INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 7(2) (2007), #A31

THE GAME OF TAKE TURN

Kennan Shelton[1]
kennan.shelton@gmail.com

Received: 12/28/05, Revised: 9/7/06, Accepted: 9/11/06

Abstract

We introduce the game of Take Turn, a vertex-deletion game played with coins placed on vertices of a graph. A move is to remove a heads-up coin and turn over any coins adjacent from the removed coin; the loser is the player who cannot remove a coin. We relate the Sprague-Grundy values for games played on directed paths and cycles to the octal game .37 and examine cases for games played on undirected paths. We also show that Take Turn played on directed graphs is PSPACE-Complete.

1. Introduction

The game of *Take Turn* is played on a graph (directed or undirected) with coins placed on the vertices of the graph. A move consists of removing a heads-up coin at some vertex v and turning over the coins on vertices adjacent from v. The loser is the player who cannot make a move. A game (or position) of Take Turn consists of the underlying graph together with the particular assignment of coin states to the vertices (heads-up or tails-up).

An example sequence of play is shown in Figure 1. The first player removes the coin at vertex 1, the second player responds by removing the coin at vertex 3, leaving no move for the first player who thus loses the game. In fact, the second player to move from this position is always able to win the game. A more interesting example starts with the position shown in Figure 2, from which the first player may force a win.

We assume that the reader is familiar with the well-established Sprague-Grundy theory of impartial games (see, for example, [1] or [4]). In this paper we will denote the Sprague-Grundy value of game A by $g(A)$, the disjunctive sum of games A and B by $A+B$, and the nim-sum of nonnegative integers a and b by $a \oplus b$ (the XOR of the binary representations of a and b). The set of games for which the first player may force a win ($g(A) > 0$) will be denoted

[1]The author's work was completed while a visiting assistant professor in the Department of Mathematics, University of Central Arkansas.

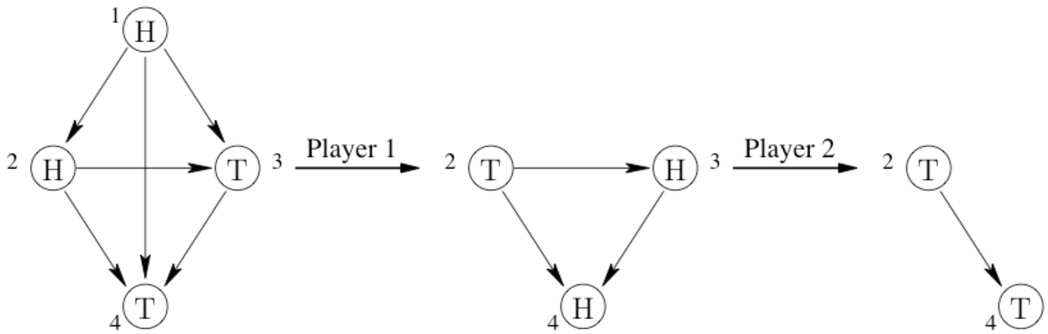

Figure 1: A sample Take Turn game

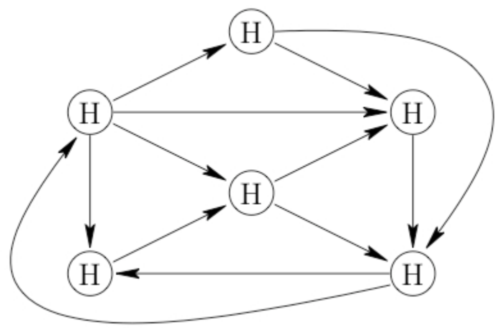

Figure 2: A winning position for Player 1

$\mathcal{N}$, while its complement will be denoted $\mathcal{P}$. Finally, mex denotes the minimally-excluded function—mex(A) is the smallest nonnegative integer not in A.

We will first investigate Take Turn played on certain directed and undirected graphs such as cycles, paths and complete bipartite graphs. Though we are able to find formulas for games played on undirected paths with particular starting positions, we will show that for Take Turn games played on undirected paths the set $\mathcal{P}$ is not recognizable by a finite automaton. We will also show that the problem of determining whether a given game on a directed graph is in $\mathcal{N}$ is PSPACE-Complete.

Take Turn is an example of a *vertex deletion* game. Nowakowski and Ottaway [5, 6] have examined a class of vertex deletion games with parity rules; these correspond in a natural way to special cases of Take Turn. We briefly explore this connection in the last section.

2. Take Turn on Directed Graphs

In this section we consider two types of directed graphs: paths and cycles. In both cases, we assume that all edges point in the same direction. While we do not find a closed-form solution for Sprague-Grundy values of general path or cycle graphs, we are able to relate the

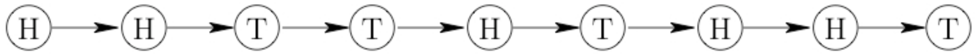

Figure 3: Take Turn on a Directed Path

values to those of the octal game **.37**.

The game **.37** is played with a heap of n beans in which players remove either one or two beans at a time. A player may remove one bean only if it is the last bean or any remaining beans are left in one heap, and the player may remove two beans and leave any remaining beans in either one or two heaps. The values of **.37** for $0 \leq n \leq 40$ ([1]) are

01201 23123 40342 13210 21451 45120 12312 34234 2.

It is not known if **.37** is ultimately periodic though values have been calculated for heaps of size up to 2^{29} [2].

2.1 Directed paths

Consider directed paths in which all edges point from left to right, such as in Figure 3. We will denote these games with an arrow placed over the configuration of coins; the game in Figure 3 is $\overrightarrow{\mathrm{HHTTHTHHT}}$. Note that in these games, removing a coin will only affect the coin immediately to the right, so any tails up coins on the left of a configuration may be ignored.

Theorem 1. The Take Turn game $\overrightarrow{\mathrm{H}^n}$ is the octal game **.37**.

Proof. The n coins in the game $\overrightarrow{\mathrm{H}^n}$ correspond to a heap of n beans. Removing the far right coin will remove only that coin and not turn over any others, leaving any remaining coins in one heap. Removing any other coin will turn the coin to the right to tails, which then becomes the far left coin of a new game. Since the tails up coin will never turn back to heads up, it is effectively removed as well. Thus removing a coin other than the far right coin results in removing *two* coins and leaves the remaining coins in at most two heaps. This is exactly the game **.37**. □

Games played on directed paths with an arbitrary assignment of coin states may be reduced to sums of games of the form $\overrightarrow{\mathrm{H}^n}$ and then to sums of positions of **.37**.

