Ramsey Theoretic Consequences of Some New Results About Algebra in the Stone-Čech Compactification

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Abstract. Recently [4] we have obtained some new algebraic results about $\beta\mathbb{N}$, the Stone-Čech compactification of the discrete set of positive integers and about βW, where W is the free semigroup over a nonempty alphabet with infinitely many variables adjoined. (The results about βW extend the Graham-Rothschild Parameter Sets Theorem.) In this paper we derive some Ramsey Theoretic consequences of these results. Among these is the following, which extends the Finite Sums Theorem.

Theorem. *Let $\mathbb{N}$ be finitely colored. Then there is a color class D which is central in $\mathbb{N}$ and*

(i) there exists a pairwise disjoint collection $\{D_{i,j} : i, j \in \omega\}$ of central subsets of D and for each $i \in \omega$ there exists a sequence $\langle x_{i,n}\rangle_{n=i}^{\infty}$ in $D_{i,i}$ such that whenever F is a finite nonempty subset of ω and $f : F \to \{1, 2, \ldots, \min F\}$ one has that $\Sigma_{n\in F}\, x_{f(n),n} \in D_{i,j}$ where $i = f(\min F)$ and $j = f(\max F)$; and

(ii) at stage n when one is chosing $(x_{0,n}, x_{1,n}, \ldots, x_{n,n})$, each $x_{i,n}$ may be chosen as an arbitrary element of a certain central subset of $D_{i,i}$, with the choice of $x_{i,n}$ independent of the choice of $x_{j,n}$.

An analogous extension of the Graham-Rothschild Theorem is established. Also included are new results about image partition regularity and kernel partition regularity of matrices.

1. Introduction

Applications of the algebra of the Stone-Čech compactification $\beta\mathbb{N}$ of the set $\mathbb{N}$ of positive integers to Ramsey Theory have fascinated the second author since he was almost young, 1975 to be precise. At that time he was made aware of the Galvin-Glazer proof of the Finite Sums Theorem – a proof that is essentially trivial given that one knows that there is an idempotent $p = p + p$ in the compact right topological semigroup $(\beta\mathbb{N}, +)$. (His original

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proof [9] was extraordinarily complicated, its only virtue being that it does not require the axiom of choice [3].) The other authors' interests are of only a slightly more recent origin.

There have been since 1975 many other Ramsey-Theoretic applications of the algebra of the Stone-Čech compactification βS of a discrete semigroup S. Among these has been the algebraic proof of the Central Sets Theorem. The notion of *central* sets in $\mathbb{N}$ is due to H. Furstenberg. He defined central sets in terms of the notions of *uniform recurrence* and *proximality* of topological dynamics and proved the following theorem. (We write $\mathcal{P}_f(A)$ for the set of finite nonempty subsets of a set A and $\omega = \mathbb{N} \cup \{0\}$.)

In [2] it was shown that there is a much simpler characterization characterization of central sets. A subset of $\mathbb{N}$ is central if and only if it is a member of an idempotent in the smallest ideal of $(\beta\mathbb{N}, +)$, and a proof of Theorem 1.1 using that description is quite simple. (Later Shi and Yang showed [16] that the algebraic and topological dynamical characterizations of central sets are equivalent in any semigroup.)

1.1 Theorem (Central Sets Theorem). *Let $n \in \mathbb{N}$ and for each $i \in \{0, 1, \ldots, n-1\}$, let $\langle y_{i,k}\rangle_{k=0}^{\infty}$ be a sequence in $\mathbb{N}$. Let C be a central subset of $\mathbb{N}$. Then there exist a sequence $\langle a_n\rangle_{n=0}^{\infty}$ in $\mathbb{N}$ and a sequence $\langle H_n\rangle_{n=0}^{\infty}$ of finite nonempty subsets of ω such that $\max H_n < \min H_{n+1}$ for each $n \in \omega$ and*

$$\big\{ \textstyle\sum_{n\in F}(a_n + \sum_{t\in H_n} y_{f(t),t}) : F \in \mathcal{P}_f(\omega) \text{ and } f : F \to \{0, 1, \ldots, k-1\}\big\} \subseteq C\,.$$

Proof. [6, Proposition 8.21]. □

Central sets in $\mathbb{N}$ have many strong combinatorial properties. For example, they contain solutions to any partition regular system of homogeneous linear equations. (See [12, Theorem 15.16]). In fact for most Ramsey Theoretic results in $\mathbb{N}$ the configurations that are guaranteed to be monochrome can be found in any central set. (Since one cell of any partition of $\mathbb{N}$ must be central, this is a stronger conclusion.)

