R}^{k-1}, \quad \mathbf{a} \mapsto \begin{cases} (\sum_{i \geq 0} a_i \gamma_1^i, \dots, \sum_{i \geq 0} a_i \gamma_s^i, \sum_{i \geq 0} a_i \gamma_{s+1}^i), & \text{if } k = 2s + 2; \\ (\sum_{i \geq 0} a_i \gamma_1^i, \dots, \sum_{i \geq 0} a_i \gamma_s^i), & \text{if } k = 2s + 1. \end{cases}$$

By construction $\mathfrak{R}(k) = \xi_k(\mathcal{R}(k))$.

The adic dynamical system $(\mathcal{R}(k), \mathcal{T}_k)$ allows us to define a dynamical system in

the Rauzy fractal $(\mathfrak{R}(k), F_k)$ such that the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{R}(k) & \xrightarrow{\mathcal{T}_k} & \mathcal{R}(k) \\
 \xi_k \downarrow & & \downarrow \xi_k \\
 \mathfrak{R}(k) & \xrightarrow{F_k} & \mathfrak{R}(k).
 \end{array} \tag{2}$$

The map F_k is defined by $F_k(x) = x + \vec{v}_i$ if $x \in \mathfrak{R}(k)_i$, where

$$\vec{v}_1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} \gamma_1 - 1 \\ \vdots \\ \gamma_{s+1} - 1 \end{pmatrix}, \quad \dots, \quad \vec{v}_k = \begin{pmatrix} \gamma_1^{k-1} - \gamma_1^{k-2} - \dots - \gamma_1 - 1 \\ \vdots \\ \gamma_{s+1}^{k-1} - \gamma_{s+1}^{k-2} - \dots - \gamma_{s+1} - 1 \end{pmatrix}.$$

Here $\vec{v}_j \in \mathbb{R}^{k-1}$, and we identify \mathbb{C}^s with \mathbb{R}^{2s} . We suppose for notational purposes that $k = 2s + 2$, with $s \geq 0$.

We say that the dynamical system (X, H) is a *piece exchange* or *domain exchange* if $X \subset \mathbb{R}^k$ and there exists a partition $\{X_1, \dots, X_m\}$ of X , i.e., $X = \cup_{i=1}^m X_i$ and their interiors in the relative topology are disjoint pairwise, such that H exchanges the elements of the partition by translations. So $(\mathfrak{R}(k), F_k)$ is a domain exchange transformation. See [1, 5, 17] for details of domain exchange transformations.

Theorem 3. *Let $\xi_k : \mathcal{R}(k) \rightarrow \mathfrak{R}(k)$ as before. Then $\xi_k(\widehat{\mathcal{R}(k)}) = \mathfrak{R}(k) = \xi_k(\mathcal{R}(k))$.*

Proof. Let us introduce some definitions and notation. Let $\widehat{\mathfrak{R}(k)} := \xi_k(\widehat{\mathcal{R}(k)})$. Let $\mathcal{R}^*(k)$ be the set of finite paths on the automaton $\mathfrak{A}(k)$, i.e.,

$$\mathcal{R}^*(k) := \{a_0 a_1 \dots a_n \in W^* : a_n \dots a_0 \text{ is a path in } \mathfrak{A}(k) \text{ for all } n \in \mathbb{N}\}$$

and $\widehat{\mathcal{R}^*(k)}$ is its dual. Let us remark that the map ξ_k is well-defined on elements of $\mathcal{R}^*(k)$ and $\widehat{\mathcal{R}^*(k)}$.

We give the proof for $k = 3$ in order to simplify the notation. But the ideas of the proof are the same for any k .

Let $a = a_0 \dots a_n \in \mathcal{R}^*$. We shall associate an element $b \in \widehat{\mathcal{R}^*}$ such that $\xi(a) = \xi(b)$. If $a \in \widehat{\mathcal{R}^*}$ then $b = a$. If $a \notin \widehat{\mathcal{R}^*}$, without loss of generality we can assume that $a_n = 1$. Let

$$l := \max_{0 \leq j \leq n} \{j : a_j = 1, a_{j-1} = 0, a_{j-2} = 0, a_{j-3} = 0\}.$$

So we define

$$b_{l-3} = 1, \quad b_{l-2} = 1, \quad b_{l-1} = 1, \quad b_l = 0, \quad b_{l+i} = a_{l+i}, \quad \text{for } 0 \leq i \leq n - l.$$

Now we repeat this process with the word $a_0 \dots a_{l-4}$, until $l-4 \geq 0$. By construction $b \in \widehat{\mathcal{R}^*}$.