Lemma 2. Let A be a (possibly empty) sequence of coins and let $n \geq 0$. Then

1. $\overrightarrow{\mathrm{A} \cdot \mathrm{TT} \cdot \mathrm{H}^n} = \overrightarrow{\mathrm{A}} + \overrightarrow{\mathrm{H}^n}$
2. $\overrightarrow{\mathrm{A} \cdot \mathrm{HT} \cdot \mathrm{H}^n} = \overrightarrow{\mathrm{A}} + \overrightarrow{\mathrm{H}^n}$

Proof. Note that the game $\overrightarrow{\mathrm{H}^{n+1}} + \overrightarrow{\mathrm{H}^n}$ has positive value since it has a next option of $\overrightarrow{\mathrm{H}^n} + \overrightarrow{\mathrm{H}^n}$. It is then easy to see that $\overrightarrow{\mathrm{HTH}^n} + \overrightarrow{\mathrm{H}^n} = 0$. To establish either formula, consider the sum of the games. For the first formula, the second player may always copy the move of the first player in either the $\overrightarrow{\mathrm{A}}$ or $\overrightarrow{\mathrm{H}^n}$ component, eventually leading to a position of the form $\overrightarrow{\mathrm{B}} + \overrightarrow{\mathrm{HTH}^k} + \overrightarrow{\mathrm{B}} + \overrightarrow{\mathrm{H}^k}$ which has value 0. For the second formula, the second player may again copy the first player's moves, except when the first player removes the H just after the sequence $\overrightarrow{\mathrm{A}}$, leaving $\overrightarrow{\mathrm{A}} + \overrightarrow{\mathrm{H}^{n+1}} + \overrightarrow{\mathrm{A}} + \overrightarrow{\mathrm{H}^n}$. But this option has positive value, so the second player has a winning response. In both cases, the sum must be 0 and so the games are equal. □

Applying Lemma 2 repeatedly to $\overrightarrow{\mathrm{HHTTHTHHT}}$ of Figure 3 we have

$$\begin{aligned}\overrightarrow{\mathrm{HHTTHTHHT}} &= \overrightarrow{\mathrm{HHTTHTH}} + \overrightarrow{\mathrm{H}^0}\\ &= \overrightarrow{\mathrm{HHTT}} + \overrightarrow{\mathrm{H}^1} + \overrightarrow{\mathrm{H}^0}\\ &= \overrightarrow{\mathrm{H}^2} + \overrightarrow{\mathrm{H}^0} + \overrightarrow{\mathrm{H}^1} + \overrightarrow{\mathrm{H}^0}\\ &= \overrightarrow{\mathrm{H}^2} + \overrightarrow{\mathrm{H}}\end{aligned}$$

and so the game has value $2 \oplus 1 = 3$ (using the known values of **.37**). Similarly, the game $\overrightarrow{\mathrm{H}^7\mathrm{T}^9\mathrm{H}^{16}\mathrm{T}^7\mathrm{H}^{10}\mathrm{T}^4}$ is equal to $\overrightarrow{\mathrm{H}^6} + \overrightarrow{\mathrm{H}^{15}} + \overrightarrow{\mathrm{H}^{10}}$ which has value $3 \oplus 1 \oplus 4 = 6$.

For a nonnegative integer m, let $\pi(m)$ denote the *parity* of m ($\pi(m) = 0$ if m is even and 1 if m is odd). Applying Lemma 2 to both break up and recombine a Take Turn position on a directed path we have

Theorem 3. For $n \geq 1$, let $h_1, \dots, h_n$ and $t_1, \dots, t_{n-1}$ be positive integers and suppose $t_n \geq 0$. Then

$$\begin{aligned}\overrightarrow{\mathrm{H}^{h_1}\mathrm{T}^{t_1}\mathrm{H}^{h_2}\mathrm{T}^{t_2}\cdots\mathrm{H}^{h_n}\mathrm{T}^{t_n}} &= \overrightarrow{\mathrm{H}^{h_1-\pi(t_1)}} + \overrightarrow{\mathrm{H}^{h_2-\pi(t_2)}} + \cdots + \overrightarrow{\mathrm{H}^{h_n-\pi(t_n)}}\\ &= \overrightarrow{\mathrm{H}^{h_1}\mathrm{T}^{t_1}} + \overrightarrow{\mathrm{H}^{h_2}\mathrm{T}^{t_2}} + \cdots + \overrightarrow{\mathrm{H}^{h_n}\mathrm{T}^{t_n}}\\ &= \overrightarrow{\mathrm{H}^{h_1}\mathrm{T}^{t_1}\cdots\mathrm{H}^{h_k}\mathrm{T}^{t_k}} + \overrightarrow{\mathrm{H}^{h_{k+1}}\mathrm{T}^{t_{k+1}}\cdots\mathrm{H}^{h_n}\mathrm{T}^{t_n}}\end{aligned}$$

where $1 \leq k \leq n-1$.

2.2 Directed Cycles

Consider a Take Turn game played on a directed cycle, such as in Figure 4, which we denote by $\mathcal{C}(\mathrm{HHTHTH})$. Note that we could also express the game as $\mathcal{C}(\mathrm{HHHTHT})$. In fact, any cycle that contains at least one heads-up coin and at least one tails-up coin may be expressed in the form $\mathcal{C}(\mathrm{H}\cdots\mathrm{T})$.

In the case of the cyclic game $\mathcal{C}(\mathrm{H}^n)$, it is immediately obvious that the only next option from $\mathcal{C}(\mathrm{H}^n)$ is $\overrightarrow{\mathrm{H}^{n-2}}$ and so

$$g(\mathcal{C}(\mathrm{H}^n)) = \begin{cases} 1 & \text{if } g\left(\overrightarrow{\mathrm{H}^{n-2}}\right) = 0\\ 0 & \text{otherwise.}\end{cases}$$

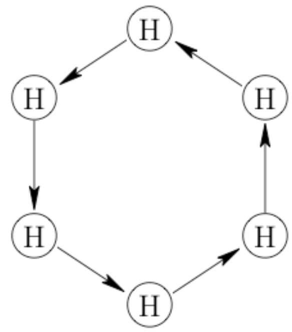

Figure 4: Take Turn on a Directed Cycle

Games on directed cycles containing at least one tails-up coin are equivalent to games on directed paths.

Theorem 4. For $n \geq 1$, let $h_1, \ldots, h_n$ and $t_1, \ldots, t_n$ be positive integers. Then

$$\mathcal{C}\left(\mathrm{H}^{h_1}\mathrm{T}^{t_1} \cdots \mathrm{H}^{h_n}\mathrm{T}^{t_n}\right) = \overrightarrow{\mathrm{H}^{h_1}\mathrm{T}^{t_1} \cdots \mathrm{H}^{h_n}\mathrm{T}^{t_n}}.$$

Proof. Denote the kth heads-tail group $\mathrm{H}^{h_k}\mathrm{T}^{t_k}$ by B_k. Suppose first that $t_n \geq 2$. We will show that the values of the next options of $\mathcal{C}(B_1 \cdots B_n)$ and of $\overrightarrow{B_1 \cdots B_n}$ are equal, so the original positions have the same value.