In Section 2 we establish the theorem which was stated in the abstract and a stronger statement involving infinitely many sequences, and infinitely many pairwise disjoint central sets.

In Section 3 we derive similar results about the free semigroup W on an arbitrary nonempty alphabet, results that extend the Graham-Rothschild Theorem.

In Section 4 we use a recent extension of the Graham-Rothschild parameter sets theorem to obtain a theorem about image partition regular matrices over arbitrary rings.

We use throughout the algebraic structure of the Stone-Čech compactification βS of a discrete semigroup S. We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of T. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

There is a natural extension of the operation of S to βS. We use the same symbol to denote the extension of the operation to βS as that used to denote the operation in S. Assume here that the operation is denoted by $\cdot$. This natural extension makes $(\beta S, \cdot)$ a compact right topological semigroup with S contained in its topological center. This says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$. Given $p, q \in \beta S$, if s and t are restricted to S, one has $p \cdot q = \lim_{s \to p}(\lim_{t \to q} s \cdot t)$.

A subset U of a semigroup S is called a left ideal if is nonempty and $S \cdot U \subseteq U$. It is called a right ideal if it is nonempty and $U \cdot S \subseteq U$. It is called a two-sided ideal, or simply an ideal, if it is both a left ideal and a right ideal. Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T and is also the union of all of the minimal right ideals of T. The intersection of any minimal left ideal and any minimal right ideal is a group. In particular there are idempotents in the smallest ideal. There is a partial ordering of the idempotents of T determined by $p \leq q$ if and only if $p = p \cdot q = q \cdot p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$. Such an idempotent is called simply "minimal". Thus central subsets of S are those which are members of minimal idempotents in βS. See [12] for an elementary introduction to the semigroup βS and for any unfamiliar algebraic facts encountered in this paper.

We note that the Ramsey theoretic results in Section 2 and Section 3 depend heavily on the brilliant contribution made to the theory of semigroup compactifications by Y. Zelenyuk, through his study of absolute coretracts [17, 18].

2. Sums in Central Subsets of $\mathbb{N}$

We introduce now a special semigroup. We shall see in Theorem 2.6 that there are copies of this semigroup close to any minimal idempotent of $\beta\mathbb{N}$.

2.1 Definition. Let A and B be nonempty sets such that $B \cap (A \times B) = \emptyset$ and let $C_{A,B} =$

$B \cup (A \times B)$. Define an operation on $C_{A,B}$ as follows for $a, c \in A$ and $b, d \in B$.

$$bd = d$$
$$b(c,d) = (c,d)$$
$$(a,b)d = (a,d)$$
$$(a,b)(c,d) = (a,d)\,.$$

If A and B are topological spaces, we assume that $A \times B$ has the product topology and that $C_{A,B}$ has the topology for which B and $A \times B$ are clopen subspaces.

It is routine to verify that the operation given above is associative.

Given a semigroup S, a subset A of S, and $x \in S$, we let $x^{-1}A = \{y \in S : xy \in A\}$. (If the operation in S is written additively, we write $-x + A = \{y \in S : x + y \in A\}$.) Note that if $p, q \in \beta S$ and $A \subseteq S$ one has $A \in pq$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$.

2.2 Lemma. *Let S be a discrete semigroup, let H be a subsemigroup of βS and let F be a finite subsemigroup of H. For each $p \in H$, let $B_p \in p$ and define $E_p = \{x \in B_p :$ for all $q \in F$, $x^{-1}B_{pq} \in q\}$. Then*

(1) for each $p \in H$, $E_p \in p$ and

(2) for all $p \in H$, all $q \in F$ and all $x \in E_p$, $x^{-1}E_{pq} \in q$.

Proof. Given $p, q \in H$ one has $B_{pq} \in pq$ so $\{x \in S : x^{-1}B_{pq} \in q\} \in p$. Therefore, since F is finite, $E_p \in p$.

