

## EXACT ENUMERATION OF GARDEN OF EDEN PARTITIONS

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*Received: 11/18/05, Accepted: 5/22/06*

### Abstract

We give two proofs for a formula that counts the number of partitions of  $n$  that have rank  $-2$  or less (which we call Garden of Eden partitions). These partitions arise naturally in analyzing the game Bulgarian solitaire, summarized in Section 1. Section 2 presents a generating function argument for the formula based on Dyson's original paper where the rank of a partition is defined. Section 3 gives a combinatorial proof of the result, based on a bijection on Bressoud and Zeilberger.

### 1. Bulgarian Solitaire

Let  $\lambda$  be a partition of the integer  $n$  having  $t$  parts written  $(\lambda_1, \dots, \lambda_t)$  in non-increasing order. "Bulgarian solitaire" is based on a function defined on partitions of  $n$  as follows:

$$D(\lambda) = (t, \lambda_1 - 1, \dots, \lambda_t - 1)$$

where any zeroes are omitted and the parts may need to be reordered to be non-increasing. In terms of the partition's Ferrers diagram,  $D$  moves the leftmost column of dots to a row in the image partition. This shift function induces a finite dynamical system on  $P(n)$ , the collection of partitions of  $n$ .

This subject came into the literature in 1982, with Brandt counting the number of partitions in cycles of each dynamical system and also the number of connected components in each system [1]. Gardner popularized the game in 1983 [4] and various aspects and mod-

ifications were addressed in several subsequent articles. See Hopkins and Jones [5] for an overview, some new results, and discussion of outstanding problems.

Here we are concerned with partitions having no preimage under  $D$ . Following the terminology of cellular automata and combinatorial game theory, we call these Garden of Eden partitions, abbreviated GE-partitions. These can be characterized using Dyson’s notion of rank [3]. Using the notation above, the rank of the partition  $\lambda = (\lambda_1, \dots, \lambda_t)$  is defined as  $\text{rank}(\lambda) = \lambda_1 - t$ .

We now state and prove a lemma which provides a key characterization of Garden of Eden partitions.

**Lemma** *A partition  $\lambda$  is a Garden of Eden partition exactly when  $\text{rank}(\lambda) \leq -2$ .*

*Proof* A partition  $\lambda = (\lambda_1, \dots, \lambda_t)$  has a preimage for each distinct  $\lambda_i \geq t-1$ , in particular

$$D((\lambda_1 + 1, \dots, \lambda_{i-1} + 1, \lambda_{i+1} + 1, \dots, \lambda_t + 1, 1^{\lambda_i - t + 1})) = \lambda$$

where the exponent  $\lambda_i - t + 1$  denotes the number of ones. This fails when the largest part satisfies  $\lambda_1 - t + 1 < 0$ , i.e., when  $\text{rank}(\lambda) = \lambda_1 - t \leq -2$ .

For example,  $(4, 4, 3, 3, 2, 2, 2) \in P(20)$  is a GE-partition, as its rank is  $-3$ ; in terms of the Ferrers diagram, no row is long enough to be moved to an initial column as there would be six remaining parts and, therefore, this partition has no preimage under the  $D$  map.

Let  $GE(n)$  denote the set of GE-partitions of  $n$  and let  $ge(n)$  be the cardinality of  $GE(n)$ . By the lemma, determining  $ge(n)$  is equivalent to counting the number of elements of  $P(n)$  with rank  $-2$  or less. GE-partitions, in addition to being partitions distinguished by the shift map, are the entryways to the dynamic system of  $P(n)$ . That is, starting from the GE-partitions leads to every partition in  $P(n)$  (for  $n \geq 3$ ), no matter how many components the system has (the dynamical systems for  $n = 1, 2$  consist of only cycle partitions, and these are the only isolated cycles; see [5] for details).