Removing a heads-up coin from B_k in $\mathcal{C}(B_1 \cdots B_n)$ produces a position of the form $\overrightarrow{C_1 B_{k+1} \cdots B_n B_1 \cdot B_{k-1} C_2}$ if $1 \leq k < n$ and of the form $\overrightarrow{C_1 B_1 \cdots B_{n-1} C_2}$ if $k = n$, where $C_1 = \mathrm{H}^p\mathrm{T}^q$ and $C_2 = \mathrm{H}^r$ for some p, q and r (which depend on the values of h_k and t_k and which coin was removed). Note that if $k = n$ then $q \geq 1$ since $t_n \geq 2$.

Using Theorem 3, we can write these positions as $\overrightarrow{B_1 \cdots B_{k-1} C_2} + \overrightarrow{C_1 B_{k+1} \cdots B_n}$ or $\overrightarrow{B_1 \cdots B_{n-1} C_2} + \overrightarrow{C_1}$. But these are exactly the next options of $\overrightarrow{B_1 \cdots B_n}$.

Now suppose $t_n = 1$. We will show that in the sum of the games $\mathcal{C}\left(\mathrm{H}^{h_1}\mathrm{T}^{t_1} \cdots \mathrm{H}^{h_n}\mathrm{T}\right)$ and $\overrightarrow{\mathrm{H}^{h_1}\mathrm{T}^{t_1} \cdots \mathrm{H}^{h_n}\mathrm{T}}$ the second player always has a winning response. Note that by Lemma 2, $\overrightarrow{\mathrm{H}^{h_1}\mathrm{T}^{t_1} \cdots \mathrm{H}^{h_n}\mathrm{T}} = \overrightarrow{\mathrm{H}^{h_1}\mathrm{T}^{t_1} \cdots \mathrm{H}^{h_n-1}}$. The first player's move is one of three types; we show that the second player has a winning response in each case.

(1) The first player removes a coin *other than* the far right heads-up coin of B_n in either the cyclic or directed game. Then the second player can respond by copying the move in the other game. By the previous argument for $t_n \geq 2$, the resulting game will be the sum of two games with the same value, and the first player has no winning option.

(2) The first player removes the far right heads-up coin of B_n in the cyclic game. Then the second player is moving in the game

$$\overrightarrow{\mathrm{H}^{h_1+1}\mathrm{T}^{t_1} \cdots \mathrm{T}^{t_{n-1}}\mathrm{H}^{h_n-1}} + \overrightarrow{\mathrm{H}^{h_1}\mathrm{T}^{t_1} \cdots \mathrm{H}^{h_n}\mathrm{T}} = \overrightarrow{\mathrm{H}^{h_1-\pi(t_1)+1}} + \overrightarrow{\mathrm{H}^{h_1-\pi(t_1)}}.$$

Since this game has a positive value, the second player has a winning move.

(3) The first player removes the far right heads-up coin of B_n in the directed game. In this case, the second player should respond by removing the resulting single heads-up coin, leaving the game

$$\mathcal{C}\left(\mathrm{H}^{h_1}\mathrm{T}^{t_1}\cdots\mathrm{H}^{h_n}\mathrm{T}^{t_n}\right)+\overrightarrow{\mathrm{H}^{h_1}\mathrm{T}^{t_1}\cdots\mathrm{H}^{h_n-1}}.$$

The first player's move in this game is now of type (1) or (2) above, so the second player will have a winning response.

Since the second player can always win the disjunctive sum of the cyclic and directed games, they have the same value. □

3. Take Turn on Undirected Graphs

In this section we assume that the underlying graph of the Take Turn game is undirected. We first present formulas for the Sprague-Grundy values of games played on complete and complete bipartite graphs, then turn to the more interesting case of games played on paths.

3.1 Complete and complete bipartite graphs

Let K_n denote the complete graph on n vertices and $K_{m,n}$ the complete bipartite graph on $m+n$ vertices with vertex partition V_1, V_2. The proof of the following theorem uses simple induction and is omitted.

Theorem 5. Let K_n^k denote the Take Turn game played on K_n with k heads-up coins $(0 \le k \le n)$. Then

$$g\left(K_n^k\right)=\begin{cases}0 & \text{if } k \le n/2\\ 1 & \text{if } k > n/2.\end{cases}$$

Let $K_{m,n}^{a,b}$ denote the Take Turn game played on $K_{m,n}$ with heads-up coins on exactly a vertices of V_1 and b vertices of V_2. Then

$$g\left(K_{m,n}^{a,b}\right)=\begin{cases}\pi(a+b) & \text{if } m,n \text{ even}\\ \pi(a)+\pi(b) & \text{if } m \text{ even, } n \text{ odd or } m \text{ odd, } n \text{ even}\\ \pi(a)\pi(b) & \text{if } m,n \text{ odd.}\end{cases}$$

3.2 Path graphs

Take Turn on an undirected path graph (a row of coins) is simple to play, but a complete analysis of this situation has not been found. We present here a few general results and the Sprague-Grundy values for some special position forms. But even simple variations can create games that have so far resisted complete analysis.

A Take Turn game played on an undirected path will be described by listing the state of the coins from left to right. In contrast to the decomposition result for directed paths (Theorem 3), no significant decomposition theorems have been found for undirected paths. However, the following theorem will allow us to limit our investigations of positions to those that end in at most one tails-up coin.

Theorem 6. Let A be a sequence of coins. Then $g(\mathrm{A} \cdot \mathrm{TT}) = g(\mathrm{A})$.

Proof. In the game $\mathrm{A} \cdot \mathrm{TT} + \mathrm{A}$, the second player may simply copy the move of the first player. Since $g(\mathrm{HT}) = 0$, if the first player removes the far right coin of A, the resulting game after the second player's response has value 0. The second player always has a response so $g(\mathrm{A} \cdot \mathrm{TT} + \mathrm{A}) = 0$ and the games are equal. □

Recall that $\pi(m)$ is the parity of a nonnegative integer m. Also, we use $\lceil x \rceil$ to denote the *ceiling* of x (the smallest integer greater than or equal to x).

Lemma 7. Let $m \geq 0$. Then

1. $g(\mathrm{TH}^m\mathrm{T}) = \pi(m)$;
2. $g(\mathrm{TH}^m) = 2\left\lceil \frac{1}{3}(m-1) \right\rceil$.