To verify (2) let $p \in H$, let $q \in F$, and let $x \in E_p$. Then for each $r \in F$, $x^{-1}B_{pqr} \in qr$ so $\{y \in S : y^{-1}(x^{-1}B_{pqr}) \in r\} \in q$. Therefore

$$x^{-1}B_{pq} \cap \textstyle\bigcap_{r \in F} \{y \in S : y^{-1}(x^{-1}B_{pqr}) \in r\} \in q\,.$$

Also $x^{-1}B_{pq} \cap \bigcap_{r \in F} \{y \in S : y^{-1}(x^{-1}B_{pqr}) \in r\} \subseteq x^{-1}E_{pq}$. (Given $r \in F$, $y \in S$, and $z \in y^{-1}(x^{-1}B_{pqr})$, one has $xyz \in B_{pqr}$ so $z \in (xy)^{-1}B_{pqr}$.) □

The proof of the following lemma is based on the proof of [13, Lemma 3.4].

2.3 Lemma. *Let A and B be nonempty sets, let M be a right topological semigroup, and let $f : M \to C_{A,B}$ be a surjective homomorphism such that $f^{-1}[B]$ and $f^{-1}[A \times B]$ are compact. Assume that q' is a minimal idempotent of $f^{-1}[B]$, q is a minimal idempotent of $f^{-1}[A \times B]$, $q \leq q'$, $f(q') = y_0$, and $f(q) = (x_0, y_0)$. Then there is a homomorphism $g : C_{A,B} \to M$ such that $f \circ g$ is the identity on $C_{A,B}$, $g[B] \subseteq K(f^{-1}[B])$, $g[A \times B] \subseteq K(f^{-1}[A \times B])$, $g(y_0) = q'$, and $g(x_0, y_0) = q$.*

Proof. We first define g on B. Let $b \in B$. Since $\{b\}$ is a left ideal of B, $f^{-1}[\{b\}]$ is a left ideal of $f^{-1}[B]$. Choose $g(b)$ to be a minimal idempotent of $f^{-1}[B]$ which is in the left ideal $f^{-1}[\{b\}]$ and the minimal right ideal $q'f^{-1}[B]$. If $b = y_0$ we can choose $g(b) = q'$. Since $q' \in K(f^{-1}[B])$, $g(b) \in K(f^{-1}[B])$. Since $g(b) \in f^{-1}[\{b\}]$, $f\big(g(b)\big) = b$. If $b_1, b_2 \in B$ then both $g(b_1)$ and $g(b_2)$ are in the minimal right ideal $q'f^{-1}[B]$ so

$$g(b_1)g(b_2) = g(b_2)$$

Since $b_2 = b_1 b_2$, this implies that the restriction of g to B is a homomorphism.

We next define g on $A \times \{y_0\}$. Let $a \in A$. Since $\{a\} \times B$ is a right ideal of $C_{A,B}$, $f^{-1}[\{a\} \times B]$ is a right ideal of $f^{-1}[A \times B]$. Choose $g(a, y_0)$ to be a minimal idempotent of $f^{-1}[A \times B]$ which is in the right ideal $q'f^{-1}[\{a\} \times B]$ and the minimal left ideal $f^{-1}[A \times B]q$. Since $q \leq q'$, we can choose $g(a, y_0) = q$ if $a = x_0$. Since $q \in K(f^{-1}[A \times B])$, $g(a, y_0) \in K(f^{-1}[A \times B])$. Notice that $f\big(g(a, y_0)\big)$ is in the right ideal $y_0(\{a\} \times B) = \{a\} \times B$ and the left ideal $(A \times B)(x_0, y_0) = A \times \{y_0\}$. Therefore, $f\big(g(a, y_0)\big) = (a, y_0)$. Since $g(a, y_0) = q'g(a, y_0)$ and $g(b)q' = q'$ for all $b \in B$,

$$g(b)g(a, y_0) = g(a, y_0)$$

for all $b \in B$. Moreover, if $a_1, a_2 \in A$ then $g(a_1, y_0)$ and $g(a_2, y_0)$ are idempotents in the same minimal left ideal implying

$$g(a_1, y_0)g(a_2, y_0) = g(a_1, y_0)$$

Finally, extend the definition of g to include all of $A \times B$ by defining $g(a, b) = g(a, y_0)g(b)$. Notice that when $b = y_0$ this definition agrees with our previous definition of $g(a, b)$ since $g(a, y_0)$ is in the left ideal $f^{-1}[A \times B]q$ and $q \leq q' = g(y_0)$. Since $g(a, y_0) \in K(f^{-1}[A \times B])$, $g(a, b) \in K(f^{-1}[A \times B])$. Also,

$$f\big(g(a, b)\big) = f\big(g(a, y_0)\big)f\big(g(b)\big) = (a, y_0)b = (a, b)$$

Checking that g is a homomorphism is now routine.