Denote by γ any root of the polynomial $x^3 - x^2 - x - 1$. Since $1 + \gamma + \gamma^2 = \gamma^3$, we get $\xi(a) = \xi(b)$. Furthermore, since $1 + \sum_{j=0}^{\infty} \gamma^{3j+1} + \gamma^{3j+2} = 0$, we have $\xi(a(0)^\infty) = \xi(b1(110)^\infty)$ and $a0^\infty \in \mathcal{R}$, $b1(110)^\infty \in \widehat{\mathcal{R}}$. The elements $a0^\infty$, $b1(110)^\infty$ are dense in \mathcal{R} and $\widehat{\mathcal{R}}$, respectively. Therefore $\mathfrak{R} \subset \widehat{\mathfrak{R}}$, the reverse inclusion is proved in a similar manner. \square

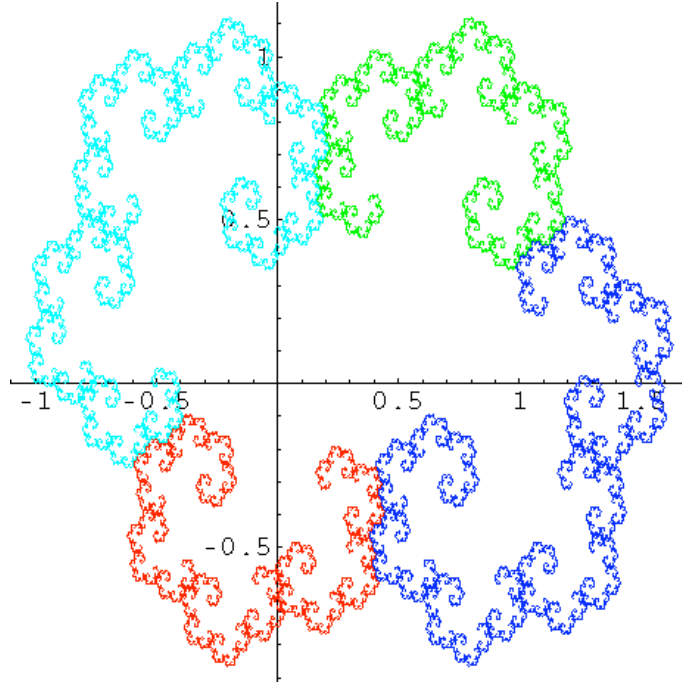


Figure 8: The set $\mathfrak{F}(3)$.

Let $\mathfrak{F}(k) := \xi_k(\mathcal{P}(k))$. Figure 8 shows the set $\mathfrak{F}(3)$ and its partition $\{\mathfrak{F}(3)_{00}, \mathfrak{F}(3)_{01}, \mathfrak{F}(3)_{11}, \mathfrak{F}(3)_{10}\}$, where $\mathfrak{F}(3)_{b_0 b_1} := \xi_3(\mathcal{P}(3)_{b_0 b_1})$ and $\mathcal{P}(3)_{b_0 b_1} := \{\mathbf{a} = a_0 a_1 \dots \in \mathcal{P}(3) : a_0 = b_0, a_1 = b_1\}$.

Corollary 4. *The sets $\mathfrak{R}(k)$ and $\mathfrak{F}(k)$ have the same center of symmetry.*

Proof. For notational purposes, let $k = 2s + 2$, with $s \geq 0$. Let $\mathbf{a} \in \mathcal{R}(k)$ or