Proof. Part (1) is easy to show using induction. It is easily checked for $m = 0, 1$ and 2. Assume the formula is true for all k less than some $m \geq 3$. The next options of $\mathrm{TH}^m\mathrm{T}$ are $\mathrm{H} + \mathrm{TH}^{m-2}\mathrm{T}$ and $\mathrm{TH}^i\mathrm{T} + \mathrm{TH}^{m-i-3}\mathrm{T}$, $(0 \leq i \leq m-3)$. Using the induction hypothesis, these options have values $1 \oplus \pi(m-2) = \pi(m+1)$ and $\pi(i) \oplus \pi(m-i-3) = \pi(m+1)$. Thus $g(\mathrm{TH}^m\mathrm{T}) = \mathrm{mex}(\pi(m+1)) = \pi(m)$.

For part (2), we again use induction after verifying the formula for $m = 0, 1, 2$. Assume (2) is true for all k less than some $m \geq 3$ and examine the next options for $\mathrm{H}^m\mathrm{T}$ for $m \geq 3$. They are:

$$\mathrm{TH}^{m-2}\mathrm{T}, \quad \mathrm{H}^i\mathrm{T} + \mathrm{TH}^{m-i-3}\mathrm{T} \quad (0 \leq i \leq m-3), \quad \mathrm{H}^{m-2}\mathrm{T} + \mathrm{H}.$$

Using the inductive hypothesis and part (1), the values of these options are

$$\pi(m-2), \quad 2\left\lceil \frac{1}{3}(i-1) \right\rceil \oplus \pi(m-i-3) \quad (0 \leq i \leq m-3), \quad 2\left\lceil \frac{1}{3}(m-3) \right\rceil \oplus 1.$$

As i ranges from 0 to $m-3$, $2\left\lceil \frac{1}{3}(i-1) \right\rceil \oplus \pi(m-i-3)$ takes on all integer values from 0 to $2\left\lceil \frac{1}{3}(m-1) \right\rceil - 1$ if $m = 3k$ or $3k+1$ and from 0 to $2\left\lceil \frac{1}{3}(m-1) \right\rceil - 2$ if $m = 3k+2$. But if $m = 3k+2$ then $2\left\lceil \frac{1}{3}(m-3) \right\rceil + 1 = 2k+1 = 2\left\lceil \frac{1}{3}(m-1) \right\rceil - 1$. Thus $g(\mathrm{TH}^m) = \mathrm{mex}(0, 1, \ldots, 2\left\lceil \frac{1}{3}(m-1) \right\rceil - 1) = 2\left\lceil \frac{1}{3}(m-1) \right\rceil$. □

Theorem 8. For $m \geq 0$, the game H^m has value 0 if m is divisible by 6 and value 1 otherwise.

Proof. The result can be immediately checked for $m = 0, 1, 2$ and 3. For $m \geq 4$, the next options from H^m are of the form

$$\begin{aligned}&\mathrm{TH}^{m-2},\\ &\mathrm{H}^i\mathrm{T} + \mathrm{TH}^{m-i-3} \quad (0 \leq i \leq m-3).\end{aligned}$$

Note that for $m \geq 4$, $g(\mathrm{TH}^{m-2}) \geq 2$. Also, the value of $\mathrm{H}^i\mathrm{T} + \mathrm{TH}^{m-i-3}$ is even and in particular not equal to 1. Thus H^m will have value 1 if $g(\mathrm{H}^i\mathrm{T} + \mathrm{TH}^{m-i-3}) = 0$ for some i and value 0 otherwise.

Consider the more general position of $\mathrm{H}^i\mathrm{T} + \mathrm{TH}^j$ for $i, j \geq 0$. We have

$$\begin{aligned}g(\mathrm{H}^i\mathrm{T} + \mathrm{TH}^j) = 0 &\iff 2\left\lceil\frac{i-1}{3}\right\rceil \oplus 2\left\lceil\frac{j-1}{3}\right\rceil = 0\\ &\iff \left\lceil\frac{i-1}{3}\right\rceil = \left\lceil\frac{j-1}{3}\right\rceil\\ &\iff k-1 < \frac{i-1}{3} \leq k \text{ and } k-1 < \frac{j-1}{3} \leq k \text{ for some } k \geq 0\\ &\iff 3k-2 < i, j \leq 3k+1 \text{ for some } k \geq 0.\end{aligned}$$

In our case, we require that $j = m - i - 3$ so $g(\mathrm{H}^m) = 1$ if and only if for some i and some k, $3k-1 \leq i, m-i-3 \leq 3k+1$. Computing m for $i = 3k-1$, $3k$, and $3k+1$ we find that $m = 6k + t$ for some t between 1 and 5. Thus $g(\mathrm{H}^m) = 0$ if and only if m is divisible by 6 and 1 otherwise. □

It is interesting to note the effect that adding a single tails up coin can have on the value of a game, as shown by Lemma 7 and Theorem 8. For any integer n, we can find a game A such that the value of A is 0, the value of $\mathrm{A} \cdot \mathrm{T}$ is greater than n and the value of $\mathrm{T} \cdot \mathrm{A} \cdot \mathrm{T}$ is again 0. In particular, we can take A to be the sequence H^m where m is any multiple of 6 greater than $3n/2$.

3.3 Other Positions on Path Graphs

We list the following games and their values without proof. Techniques of proof are similar to Lemma 7 and Theorems 6 and 8.

Theorem 9.

1. Given a row of coins A, let A^R denote the reverse of A (for example, $(\mathrm{HHTH})^R$ is HTHH). Then

   (a) $g(\mathrm{A} \cdot \mathrm{T} \cdot \mathrm{A}^R) = 0$

(b) $g(\mathrm{A} \cdot \mathrm{H} \cdot \mathrm{A}^R) = 1$

(c) $g(\mathrm{A} \cdot \mathrm{HT} \cdot \mathrm{A}^R) = 0$.

2. For integers $a_i \geq 1$, the positions

(a) $\mathrm{HT}^{a_1}\mathrm{HT}^{a_2} \cdots \mathrm{HT}^{a_n}\mathrm{H}$

(b) $\mathrm{THT}^{a_1}\mathrm{HT}^{a_2} \cdots \mathrm{HT}^{a_n}\mathrm{HT}$

(c) $\mathrm{HHT}^{a_1}\mathrm{HT}^{a_2} \cdots \mathrm{HT}^{a_n}\mathrm{H}$

(d) $\mathrm{HHT}^{a_1}\mathrm{HT}^{a_2} \cdots \mathrm{HT}^{a_n}\mathrm{HH}$

have value $\pi(a_1 + a_2 + \cdots + a_n + 1)$ while the positions

(a) $\mathrm{HT}^{a_1}\mathrm{HT}^{a_2} \cdots \mathrm{HT}^{a_n}\mathrm{HT}$

(b) $\mathrm{HHT}^{a_1}\mathrm{HT}^{a_2} \cdots \mathrm{HT}^{a_n}\mathrm{HT}$

have value $\pi(a_1 + a_2 + \cdots + a_n)$.