For $a \in A$ and $b, b' \in B$

$$g(a, b)g(b') = g(a, y_0)g(b)g(b') = g(a, y_0)g(b') = g(a, b') = g\big((a, b)b'\big)$$

and

$$g(b')g(a, b) = g(b')g(a, y_0)g(b) = g(a, y_0)g(b) = g(a, b) = g\big(b'(a, b)\big)$$

For $a_1, a_2 \in A$ and $b_1, b_2 \in B$

$$\begin{aligned} g(a_1 b_1) g(a_2, b_2) &= g(a_1, y_0) g(b_1) g(a_2, y_0) g(b_2) \\ &= g(a_1, y_0) g(a_2, y_0) g(b_2) \\ &= g(a_1, y_0) g(b_2) \\ &= g(a_1, b_2) \\ &= g\big((a_1, b_1)(a_2, b_2)\big) \end{aligned}$$

□

The following elementary topological lemma will be needed later.

2.4 Lemma. *Let X and Y be infinite discrete spaces, let $g : X \to Y$ and denote also by g its continuous extension taking βX to βY. If D is a compact G_δ subset of βX, then $g[D]$ is a compact G_δ subset of βY.*

Proof. For each $U \subseteq X$, $g[\,\overline{U}\,] = \overline{g[U]}$. (See for example [12, Lemma 3.30].) Thus g is an open map. Let D be a compact G_δ subset of βX and pick open subsets U_n of βX for each $n \in \mathbb{N}$ such that $D = \bigcap_{n=1}^\infty U_n$. Since g is continuous, $g[D]$ is compact.

Using the fact that D is compact, inductively choose clopen V_n for each $n \in \mathbb{N}$ such that $D \subseteq V_n \subseteq U_n$ and, if $n > 1$, $V_n \subseteq V_{n-1}$. Then $D = \bigcap_{n=1}^\infty V_n$ and so $g[D] \subseteq \bigcap_{n=1}^\infty g[V_n]$. Since g is an open map, it suffices to show that $\bigcap_{n=1}^\infty g[V_n] \subseteq g[D]$. To this end let $p \in \bigcap_{n=1}^\infty g[V_n]$ and for each $n \in \mathbb{N}$, pick $x_n \in V_n$ such that $p = g(x_n)$. If y is a cluster point of the sequence $\langle x_n \rangle_{n=1}^\infty$, then $y \in D$ and $p = g(y)$. □

Given $n \in \mathbb{N}$, we define $\mathrm{supp}(n) \subseteq \omega$ by $n = \sum_{t \in \mathrm{supp}(n)} 2^t$ and we let $\mathrm{supp}(0) = \emptyset$. We write $\mathbb{H} = \bigcap_{n=1}^\infty c\ell_{\beta\mathbb{N}}(\mathbb{N}2^n)$. Note that by [12, Lemma 6.6] all of the idempotents of $\beta\mathbb{N}$ are in $\mathbb{H}$.

2.5 Lemma. *Let p be an idempotent in $\beta\mathbb{N}$ and let D be a compact G_δ subset of $\beta\mathbb{N}$ such that $p \in D$. There is a compact G_δ subsemigroup V of $\beta\mathbb{N}$, with $p \in V \subseteq D \cap \mathbb{H}$, which contains idempotents q and q' with the following properties:*

(1) $q \in K(V)$;

(2) $q \leq q'$; and

(3) for every compact G_δ subsemigroup L of V which contains q and q', there exist a compact G_δ subsemigroup M of L which contains q and q' and compact subspaces A and B of $\beta\omega$, with $|A| = |B| = 2^{\mathfrak{c}}$, such that there is a continuous surjective homomorphism $f : M \to C_{A,B}$. Furthermore, $q \in K(f^{-1}[A \times B]) \subseteq K(V)$ and $q' \in K(f^{-1}[B])$.