$\mathbf{a} \in \mathcal{P}(k)$. Then

$$\begin{aligned} \xi_k(\mathbf{a}) + \xi_k(\hat{\mathbf{a}}) &= \left(\sum_{i \geq 0} a_i \gamma_1^i, \dots, \sum_{i \geq 0} a_i \gamma_s^i, \sum_{i \geq 0} a_i \gamma_{s+1}^i \right) \\ &\quad + \left(\sum_{i \geq 0} \hat{a}_i \gamma_1^i, \dots, \sum_{i \geq 0} \hat{a}_i \gamma_s^i, \sum_{i \geq 0} \hat{a}_i \gamma_{s+1}^i \right) \\ &= \left(\sum_{i \geq 0} a_i \gamma_1^i, \dots, \sum_{i \geq 0} a_i \gamma_s^i, \sum_{i \geq 0} a_i \gamma_{s+1}^i \right) \\ &\quad + \left(\sum_{i \geq 0} (1 - a_i) \gamma_1^i, \dots, \sum_{i \geq 0} (1 - a_i) \gamma_s^i, \sum_{i \geq 0} (1 - a_i) \gamma_{s+1}^i \right) \\ &= \left(\sum_{i \geq 0} \gamma_1^i, \dots, \sum_{i \geq 0} \gamma_s^i, \sum_{i \geq 0} \gamma_{s+1}^i \right) \\ &= \left(\frac{1}{1 - \gamma_1}, \dots, \frac{1}{1 - \gamma_s}, \frac{1}{1 - \gamma_{s+1}} \right). \end{aligned}$$

Since $\xi_k(\mathbf{a})$, and $\xi_k(\hat{\mathbf{a}})$ are in $\mathfrak{R}(k)$, or in $\mathfrak{F}(k)$, then the center of symmetry of these sets is $(1/2(1 - \gamma_1), \dots, 1/2(1 - \gamma_{s+1}))$ if $k = 2s + 2$ or $(1/2(1 - \gamma_1), \dots, 1/2(1 - \gamma_s))$ if $k = 2s + 1$. □

Corollary 5. *The center of symmetry of $\mathfrak{R}(k)$ and $\mathfrak{F}(k)$ is in $\mathfrak{R}(k)$, but not in $\mathfrak{F}(k)$.*

Proof. By Proposition 4, the center of symmetry of $\mathfrak{R}(k)$ and $\mathfrak{F}(k)$ is $(1/2(1 - \gamma_1), \dots, 1/2(1 - \gamma_{s+1}))$ if $k = 2s + 2$ or $(1/2(1 - \gamma_1), \dots, 1/2(1 - \gamma_s))$ if $k = 2s + 1$.

Since $1 + \gamma_i + \dots + \gamma_i^{k-1} = \gamma_i^k$, we have:

$$\frac{1}{2(1 - \gamma_i)} = \frac{1}{2} \sum_{j=0}^{\infty} \gamma_i^j = \frac{1}{2} (2\gamma_i^k + 2\gamma_i^{2k+1} + 2\gamma_i^{3k+2} + \dots) = \sum_{j=0}^{\infty} \gamma_i^{(k+1)j+k}.$$

So the center of symmetry of $\mathfrak{R}(k)$ is $\xi_k(\mathbf{b})$, where $\mathbf{b} = (\underbrace{0 \dots 0}_k 1)^\infty$. Clearly \mathbf{b} is in

$\mathcal{R}(k)$, so $\xi_k(\mathbf{b}) \in \mathfrak{R}(k)$.

It is clear that \mathbf{b} is not in $\mathcal{P}(k)$. In order to prove that $\xi_k(\mathbf{b}) \notin \mathfrak{F}(k)$, we shall show that the image under ξ_k of a neighbourhood of the point \mathbf{b} is disjoint from $\mathfrak{F}(k)$. For simplicity we assume that $k = 3$. Let

$$U_{\mathbf{b}} := \{\mathbf{a} \in \mathcal{R}(3) : a_3 = 1, a_i = 0 \text{ for } i = 0, 1, 2, 4, 5, 6\}$$

and $\mathcal{U} := \xi_3(U_{\mathbf{b}})$, so $\mathcal{U} = \gamma^3 + \gamma^7(\mathfrak{R}(3))$. We have $\sum_{i \geq 0} a_i \gamma^i = \sum_{i \geq 0} a'_i \gamma^i$ for $\mathbf{a}, \mathbf{a}' \in \mathcal{R}(3)$ if and only if we can pass from one expression to the other using the relation $1 + \gamma + \gamma^2 = \gamma^3$. Considering the elements of \mathcal{U} are of the form $\sum_{i \geq 0} a_i \gamma^i$ with $a_3 = 1$ and $a_i = 0$ for $i = 0, 1, 2, 4, 5, 6$, it is not possible to rewrite them in any of the following forms:

$$1 + \gamma + \gamma^4(x), \quad 1 + \gamma^2 + \gamma^4(x), \quad \gamma + \gamma^4(x), \quad \gamma^2 + \gamma^4(x),$$

where x is an element of $\mathfrak{R}(3)$, which are the allowed expressions for the elements of $\mathfrak{F}(3)$. Therefore \mathcal{U} and $\mathfrak{F}(3)$ are disjoint. \square