5. For $m, n \geq 0$,

(a) $g(\mathrm{TH}^m\mathrm{TH}^n\mathrm{T}) = 0$ if $|m - n| \leq 1$

(b) $g(\mathrm{TH}^m\mathrm{TH}) = g(\mathrm{TH}^m) + 1$

(c) $g(\mathrm{TH}^m\mathrm{THH}) = g(\mathrm{TH}^m) + 1$

(d) $g(\mathrm{H}^m\mathrm{TH}) = 2$ if $m + 1$ is divisible by 6, 0 otherwise

(e) $g(\mathrm{H}^m\mathrm{THH}) = 2$ if m or $m + 1$ is divisible by 6, 0 otherwise

(f) $g(\mathrm{H}^m\mathrm{TH}^n) = 0$ if $|m - n| \leq 1$.

(g) $g(\mathrm{H}^m\mathrm{TTH}^m) = 0$ for $m \geq 6$.

6. For $m, n \geq 0$,

$$g(\mathrm{TH}^m\mathrm{TTH}^n\mathrm{T}) = \begin{cases} \pi(m+n) & \text{if } m \neq 1 \text{ and } n \neq 1, \\ g(\mathrm{TH}^m) + 1 & \text{if } n = 1, \\ g(\mathrm{TH}^n) + 1 & \text{if } m = 1. \end{cases}$$

7. For $m \geq 5$, $n \geq 3$, $g(\mathrm{TH}^m\mathrm{TTH}^n) = g(\mathrm{TH}^{n-3}) + \pi(m + 1)$.

8. For $m, n \geq 9$, $|m - n| \geq 3$, $g(\mathrm{H}^m\mathrm{TTH}^n) > 0$.

A natural form to consider next is $\mathrm{H}^m\mathrm{T}^k\mathrm{H}^n$. However, the problem of computing the Sprague-Grundy values of these positions even when $k = 1$ does not appear to have a simple solution. Tables 1, 2 and 3 show values of these positions for $t = 1, 2$ and 3 and $0 \leq m, n \leq 20$. (These values were calculated using the Combinatorial Game Suite [8].)

m,n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	2	2	2	4	4	4	6	6	6	8	8	8	10	10	10	12	12	12	14
1		0	0	0	0	2	0	0	0	0	0	2	0	0	0	0	0	2	0	0	0
2			0	0	0	2	2	0	0	0	0	2	2	0	0	0	0	2	2	0	0
3				0	0	2	2	0	0	0	0	2	2	0	0	0	0	2	2	0	0
4					0	0	2	0	0	0	0	0	1	2	2	0	0	0	2	2	2
5						0	0	2	2	2	4	4	4	6	6	6	8	8	8	10	10
6							0	0	2	2	2	4	4	4	6	6	6	8	8	8	10
7								0	0	0	0	2	0	0	0	0	0	1	0	8	8
8									0	0	0	9	9	0	0	0	0	2	2	0	0
9										0	0	8	6	2	8	0	8	8	2	10	8
10											0	0	2	2	8	0	8	8	2	10	8
11												0	0	6	10	2	2	1	1	2	2
12													0	0	10	2	2	2	0	0	2
13														0	0	3	2	2	2	0	1
14															0	0	2	2	2	4	4
15																0	0	2	2	2	0
16																	0	0	2	2	2
17																		0	0	2	2
18																			0	0	6
19																				0	0
20																					0

Table 1: Values of $\mathrm{H}^m\mathrm{TH}^n$

Computer experiments with these and related forms (generally the next options from $\mathrm{H}^m\mathrm{T}^k\mathrm{H}^n$) have led to the following conjectures:

Conjecture 12.

1. For any m, n, eventually $g(\mathrm{TH}^m\mathrm{TH}^n)$ will be periodic in m either of period 2 or of period 3 with saltus 1.

2. For $m, n \geq 9$, $g(\mathrm{H}^m\mathrm{TTH}^n) = 0$ if and only if $|m - n| \leq 1$ or $(m, n) = (3k + 1, 3k - 1)$ or $(m, n) = (3k - 1, 3k + 1)$ for some k.

3. For $(m, n) \neq (5, 12 + 6k)$ or $(12 + 6k, 5)$, $g(\mathrm{H}^m\mathrm{TTH}^n) = g(\mathrm{H}^m\mathrm{TTTTH}^n)$, otherwise $g(\mathrm{H}^m\mathrm{TTH}^n) = g(\mathrm{H}^m\mathrm{TTTTH}^n) \oplus 1$.

4. For all m, n, $g(\mathrm{H}^m\mathrm{TTTH}^n) = g(\mathrm{H}^m\mathrm{TTTTTH}^n)$. Thus for any $k \geq 3$, $g(\mathrm{H}^m\mathrm{T}^k\mathrm{H}^n) = g(\mathrm{H}^m\mathrm{T}^{k+2}\mathrm{H}^n)$.

The last conjecture is of particular interest. If true, it may be useful to attempt to prove a similar statement for arbitrary sequences A and B: For k large enough, for any coin sequences A and B, $g(\mathrm{A} \cdot \mathrm{T}^k \cdot \mathrm{B}) = g(\mathrm{A} \cdot \mathrm{T}^{k+2} \cdot \mathrm{B})$.

m,n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	1	1	1	1	1	0	1	1	1	1	1	0	1	1	1	1	1	0	1	1
1		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	3	3	3	5	5	5	7	7	7	9	9	9	11	11	11	13	13
3				1	3	3	3	5	5	5	7	7	7	9	9	9	11	11	11	13	13
4					1	1	1	1	3	3	3	5	5	5	7	7	7	9	9	9	11
5						1	1	1	1	1	1	5	5	4	5	5	5	9	9	8	9
6							0	0	6	6	5	5	4	4	10	10	9	9	8	8	14
7								0	6	6	5	3	3	3	1	1	1	7	7	7	5
8									0	0	0	3	3	9	7	7	7	1	1	1	3
9										0	0	2	2	2	12	12	11	9	14	14	8
10											0	1	1	1	3	3	3	9	13	13	8
11												0	0	0	1	1	1	12	12	11	11
12													0	0	14	14	13	12	12	11	11
13														0	14	14	13	12	12	11	16
14															0	0	0	2	2	2	4
15																0	0	2	2	2	4
16																	0	1	1	1	3
17																		0	0	0	1
18																			0	0	6
19																				0	6
20																					0