Proof. As in the proof of Lemma 2.4, D is the intersection of a decreasing sequence of clopen sets. So, there is a decreasing sequence $\langle D_n\rangle_{n=0}^{\infty}$ of subsets of $\mathbb{N}$ such that $D = \bigcap_{n=0}^{\infty} c\ell_{\beta\mathbb{N}}(D_n)$. For each $n \in \omega$, let $D_n^\star = \{x \in D_n : -x + D_n \in p\}$. By [11, Lemma 4.14], $D_n^\star \in p$ and, for each $x \in D_n^\star$, $-x + D_n^\star \in p$. For each $n \in \omega$ and each $t \in \mathbb{N}$, let

$$Q_{n,t} = 2^t\mathbb{N} \cap D_n^\star \cap \bigcap\{-x + D_n^\star : x \in D_n^\star \cap \{1, 2, \dots, t\}\}$$

and let $V_n = \bigcap_{t=1}^{\infty} c\ell_{\beta\mathbb{N}}(Q_{n,t})$. It is routine to verify that V_n is a subsemigroup of $\beta\mathbb{N}$ such that $p \in V_n \subseteq c\ell_{\beta\mathbb{N}}(D_n) \cap \mathbb{H}$. (For the details, see the proof of [10, Theorem 2.12].)

Let $V = \bigcap_{n=0}^{\infty} V_n$. Observe that V is a G_δ subsemigroup of $\beta\mathbb{N}$ such that $p \in V \subseteq D \cap \mathbb{H}$.

We can inductively choose a sequence $\langle s_r\rangle_{r=1}^{\infty}$ in $\mathbb{N}$ so that, for each $r \in \mathbb{N}$,

$$s_r \in \textstyle\bigcap_{n=0}^{r} \bigcap_{t=1}^{r} Q_{n,t} \text{ and } \max\big(\mathrm{supp}(s_r)\big) < \min\big(\mathrm{supp}(s_{r+1})\big)\,.$$

Observe that $c\ell_{\beta\mathbb{N}}(\{s_r : r \in \omega\}) \cap \mathbb{N}^\star \subseteq V$.

Define $\phi : \mathbb{N} \to \omega$ by $\phi(n) = \max \mathrm{supp}(n)$. Let $E = \bigcup_{n=0}^{\infty} \mathrm{supp}(s_{2n+1})$, let $F = \{n \in \mathbb{N} : \mathrm{supp}(n) \subseteq E\}$ and let $G = \mathbb{N} \setminus F$. We put $V_0 = V \cap c\ell_{\beta\mathbb{N}} F$ and $V_1 = V \cap c\ell_{\beta\mathbb{N}} G = V \setminus V_0$. Note that $c\ell_{\beta\mathbb{N}}\{s_{2n+1} : n \in \omega\} \setminus \mathbb{N} \subseteq V_0$ and $c\ell_{\beta\mathbb{N}}\{s_{2n} : n \in \omega\} \setminus \mathbb{N} \subseteq V_1$, and so V_0 and V_1 are nonempty.

Suppose that $m, n \in \mathbb{N}$ and $\max\big(\mathrm{supp}(m)\big) < \min\big(\mathrm{supp}(n)\big)$. If $m \in G$ or $n \in G$, then $m+n \in G$. If $m \in F$ and $n \in F$, then $m+n \in F$. Recalling that $x+y = \lim\limits_{m\to x}\lim\limits_{n\to y}(m+n)$, it follows that V_1 is an ideal of V and V_0 is a subsemigroup of V. Consequently $K(V_1) = K(V)$ by [12, Theorem 1.65]. Let q' be any minimal idempotent of V_0 and pick by [12, Theorem 1.60] a minimal idempotent q of V such that $q \le q'$. Note that $q \in V_1$.

Define $\theta : \mathbb{N} \to \omega$ by

$$\theta(n) = \begin{cases} \min(\mathrm{supp}(n) \setminus E) & \text{if } n \in G \\ 1 & \text{if } n \in \mathbb{N} \setminus G\,. \end{cases}$$

Denote also by θ and ϕ the continuous extensions of these functions taking $\beta\mathbb{N}$ to $\beta\omega$.

Now let L be a compact G_δ subsemigroup of V which contains q and q'. Put $B = \phi[L \cap V_0] \cap \phi[L \cap V_1]$ and let $M = \phi^{-1}[B] \cap L$. Note that $B = \phi[M \cap V_0]$. Put $A = \theta[M \cap V_1]$. Since $q = q + q'$, $\phi(q) = \phi(q')$ by [12, Lemma 6.8] and so q and q' are in M. By Lemma 2.4, A and B are compact G_δ subsets of $\beta\mathbb{N}$. Since $q \in \mathbb{H}$, $\theta(q)$ and $\phi(q)$ are in $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. Since A and B meet $\mathbb{N}^*$, it follows from [12, Theorems 3.36 and 3.59], that $|A| = |B| = 2^{\mathfrak{c}}$. Let A and B have the relative topologies induced by $\beta\omega$.