4.1. Some Remarks

- (i) From Theorem 3 and Corollary 4 it follows that the duality on the Rauzy fractal, $\mathfrak{R}(k)$ and on the subset $\mathfrak{F}(k)$, acts as a reflection in the center of symmetry.
- (ii) In the special case of $k = 3$, the boundary of the set $\mathfrak{R}(3)$ is contained in the $\mathfrak{F}(3)$, since the characterization of the boundary of $\mathfrak{R}(3)$ given in [16] does not admit points whose symbolic representation has more than three consecutives 0s. However, the inclusion is proper since the Hausdorff dimension of $\mathfrak{F}(3)$ is $2 \log \rho(2) / \log \rho(3) \approx 1.579$, where $\rho(k)$ is the root outside the unit circle of the polynomial $x^k - x^{k-1} - \dots - x - 1$ for $k = 2, 3$ [23]; and the Hausdorff dimension of the boundary of $\mathfrak{R}(3)$ is $2 \log \theta / \log \rho(3) \approx 1.093$ where θ is the root outside the unit circle of the polynomial $x^4 - 2x - 1$ [14, 24]. We conjecture that for $k \geq 4$, the boundary of $\mathfrak{R}(k)$ is properly contained in $\mathfrak{F}(k)$.
- (iii) The map $F_k : \mathfrak{R}(k) \rightarrow \mathfrak{R}(k)$, was defined on each element of the natural decomposition of $\mathfrak{R}(k)$. In a similar way, we will use the symbolic dynamics to define a map $S_k : \mathfrak{F}(k) \rightarrow \mathfrak{F}(k)$, such that it makes commutative the diagram:

$$\begin{array}{ccc}
 \mathcal{P}(k) & \xrightarrow{S_k} & \mathcal{P}(k) \\
 \xi_k \downarrow & & \downarrow \xi_k \\
 \mathfrak{F}(k) & \xrightarrow{S_k} & \mathfrak{F}(k).
 \end{array}$$

The set $\mathcal{P}(k)$ admits the natural decomposition

$$\left\{ \mathcal{P}(k)_{01}, \mathcal{P}(k)_{001}, \dots, \mathcal{P}(k)_{\underbrace{0\dots 0}_{k-1}}, \mathcal{P}(k)_{10}, \mathcal{P}(k)_{110}, \dots, \mathcal{P}(k)_{\underbrace{1\dots 1}_{k-1}} \right\},$$

where

$$\mathcal{P}(k)_{b_0\dots b_m} := \{ \mathbf{a} = a_0 a_1 \dots \in \mathcal{P}(k) : a_i = b_i, \text{ for } 0 \leq i \leq m \},$$

so the natural partition of $\mathfrak{F}(k)$ is given by the image under the map ξ_k of the elements of the natural decomposition of $\mathcal{P}(k)$.

From Theorem 1 we would expect that the dynamical system $(\mathfrak{F}(3), S_3)$ is semiconjugate to $(\mathfrak{R}(2), F_2)$; however, this is not true, since the map ϕ_3 does not preserves the identification of points under the map ξ_2 . For example, consider $\mathbf{a} = 001101101101101\dots$ and $\mathbf{a}' = 10101101101\dots$; they are mapped

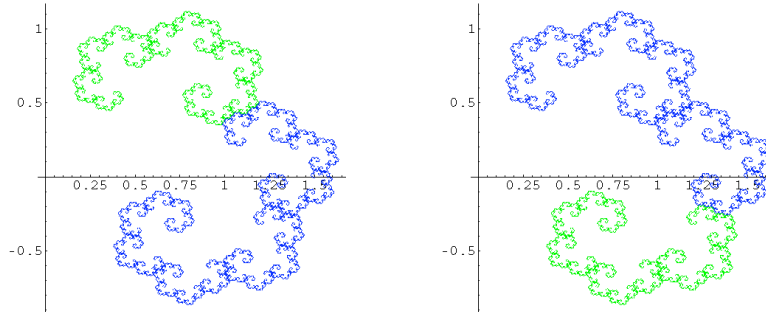


Figure 9: At the left the set $\widetilde{\mathfrak{F}}(3)$ and its partition; at the right the image of the partition under \widetilde{S}_3 .

to the same point in $\mathfrak{F}(3)$, $\xi_3(\mathbf{a}) = \xi_3(\mathbf{a}')$; cf. [24] or [16]. However $\phi_3(\mathbf{a}) = 10100100100\dots$ and $\phi_3(\mathbf{a}') = 00000100100\dots$, and their images under ξ_2 are different points in $\mathfrak{R}(2)$, since a necessary condition for points in $\mathfrak{R}(2)$ to be mapped to the same point under ξ_2 is that their sequences are eventually periodic of the form $01010101\dots$.