Our goal in the remainder of the paper is to prove the following theorem which satisfies the request made in [5] that an exact formula be found for  $ge(n)$ . Write  $p(n)$  for the cardinality of  $P(n)$ .

**Theorem** *For all  $n \geq 1$ ,*

$$ge(n) = p(n - 3) - p(n - 9) + p(n - 18) \cdots = \sum_{j \geq 1} (-1)^{j+1} p(n - b(j)) \tag{1}$$

where  $b(j) = (3j^2 + 3j)/2$  is three times the  $j^{\text{th}}$  triangular number.

Note that the right-hand side of (1) is finite: if  $b(j) > n$ , then  $p(n - b(j)) = 0$ .

We provide two proofs of the theorem below, one based on generating functions, another

combinatorial in nature.

## 2. Generating Function Proof

We begin this section with a generating function result that Dyson proves in his original paper on the rank of a partition [3]. Namely, he shows that the generating function for the number of partitions of  $n$  with rank equal to a fixed integer  $m$  is

$$\left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} (-1)^{r-1} \left(q^{(3r^2-r)/2} - q^{(3r^2+r)/2}\right) q^{mr}.$$

Next, we note that  $ge(n)$  is equivalent to counting the elements of  $P(n)$  with rank 2 or more (which can be readily seen via conjugation of partitions). Therefore, we know that the generating function for  $ge(n)$  is given by

$$\sum_{n=0}^{\infty} ge(n)q^n = \sum_{m=2}^{\infty} \left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} (-1)^{r-1} \left(q^{(3r^2-r)/2} - q^{(3r^2+r)/2}\right) q^{mr}.$$

We can simplify this sum to obtain the following:

$$\begin{aligned} \sum_{n=0}^{\infty} ge(n)q^n &= \sum_{m=2}^{\infty} \left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} (-1)^{r-1} \left(q^{(3r^2-r)/2} - q^{(3r^2+r)/2}\right) q^{mr} \\ &= \left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} (-1)^{r-1} \left(q^{(3r^2-r)/2} - q^{(3r^2+r)/2}\right) \sum_{m=2}^{\infty} q^{mr} \\ &= \left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} (-1)^{r-1} \left(q^{(3r^2-r)/2} - q^{(3r^2+r)/2}\right) \left(\frac{1}{1-q^r} - 1 - q^r\right) \\ &= \left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} (-1)^{r-1} \left(q^{(3r^2-r)/2} - q^{(3r^2+r)/2}\right) \left(\frac{q^{2r}}{1-q^r}\right) \\ &= \left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{1-q^r} \left(q^{(3r^2+3r)/2} - q^{(3r^2+5r)/2}\right) \\ &= \left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{1-q^r} q^{(3r^2+3r)/2} (1-q^r) \\ &= \left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} (-1)^{r-1} q^{(3r^2+3r)/2}. \end{aligned}$$

Thus, we have

$$\sum_{n=0}^{\infty} ge(n)q^n = \left( \prod_{i=1}^{\infty} \frac{1}{1-q^i} \right) \sum_{r=1}^{\infty} (-1)^{r-1} q^{(3r^2+3r)/2}. \tag{2}$$

Since

$$\prod_{i=1}^{\infty} \frac{1}{1-q^i} = \sum_{n=0}^{\infty} p(n)q^n$$

is the generating function for  $p(n)$ , comparison of the coefficients of  $q^n$  on each side of (2) yields the result of the theorem and the proof is complete.

We close this section by noting that similar generating function manipulations can be used to obtain formulas for the numbers of partitions with positive rank and nonnegative rank. Other values, such as all partitions with rank 3 or more, work out to involve values  $p(n - a)$  where the  $a$  parameters come from two quadratic sequences.