Define $f : M \to C_{A,B}$ by

$$f(x) = \begin{cases} \phi(x) & \text{if } x \in M \cap V_0 \\ \big(\theta(x), \phi(x)\big) & \text{if } x \in M \cap V_1\,. \end{cases}$$

Notice that f is continuous because θ and ϕ are continuous. We claim that f is a surjective homomorphism. To see that f is a homomorphism it suffices to show that

(1) if $x, y \in V$, then $\phi(x+y) = \phi(y)$;

(2) if $x \in V_1$ and $y \in V$, then $\theta(x+y) = \theta(x)$; and

(3) if $x \in V_1$ and $y \in V_0$, then $\theta(y+x) = \theta(x)$.

For (1) see [12 Lemma 6.8]. To verify (2) it suffices to show that $\theta \circ \rho_y$ and θ agree on G, which is a member of x. So let $n \in G$ and pick $m \in \mathbb{N}$ such that $2^m > n$. Then $\theta \circ \rho_y(n) = \theta(n+y) = \theta \circ \lambda_n(y)$ and $\theta \circ \lambda_n$ is constantly equal to $\theta(n)$ on $2^m\mathbb{N}$, a member of y. To verify (3) it suffices to show that $\theta \circ \rho_x$ is constantly equal to $\theta(x)$ on F, a member of y. So let $n \in F$ and pick $m \in \mathbb{N}$ such that $2^m > n$. Then $\theta \circ \rho_x(n) = \theta(n+x) = \theta \circ \lambda_n(x)$ so it suffices to show that $\theta \circ \lambda_n$ and θ agree on $2^m\mathbb{N} \cap G$ which is a member of x. So let $k \in 2^m\mathbb{N} \cap G$. Then $\mathrm{supp}(n+k) \setminus E = \mathrm{supp}(k) \setminus E$ so $\theta \circ \lambda_n(k) = \theta(k)$.

To see that f is surjective let $a \in A$ and $b \in B$. Pick $x \in M \cap V_1$ and $y \in M \cap V_0$ such that $\theta(x) = a$ and $\phi(y) = b$. Then $f(y) = b$ and $f(x+y) = f(x)f(y) = \big(\theta(x), \phi(x)\big)\phi(y) = \big(\theta(x), \phi(y)\big) = (a, b)$.

Finally, observe that $f^{-1}[B] \subseteq V_0$ and $f^{-1}[A \times B] \subseteq V_1$. Since $q' \in K(V_0)$ and $q \in K(V_1)$, $q' \in K(f^{-1}[B])$ and $q \in K(f^{-1}[A \times B])$. □

2.6 Theorem. *Let D be a central subset of $\mathbb{N}$. There exists a sequence $\langle p_i \rangle_{i=0}^{\infty}$ of idempotents in $c\ell_{\beta\mathbb{N}} D \cap K(\beta\mathbb{N})$ such that $p_i + p_j \neq p_l + p_m$ whenever $(i,j) \neq (l,m)$, $p_i + p_j + p_l = p_i + p_l$ for all $i, j, l \in \omega$, and $\{p_i + p_j : i, j \in \omega\}$ is discrete.*

Proof. Let p be a minimal idempotent in $\beta\mathbb{N}$ for which $D \in p$. By Lemma 2.5, there is a compact subsemigroup V of $\beta\mathbb{N}$ with $p \in V \subseteq c\ell_{\beta\mathbb{N}} D$, and there are a compact subsemigroup M of V and compact sets A and B, with $|A| = |B| = 2^{\mathfrak{c}}$, for which there exists a continuous surjective homomorphism $f : M \to C_{A,B}$. Furthermore,

$$K(f^{-1}[A \times B]) \subseteq K(V)$$

and therefore, since V meets $K(\beta\mathbb{N})$, $K(f^{-1}[A \times B]) \subseteq K(\beta\mathbb{N})$ [12, Theorem 1.65]. By Lemma 2.3, there is an injective homomorphism $g : C_{A,B} \to M$ for which $g[A \times B] \subseteq K(f^{-1}[A \times B]) \subseteq K(\beta\mathbb{N})$.