(iv) The involution Ψ allows us to define an equivalent relation in $\mathfrak{F}(3)$. In Theorem 2 of [16], necessary and sufficient conditions are given when two points of $\mathfrak{R}(3)$ are mapped to the same point in $\mathfrak{R}(3)$ under the map ξ_3 . These conditions are invariant under the involution Ψ , so if \mathbf{a} and $\mathbf{a}' \in \mathcal{P}(3)$, such that $\xi_3(\mathbf{a}) = \xi_3(\mathbf{a}')$, then $\xi_3(\Psi(\mathbf{a})) = \xi_3(\Psi(\mathbf{a}'))$. So the set $\widetilde{\mathfrak{F}}(3) := \xi_3(\mathcal{P}(3)/\Psi)$ and the map $\widetilde{S}_3 := S_3/\Psi$ are well-defined. The natural decomposition of $\widetilde{\mathfrak{F}}(3)$ induces a natural decomposition on the set $\widetilde{\mathfrak{F}}(3)$: $\{\widetilde{\mathfrak{F}}(3)_0, \widetilde{\mathfrak{F}}(3)_1\}$ where $\widetilde{\mathfrak{F}}(3)_0 = \xi_3(\mathcal{P}(3)_{01}/\Psi) = \xi_3(\mathcal{P}(3)_{10}/\Psi)$ and $\widetilde{\mathfrak{F}}(3)_1 = \xi_3(\mathcal{P}(3)_{00}/\Psi) = \xi_3(\mathcal{P}(3)_{11}/\Psi)$.

(v) The set $\mathcal{P}(3)/\Psi$ admits the partition $\{(\mathcal{P}(3)/\Psi)_0, (\mathcal{P}(3)/\Psi)_1\}$, where

$$(\mathcal{P}(3)/\Psi)_0 = \mathcal{P}(3)_{01}/\Psi = \mathcal{P}(3)_{10} / \Psi, \quad (\mathcal{P}(3)/\Psi)_1 = \mathcal{P}(3)_{00}/\Psi = \mathcal{P}(3)_{11} / \Psi.$$

These sets can be obtained as images of the maps $\mathcal{H}_j : \mathcal{P}(3) \rightarrow \mathcal{P}(3)$, for $j = 0, 1$, where

$$\mathcal{H}_0(\mathbf{a}) := \begin{cases} 0\Psi(\mathbf{a}) & \text{if } \mathbf{a} = 0\dots \\ 1\Psi(\mathbf{a}) & \text{if } \mathbf{a} = 1\dots \end{cases}, \quad \mathcal{H}_1(\mathbf{a}) := \begin{cases} 00\Psi(\mathbf{a}) & \text{if } \mathbf{a} = 0\dots \\ 11\Psi(\mathbf{a}) & \text{if } \mathbf{a} = 1\dots \end{cases}.$$

These maps are well-defined since $\mathcal{H}_j(\mathbf{a})$ is a path in the automaton $\mathfrak{A}(3) \times \widetilde{\mathfrak{A}}(3)$, for $\mathbf{a} \in \mathcal{P}(3)$. It can be easily checked that $\phi_3(\mathcal{H}_0(\mathbf{a})) = 0\phi_3(\mathbf{a})$ and

$\phi_3(\mathcal{H}_1(\mathbf{a})) = 10\phi_3(\mathbf{a})$. So the maps $\{\phi_3 \circ \mathcal{H}_0, \phi_3 \circ \mathcal{H}_1\}$ define the iterated function system (IFS) [9], whose fixed point is $\mathfrak{R}(2)$ [23]. This construction generalizes in a straightforward manner to $\mathcal{P}(k)$, for $k \geq 3$.