### 3. Combinatorial Proof

In this section, we provide an alternate proof of the theorem using a bijection on certain partitions. We consider a map

$$\varphi : \left( GE(n) \cup \bigcup_{j \geq 2, \text{ even}} P(n - b(j)) \right) \iff \bigcup_{j \text{ odd}} P(n - b(j))$$

detailed below. Notice that  $GE(n) \subset P(n)$ , which corresponds to  $j = 0$ . For  $\lambda \in P(n - b(j))$  or  $\lambda \in GE(n)$  with  $j = 0$ , let  $\lambda = (\lambda_1, \dots, \lambda_t)$  and define  $\varphi$  by

$$\varphi(\lambda) = \begin{cases} (\lambda_2 + 1, \dots, \lambda_t + 1, 1^{\lambda_1 - t + 3j + 1}) \in P(n - b(j - 1)) & \text{if } \lambda_1 \geq t - 3j - 1 \\ (t - 3j + 3, \lambda_1 - 1, \dots, \lambda_t - 1) \in P(n - b(j + 1)) & \text{if } \lambda_1 \leq t - 3j - 2. \end{cases}$$

By the lemma,  $GE(n)$  consists of the partitions of  $n$  with  $\lambda_1 - t \leq -2$ , so  $\varphi$  sends all of these into  $P(n - 3)$ . The map is defined for all other partitions in all  $P(n - b(j))$ , and applying  $\varphi$  twice gives the identity map.

Consider the case  $n = 20$ . The map gives a bijection between  $GE(20) \cup P(11)$  and  $P(17) \cup P(2)$ . In particular,  $\varphi$  sends  $GE(20)$  into  $P(17)$ , sends some of  $P(17)$  onto  $GE(20)$  and the rest into  $P(11)$ , etc. It is interesting to notice the similarities between  $\varphi$  and the shift map  $D$ . For example, consider  $(4, 4, 3, 3, 2, 2, 2) \in GE(20)$ , so  $j = 0$ .

$$D((4, 4, 3, 3, 2, 2, 2)) = (7, 3, 3, 2, 2, 1, 1, 1) \in P(20),$$

$$\varphi((4, 4, 3, 3, 2, 2, 2)) = (4, 3, 3, 2, 2, 1, 1, 1) \in P(17).$$

The images of  $D$  and  $\varphi$  differ by  $3(j+1) = 3$  in the first row. For an example of the other case of  $\varphi$ , consider  $(6, 5) \in P(11)$ , so  $j = 2$ .

$$D((6, 1^5)) = (6, 5) \text{ with } (6, 1^5) \in P(11), \varphi((6, 5)) = (6, 1^{11}) \in P(17).$$

The image of  $\varphi$  and the given preimage under  $D$  differ by  $3j = 6$  in the first column. A similar relation to  $D$  holds for a bijection of Bressoud and Zeilberger [2] upon which  $\varphi$  is based (their  $\phi$  gives a bijective proof of Euler's pentagonal number theorem).

Similar bijections can be detailed to give combinatorial proofs for the numbers of partitions with positive rank and nonnegative rank mentioned at the end of the previous section.

## Acknowledgments

We would like to thank Michael A. Jones for continued discussion of Bulgarian solitaire. Also, the authors appreciate Bruce Landman's organization of the 2005 INTEGERS Conference, where this collaboration began.

## References

- [1] Jørgen Brandt. Cycles of partitions, Proc. Amer. Math. Soc. 85 (1982) 483-486.
- [2] David Bressoud and Doron Zeilberger. Bijectioning Euler's partitions-recurrence, Amer. Math. Monthly 92 (1985) 54-55.
- [3] Freeman Dyson. Some guesses in the theory of partitions, Eureka 8 (1944) 10-15. Also available in *Selected Papers of Freeman Dyson: With Commentary*, American Mathematical Society 1986.
- [4] Martin Gardner. Bulgarian solitaire and other seemingly endless tasks, Sci. Amer. 249 (1983) 12-21.
- [5] Brian Hopkins and Michael A. Jones. Shift-induced dynamical systems on partitions and compositions, submitted.