Table 2: Values of $\mathrm{H}^m\mathrm{TTH}^n$

m,n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	2	2	2	4	4	4	6	6	6	8	8	8	10	10	10	12	12	12	14
1		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2			0	0	0	0	2	0	0	0	0	0	2	0	0	0	0	0	2	0	0
3				0	0	0	2	0	0	0	0	0	2	0	0	0	0	0	2	0	0
4					0	0	4	0	0	0	0	0	2	0	0	0	0	0	4	0	0
5						0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0
6							0	4	6	2	2	4	4	8	10	6	6	8	8	12	14
7								0	0	0	0	0	4	0	0	0	0	0	4	0	0
8									0	0	0	0	6	0	0	0	0	0	6	0	0
9										0	0	0	6	0	0	0	0	0	2	0	0
10											0	0	6	0	0	0	0	0	2	0	0
11												0	0	0	0	0	0	0	4	0	0
12													0	8	10	2	2	4	4	12	14
13														0	0	0	0	0	8	0	0
14															0	0	0	0	10	0	0
15																0	0	0	6	0	0
16																	0	0	6	0	0
17																		0	0	0	0
18																			0	12	14
19																				0	0
20																					0

Table 3: Values of $\mathrm{H}^m\mathrm{TTTH}^n$

3.4 The Set $\mathcal{P}$ is Not Regular

Recall that $\mathcal{P}$ is the set of positions for which the second player may force a win: $\mathcal{P} = \{A : g(A) = 0\}$. We will use the Pumping Lemma to show that $\mathcal{P}$ is not a regular language—the sequences in $\mathcal{P}$ cannot be recognized by a finite automaton. See [7] for information on automata, regular languages and the Pumping Lemma.

Lemma 10. (Pumping Lemma) If $\mathcal{A}$ is a regular language then there is some number p (the *pumping length*) such that if $A \in \mathcal{A}$ then A can be written as the concatenation of three sequences: $A = xyz$ such that y is nonempty, the length of xy is at most p, and for any $i \geq 0$, $xy^iz \in \mathcal{A}$.

We will use two of the formulas stated in Theorem 9: $g(\mathrm{H}^n\mathrm{TTH}^n) = 0$ for $n \geq 6$ and $g(\mathrm{H}^m\mathrm{TTH}^n) > 0$ if $|m - n| \geq 3$ and $m, n \geq 9$.

Theorem 11. The set $\mathcal{P}$ does not form a regular language (over the alphabet $\{\mathrm{H}, \mathrm{T}\}$).

Proof. Suppose $\mathcal{P}$ is regular and let p be its pumping length. Let n be such that $n \geq p + 1$ and $n \geq 9$ and consider the sequence $\mathrm{H}^n\mathrm{TTH}^n \in \mathcal{P}$. Then the sequences x, y and z from the Pumping Lemma have the form $x = \mathrm{H}^s$, $y = \mathrm{H}^t$ and $z = \mathrm{H}^{n-s-t}\mathrm{TTH}^n$ for some $s \geq 0$ and $t \geq 1$ with $s+t \leq p$. So $xy^iz = \mathrm{H}^{n+(i-1)t}\mathrm{TTH}^n$. But for i large enough, $|n + (i - 1)t - n| \geq 3$ and so $g(\mathrm{H}^{n+(i-1)^t}\mathrm{TTH}^n) > 0$. Thus xy^iz is not in $\mathcal{P}$, which contradicts the Pumping Lemma. Therefore $\mathcal{P}$ is not regular. □

4. Take Turn is PSPACE-Complete

We now turn to a more general question than computing the exact Sprague-Grundy value of a Take Turn game and ask: Given a Take Turn game on a directed graph, can the first player force a win? This is equivalent to asking if the Sprague-Grundy value is positive. We will call the question the *Take Turn Decision Problem* and denote it by *TT*. Note that *TT* is decidable—the simple algorithm that actually computes the Sprague-Grundy value of a given game will answer *TT*—but here we are interested in the amount of *memory* required to answer the question.

In general, a decision problem Π is a 'yes-no' question about some given input. For example, given an undirected graph, is it Eulerian? Or, does a given graph have a Hamiltonian circuit? A decision problem Π is in *PSPACE* if there is an algorithm that will answer (decide) Π using at most a polynomially bounded amount of memory (as a function of the size of the input). A decision problem Π is *PSPACE-Hard* if for any decision problem Π' in PSPACE there is an algorithm that uses a polynomially bounded amount of time to reduce Π' to Π—input to Π' can be transformed to an input to Π so that a decision of Π corresponds to a decision of Π'. A decision problem is *PSPACE-Complete* if it is PSPACE-Hard and is

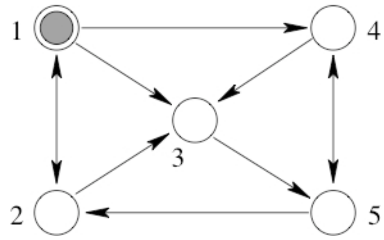

Figure 5: Example Generalized Geography game

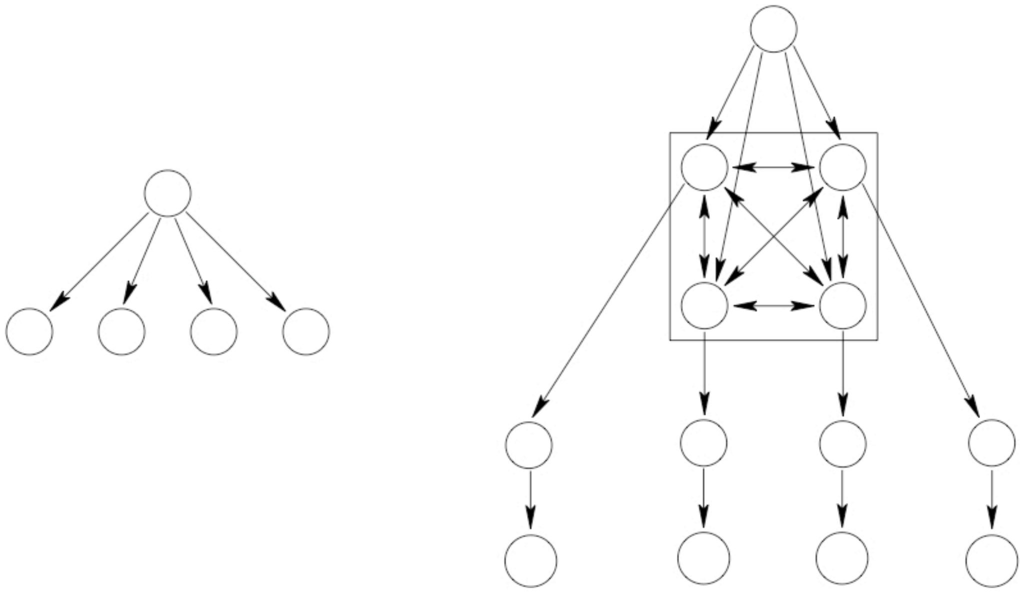

Figure 6: Converting a Geography graph G to a Take Turn graph H

itself in PSPACE. For more information on complexity classes, including PSPACE, see [3] and [7]

Since any game of Take Turn played on a graph with N vertices ends in at most N moves, TT is in PSPACE. We will show that TT is PSPACE-Complete by reducing an instance of the decision problem for the game Generalized Geography, denoted GG, to an instance of TT. The decision problem GG is known to be PSPACE-Complete, so we will have shown that any PSPACE problem can be reduced to TT (via GG). With a slight abuse of terminology, we may also say that the game Take Turn is PSPACE-Complete.