The partition $\{\widetilde{\mathfrak{F}}(3)_0, \widetilde{\mathfrak{F}}(3)_1\}$ of the set $\widetilde{\mathfrak{F}}(3)$ given in remark (iv) can be obtained as follows: $\widetilde{\mathfrak{F}}(3)_j = G_j(\mathcal{P}(3))$, where $G_j = \xi_3 \circ \mathcal{H}_j$, for $j = 0, 1$. Using techniques of fractal geometry it can be proved that $\mathcal{M}_s(\widetilde{\mathfrak{F}}(3)_0)/\mathcal{M}_s(\widetilde{\mathfrak{F}}(3)_1) = (\sqrt{5} - 1)/2$, where \mathcal{M}_s is the s -dimensional Hausdorff measure and s is the Hausdorff measure of $\widetilde{\mathfrak{F}}(3)$; for more details see [23]. Hence the ratio $\mathcal{M}_s(\widetilde{\mathfrak{F}}(3)_0)/\mathcal{M}_s(\widetilde{\mathfrak{F}}(3)_1)$ is equal to $m(\mathfrak{R}(2)_0)/m(\mathfrak{R}(2)_1)$, where m is the 1-dimensional Lebesgue measure.

Proposition 6. *The dynamical system $(\widetilde{\mathfrak{F}}(3), \widetilde{S}_3)$ exchanges the elements of the partition $\{\widetilde{\mathfrak{F}}(3)_0, \widetilde{\mathfrak{F}}(3)_1\}$.*

Proof. Let

$$\begin{aligned} \widetilde{\mathfrak{F}}(3)_{00} &:= \xi_3(\mathcal{P}(3)_{010}/\Psi) = \xi_3(\mathcal{P}(3)_{101}/\Psi) \quad \text{and} \\ \widetilde{\mathfrak{F}}(3)_{01} &:= \xi_3(\mathcal{P}(3)_{011}/\Psi) = \xi_3(\mathcal{P}(3)_{100}/\Psi). \end{aligned}$$

So $\{\widetilde{\mathfrak{F}}(3)_{00}, \widetilde{\mathfrak{F}}(3)_{01}\}$ is a partition of $\widetilde{\mathfrak{F}}(3)_0$. By the symbolic dynamics, it follows that

$$\widetilde{S}_3(\widetilde{\mathfrak{F}}(3)_{00}) = \widetilde{\mathfrak{F}}(3)_{10}, \quad \widetilde{S}_3(\widetilde{\mathfrak{F}}(3)_{01}) \subset \widetilde{\mathfrak{F}}(3)_{00} \quad \text{and} \quad \widetilde{S}_3(\widetilde{\mathfrak{F}}(3)_{10}) \subset \widetilde{\mathfrak{F}}(3)_0.$$

So the action of \widetilde{S}_3 on the natural decomposition of $\widetilde{\mathfrak{F}}(3)$ is a permutation of its elements. □

Using the same ideas we can generalize the previous result to $\widetilde{\mathfrak{F}}(k)$, for $k \geq 3$. So in a similar manner we define the dynamical system $(\widetilde{\mathfrak{F}}(k), \widetilde{S}_k)$.

Proposition 7. *Let $k \geq 3$. The dynamical system $(\widetilde{\mathfrak{F}}(k), \widetilde{S}_k)$ acts as a permutation on the $k - 1$ pieces of the natural partition of $\widetilde{\mathfrak{F}}(k)$.*

While we have shown that $(\widetilde{\mathfrak{F}}(3), \widetilde{S}_3)$ is not semi-conjugate to the dynamical system $(\mathfrak{R}(2), F_2)$; hence $(\widetilde{\mathfrak{F}}(3), \widetilde{S}_3)$ is not conjugate to $(\mathfrak{R}(2), F_2)$. We pointed out the behaviour of the system $(\widetilde{\mathfrak{F}}(3), \widetilde{S}_3)$ “resembles” the system $(\mathfrak{R}(2), F_2)$, from the metrical and dynamical point of view.