Let $\{a_i : i \in \omega\}$ and $\{b_i : i \in \omega\}$ be discrete subsets of A and B respectively, with $a_i \neq a_j$ and $b_i \neq b_j$ when $i \neq j$. For $i \in \omega$, let $p_i = g(a_i, b_i)$. Then for $i, j \in \omega$, we have $p_i + p_j = g(a_i, b_j)$. If $i, j, l, m \in \omega$ and $(i,j) \neq (l,m)$, then $p_i + p_j \neq p_l + p_m$. Further, for any $i, j, l \in \omega$, $p_i + p_j + p_l = p_i + p_l$.

Given $i \in \omega$, let R_i and U_i be neighborhoods of a_i and b_i in A and B respectively such that $R_i \cap \{a_j : j \in \omega \setminus \{i\}\} = \emptyset$ and $U_i \cap \{b_j : j \in \omega \setminus \{i\}\} = \emptyset$. Then, since f is continuous and $f \circ g$ is the identity on $C_{A,B}$, we have that for any $i, j \in \omega$, $f^{-1}[R_i \times U_j]$ is a neighborhood of $p_i + p_j$ in V_1 which misses $\{p_k + p_l : (k, l) \in \omega \times \omega \setminus \{(i, j)\}\}$. $\square$

To indicate precisely the ability to make many choices for the terms of our sequences we formalize the notion of a tree. We are treating members of ω as ordinals, so $0 = \emptyset$ and for each $n \in \mathbb{N}$, $n = \{0, 1, \ldots, n-1\}$.

2.7 Definition.

(a) T is a *tree* if and only if T is a nonempty set of functions, for each $g \in T$, domain$(g) \in \omega$, and if domain$(g) = n > 0$, then $g_{|n-1} \in T$.

(b) Let g be a function with domain$(g) = n \in \omega$ and let x be given. Then $g^\frown x = g \cup \{(n, x)\}$.

(c) Given a tree T and $g \in T$, $B_g = \{x : g^\frown x \in T\}$.

(d) Given a tree T, g is a *path through* T if and only if g is a function, domain$(g) = \omega$, and for each $n \in \omega$, $g_{|n} \in T$.

Given g in a tree T, B_g will be referred to as the *set of successors of* g.

The following theorem says very roughly that, given any central subset D of $\mathbb{N}$ there exists a pairwise disjoint collection $\{D_{i,j} : i, j \in \omega\}$ of central subsets of D and for each $i \in \omega$ there exist very many sequences $\langle x_{i,n}\rangle_{n=i}^\infty$ in $D_{i,i}$ such for any $F \in \mathcal{P}_f(\mathbb{N})$ and any $f : F \to \{1, 2, \ldots, \min F\}$, all sums of the form $\sum_{n\in F} x_{f(n),n}$ lie in a $D_{i,j}$ which is determined only by the first term and the last term in the sum. Somewhat less roughly it says that

(i) given any central subset D of $\mathbb{N}$, there exists a pairwise disjoint collection $\{D_{i,j} : i, j \in \omega\}$ of central subsets of D and for each $i \in \omega$ there exists a sequence $\langle x_{i,n}\rangle_{n=i}^\infty$ in $D_{i,i}$ such that whenever $F \in \mathcal{P}_f(\omega)$ and $f : F \to \{1, 2, \ldots, \min F\}$ one has that $\sum_{n\in F} x_{f(n),n} \in D_{i,j}$ where $i = f(\min F)$ and $j = f(\max F)$; and

(ii) at stage n when one is chosing $(x_{0,n}, x_{1,n}, \ldots, x_{n,n})$, each $x_{i,n}$ may be chosen from a central subset of $D_{i,i}$, with the choice of $x_{i,n}$ independent of the choice of $x_{j,n}$.

Notice in particular that if $\mathbb{N}$ is finitely colored, the central set D can be chosen to be one of the color classes.