The game of Generalized Geography is played on a directed graph. One vertex is designated as the starting vertex; a token is placed on that vertex at the start of play. Players move by moving the token to an adjacent vertex (respecting the directions of the edges) subject to the condition that no vertex may be visited twice. The player who cannot make a move loses the game. An example game is shown in Figure 5; the token starts in vertex 1. In this game, the first player can force a win by moving to vertex 3.

We need to show how to transform a Generalized Geography game into a Take Turn game so that the first player can force a win in the Generalized Geography game if and only if the first player can force a win in the corresponding game of Take Turn.

Let G be the underlying directed graph for a Generalized Geography game. We will construct a new graph H and play Take Turn on H. For a vertex v in G, let m_v be the out-degree of v. Construct vertices a_v and $b_v(i), 1 \leq i \leq m_v$ in H. Also construct in H the complete directed graph K_{m_v}. Note that if $m_v = 0$ then only a_v will be constructed.

Next construct a directed edge from a_v to each vertex of K_{m_v} and a directed edge from the ith vertex of K_{m_v} to vertex $b_v(i)$. Let $w_i, 1 \leq i \leq m_v$ be the followers of v in G. For each vertex $b_v(i)$, construct a directed edge in H from $b_v(i)$ to a_{w_i}. Figure 6 shows an example of this construction for a vertex v of G.

After constructing a_v and the $b_v(i)$ for each v in G, construct vertex s in H and a directed edge from s to the vertex a_v where v is the starting vertex of in the Geography game. To obtain a game of Take Turn, place coins on the vertices of H so that only the coin at vertex s is heads-up.

Figure 7 shows the converted graph of the Generalized Geography game in Figure 5. The dotted boxes indicate those constructed vertices corresponding to the original vertices of G. For ease of drawing, the complete graphs are represented by the boxes K_m; the directed edge entering the box represents m edges, one going to each vertex of the complete graph, while the edges coming out of the K_m box each come from a unique vertex of the complete graph. There are tails-up coins at each vertex except s.

It is not difficult to see how the resulting Take Turn game matches the Geography game. When the previous player to move takes the coin from the a_v vertex, all of the coins in the complete graph K_{m_v} turn to heads-up, allowing the next player a choice of moves. This corresponds to that player being able to choose which vertex w to move to from v in the Geography game. After making a choice, the remaining coins in the complete graph are turned back to tails-up so may not be removed; a single heads-up coin remains at some b_v vertex. The forced move from that b_v vertex to the a_w vertex acts to switch the order of the players so that the previous player may choose from the complete graph K_{m_w}, simulating the move from vertex w in the Geography game.

If the coin at vertex a_w has already been removed then the player who chooses the vertex b_v leading to a_w will put himself at a disadvantage—his opponent may then remove the coin at b_v and, since no coins will turn over, there will be no other heads up coins to remove. Thus, assuming rational play, no player will make such a choice. This enforces the requirement in Geography that no vertex be visited twice.

The preceding discussion shows

Theorem 13. The decision problem *TT*, and thus the game of Take Turn on directed graphs, is PSPACE-Complete.

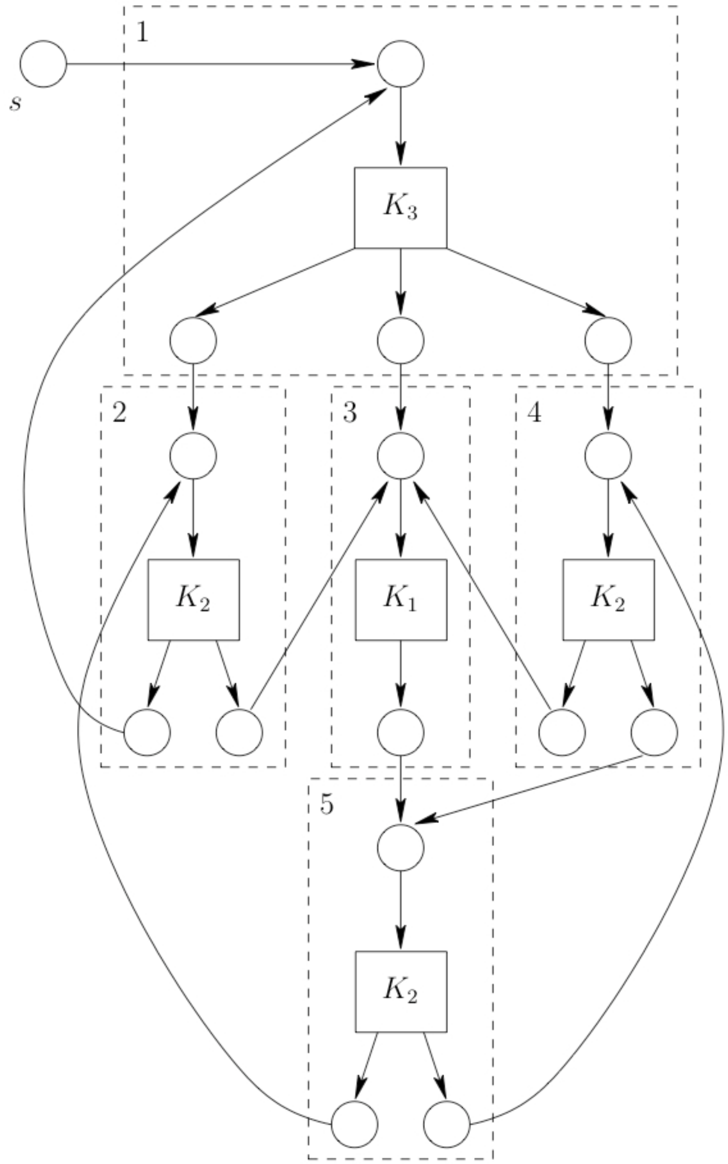

Figure 7: Converted Take Turn game from Figure 5. Each vertex contains a coin; only the coin at vertex s is heads-up.

5. Take Turn as a Parity Vertex Deletion Game

Nowakowski and Ottaway ([5], [6]) have investigated vertex deletion games in which the players are allowed remove vertices based on the parity of the vertex degree. In the game of Even/Even, players are allowed to remove only vertices of even degree; in the game Odd/Odd, players remove vertices of odd degree. For directed graphs, use the in-degree of a vertex.