- (vi) We would like to remark that if we consider the automaton of paths of length l , with $l > 1$, of the k -bonacci system, we obtain the same set $\mathcal{P}(k)$ as the maximal invariant subset of $\mathcal{R}(k)$ under the mirror involution. In fact: Let

$\mathfrak{C}(l, k)$ be the automaton of paths of length l from $\mathfrak{A}(k)$, the k -bonacci automaton. The set of infinite paths on $\mathfrak{C}(l, k)$ is the same set as the set of infinite paths on $\mathfrak{A}(k)$, i.e., $\mathcal{R}(k)$. The transition matrix of the automaton $\mathfrak{C}(k - 1, k)$ is calculated in [22]. The labels of $\mathfrak{C}(l, k)$ are words of length l in $W = \{0, 1\}$. So we can consider the dual automaton of $\mathfrak{C}(l, k)$ and get its canonical product. Let $\widehat{\mathcal{P}(l, k)}$ be the set of infinite paths on the canonical product of $\mathfrak{C}(l, k)$. Since $\widehat{\mathcal{P}(l, k)}$ and $\mathcal{P}(k)$ are, by construction, the maximal invariant set of $\mathcal{R}(k)$ under the the mirror involution, we get $\widehat{\mathcal{P}(l, k)} = \mathcal{P}(k)$.

- (vii) Here we shall consider the domain exchange transformation on $\widehat{\mathfrak{A}(k)}$ and its relation on the domain exchange transformation on $\mathfrak{A}(k)$.

Let $\{\mathfrak{A}(k)_1, \dots, \mathfrak{A}(k)_k\}$ be the natural decomposition of $\mathfrak{A}(k)$ defined in (1), i.e., $\mathfrak{A}(k)_j = \xi_k(\mathcal{R}(k)_j)$, where

$$\mathcal{R}(k)_j := \left\{ \mathbf{a} \in \mathcal{R}(k) : \mathbf{a} = \underbrace{1 \dots 1}_{j-1} 0 \dots \right\}.$$

Let $\widehat{\mathfrak{A}(k)}_j := \xi_k(\widehat{\mathcal{R}(k)}_j)$, so the natural decomposition of $\widehat{\mathfrak{A}(k)}$ is $\{\widehat{\mathfrak{A}(k)}_1, \dots, \widehat{\mathfrak{A}(k)}_k\}$. Let $\psi_k : \mathbb{R}^{k-1} \rightarrow \mathbb{R}^{k-1}$ be the reflection map by the center of symmetry of $\mathfrak{A}(k)$, mentioned in Remark (i), i.e., $\psi_k(x) = 2\omega_k - x$, where ω_k is the center of symmetry of $\mathfrak{A}(k)$ described in Corollary 4. According to the proof of this corollary, $\xi_k(\hat{\mathbf{a}}) = 2\omega_k - \xi_k(\mathbf{a}) = \psi_k(\xi_x(\mathbf{a}))$, so $\widehat{\mathfrak{A}(k)}_j = \psi_k(\mathfrak{A}(k)_j)$.

In Section 4 we introduced F_k , the piece exchange map on $\mathfrak{A}(k)$, so that the diagram (2) commutes. Since in $\widehat{\mathcal{R}(k)}$ there is an adic map \widehat{T}_k , it induces a map on $\widehat{\mathfrak{A}(k)}$ so that the following diagram is commutative:

$$\begin{array}{ccc} \widehat{\mathcal{R}(k)} & \xrightarrow{\widehat{T}_k} & \widehat{\mathcal{R}(k)} \\ \xi_k \downarrow & & \downarrow \xi_k \\ \widehat{\mathfrak{A}(k)} & \xrightarrow{\widehat{F}_k} & \widehat{\mathfrak{A}(k)}. \end{array}$$

In Section 2 we pointed out that the dual adic map relates to the adic map, so that $\mathcal{T}(\widehat{\mathcal{T}(\hat{\mathbf{a}})}) = \mathbf{a}$, or equivalently $\widehat{\mathcal{T}}(\widehat{\mathcal{T}(\mathbf{a})}) = \hat{\mathbf{a}}$. Hence, $F_k(\psi_k(\widehat{F}_k(\psi_k(x)))) = x$ for any $x \in \mathfrak{A}(k)$. Therefore $\widehat{F}_k(\widehat{\mathfrak{A}(k)}_j) = \psi_k(F_k^{-1}(\mathfrak{A}(k)_j))$. We can conclude that the map \widehat{F}_k is a piece exchange on the decomposition $\{\widehat{\mathfrak{A}(k)}_1, \dots, \widehat{\mathfrak{A}(k)}_k\}$ of $\widehat{\mathfrak{A}(k)}$.

By Theorem 3 we have that $\widehat{\mathfrak{R}(k)} = \mathfrak{R}(k)$. We can conclude that there are two decompositions on this set, that define different piece exchange transformations. They are related by the symmetry map ψ_k .

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