The connection to Take Turn is obvious. To simulate an Odd/Odd game with Take Turn, we require that heads-up coins be placed only on vertices of odd degree (and tails-up coins only on vertices of even degree). Reverse the placement to simulate an Even/Even game. Removing a heads-up coin will change the degree parity of any adjacent vertices while also turning over the coins, thus preserving the correspondence with the vertex degree.

The following result on Even/Even may be found in [6]. It is restated here in terms of Take Turn.

Proposition 14. Suppose G is an undirected graph or a directed forest. Let A be a game of Take Turn with underlying graph G such that a heads-up coin is placed on a vertex v if and only if v has even degree (in-degree for a directed graph). Then

$$g(A) = \begin{cases} 0 & \text{if } |V(G)| \text{ is even,} \\ 1 & \text{if } |V(G)| \text{ is odd} \end{cases}$$

where $|V(G)|$ is the number of vertices in G.

Though the games Even/Even and Odd/Odd are restricted forms of Take Turn, they are just as hard to solve. We can adapt Theorem 13 to show

Theorem 15. The games of Even/Even and Odd/Odd played with directed graphs are PSPACE-Complete.

Proof. To reduce a game of Generalized Geography to a game of Even/Even, construct the graph H as described in Section 4. If vertex v of H ($v \neq s$) has even in-degree, construct two other vertices x and y with edges from x to v, x to y and y to x. Then v, x, and y each have odd in-degree. To reduce Geography to Odd/Odd, add a vertex t with an edge from t to s and for each vertex v in H of odd in-degree add a vertex x with an edge from x to v. The translation of GG to the appropriate decision problem for Even/Even or Odd/Odd is immediate. □

Recall that we can construct Take Turn positions with arbitrarily large values:

$$g(\mathrm{H}^m\mathrm{T}) = 2\left\lceil \frac{1}{3}(m-1) \right\rceil$$
$$g(\mathrm{HTH}^m\mathrm{T}) = 2\left\lceil \frac{1}{3}(m-1) \right\rceil + 1$$

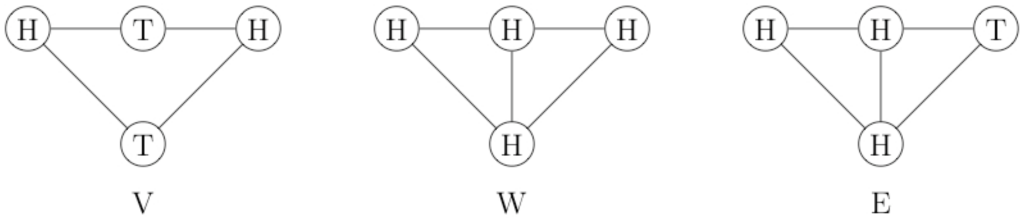

Figure 8: Games V, W and E

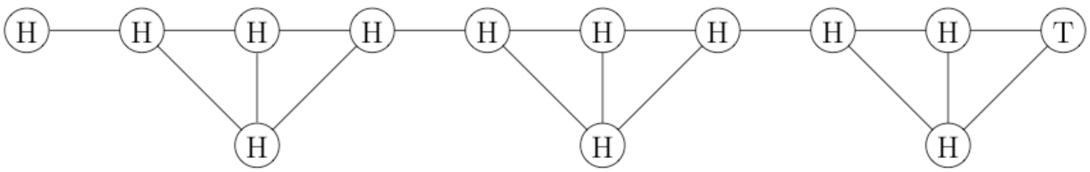

Figure 9: Take Turn game corresponding to an Odd/Odd game with value 6

(Lemma 7 and Theorem 9.5b). While these forms do not yield Odd/Odd games directly, we can use them as guides to create Odd/Odd games.

More precisely, let V, W and E denote the games shown in Figure 8. These do not correspond to Odd/Odd games, but we can adjoin copies of each to create larger games of the form H–W^n–E and H–V–W^n–E by connecting vertices along the top row; the resulting games will be Odd/Odd. For example, the game in Figure 9 is H–W^2–E and the game in Figure 10 is H–V–W–E. It is not difficult to show that these games have values 6 and 5 respectively.

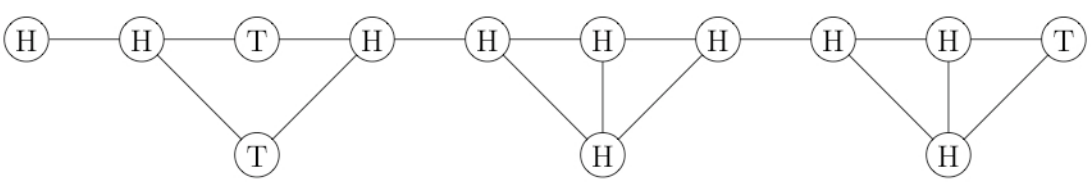

Figure 10: Take Turn game corresponding to an Odd/Odd game with value 5

Based on computer calculations of game values of H–V^m–W^n–E for $0 \leq m, n \leq 5$, we make the following conjecture.

Conjecture 16. For $m, n \geq 0$, $g\,(\text{H–V}^m\text{–W}^n\text{–E}) = 2(n+1) + \pi(m)$.

If this conjecture holds then it would answer, in the affirmative, a question originally posed by Nowakowski and Ottaway: are there Odd/Odd games on connected graphs for any Sprague-Grundy value?

Note that the games $g\,(\text{H–V}^m\text{–W}^n\text{–E})$ cannot be played on a grid graph. Other computer calculations lead us to also conjecture that this is a necessary property of Odd/Odd games with large values.

Conjecture 17. The Sprague-Grundy value of an Odd/Odd game played on a grid graph is either 0 or 1.

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays*, 2 ed., vol. 1, A K Peters, 2000.

[2] A. Flammenkamp, *Octal Games*, `http://wwwhomes.uni-bielefeld.de/achim/octal.html`, August 2005.

[3] M. R. Garey and D. S. Johnson, *Computers and Intractibility: a Guide to the Theory of NP-Completeness*, W.H. Freeman, 1979.

[4] R. K. Guy, *Impartial Games*, Games of No Chance (R. Nowakowski, ed.), Cambridge Univ. Press, 1998, pp. 61–78.

[5] R. Nowakowski and P. Ottaway, Vertex Deletion Games with Parity Rules, *INTEGERS* 5(2) (2005), #A15.

[6] P. Ottaway, *Analysis of Three New Combinatorial Games*, M.Sc. Thesis, Dalhousie University, 2003.

[7] M. Sipser, *Introduction to the Theory of Computation*, Course Technology, 2005.

[8] CGSuite, A java based toolkit for evaluating games, v. 0.6, `http://cgsuite.sourceforge.net/`.