2.8 Theorem. *Let D be a central subset of $\mathbb{N}$. Then there exist a choice of $D_{i,j}$ for each $i, j \in \omega$ and a tree T such that*

(1) for each $i, j \in \omega$, $D_{i,j}$ is a central subset of D;

(2) if $i, j, l, m \in \omega$ and $(i, j) \neq (l, m)$, then $D_{i,j} \cap D_{l,m} = \emptyset$;

(3) for each $g \in T$, if domain$(g) = n$, *then there exist* $U_0, U_1, \ldots, U_n$ *such that* $B_g = U_0 \times U_1 \times \ldots \times U_n$, *each* U_i *is central, and each* $U_i \subseteq D_{i,i}$*; and*

(4) if g *is a path through* T *and for each* $n \in \omega$, $g(n) = (x_{0,n}, x_{1,n}, \ldots, x_{n,n})$, *then whenver* $F \in \mathcal{P}_f(\omega)$, $f : F \to \{0, 1, \ldots, \min F\}$, $i = f(\min F)$, *and* $j = f(\max F)$, *one has* $\sum_{n \in F} x_{f(n),n} \in D_{i,j}$.

Proof. Pick by Theorem 2.6 a sequence $\langle p_i \rangle_{i=0}^{\infty}$ of idempotents in $c\ell_{\beta\mathbb{N}} D \cap K(\beta\mathbb{N})$ such that $p_i + p_j \neq p_l + p_m$ whenever $(i, j) \neq (l, m)$, $p_i + p_j + p_l = p_i + p_l$ for all $i, j, l \in \omega$, and $\{p_i + p_j : i, j \in \omega\}$ is discrete.

For each $i, j \in \omega$, pick $D_{i,j} \in p_i + p_j$ such that $D_{i,j} \subseteq D$ and, if $(i, j) \neq (l, m)$, then $D_{i,j} \cap D_{l,m} = \emptyset$. These sets then satisfy conclusions (1) and (2).

For $i, j, k \in \omega$, let

$$E_{k,p_i+p_j} = \{x \in D_{i,j} : \text{ for all } l, m \in \{0, 1, \ldots, k\}, -x + D_{i,m} \in p_l + p_m\}.$$

By Lemma 2.2 (with $H = \big\{p_l + p_m : l, m \in \{0, 1, \ldots, k\} \cup \{i, j\}\big\}$) we have that each $E_{k,p_i+p_j} \in p_i + p_j$. Also, for all $l, m \in \{0, 1, \ldots, k\}$ and all $x \in E_{k,p_i+p_j}$, one has that $-x + E_{k,p_i+p_m} \in p_l + p_m$.

We define the tree T by defining $T_n = \{g \in T : \text{domain}(g) = n\}$ inductively. Let $T_0 = \{\emptyset\}$ (of course). We let $B_\emptyset = E_{0,p_0}$ so that $T_1 = \big\{\{(0, x)\} : x \in E_{0,p_0}\big\}$, i.e., $g \in T_1$ if and only if domain$(g) = 1 = \{0\}$ and $g(0) \in E_{0,p_0}$. Now let $s \in \mathbb{N}$ and assume that T_m has been defined for $m \in \{0, 1, \ldots, s\}$ so that

(a) if $m \in \{0, 1, \ldots, s - 1\}$ and $g \in T_m$ then there exist $U_0, U_1, \ldots, U_m$ such that $B_g = U_0 \times U_1 \times \ldots \times U_m$ and for each $j \in \{0, 1, \ldots, m\}$, $U_j \in p_j$ and $U_j \subseteq D_{j,j}$ and

(b) if $m \in \{1, 2, \ldots, s\}$, $g \in T_m$, and for each $n \in \{0, 1, \ldots, m - 1\}$, $g(n) = (x_{0,n}, x_{1,n}, \ldots, x_{n,n})$, then whenever $\emptyset \neq F \subseteq \{0, 1, \ldots, m - 1\}$, $f : F \to \{0, 1, \ldots, \min F\}$, $i = f(\min F)$, and $j = f(\max F)$, one has $\sum_{n \in F} x_{f(n),n} \in E_{\min F, p_i + p_j}$.

Both hypotheses are satisfied for $s = 1$. We define T_{s+1} by defining B_g for each $g \in T_s$. (Then one has $T_{s+1} = \{g \frown \vec{x} : \vec{x} \in B_g\}$.) So let $g \in T_s$ be given. For each $n \in \{0, 1, \ldots, s-1\}$ let $g(n) = (x_{0,n}, x_{1,n}, \ldots, x_{n,n})$. Let

$$Y_g = \big\{ \textstyle\sum_{n \in F} x_{f(n),n} : \emptyset \neq F \subseteq \{0, 1, \ldots, s - 1\} \text{ and } f : F \to \{0, 1, \ldots, \min F\}\big\}$$

and note that Y_g is finite. Let $U_{g,s} = E_{s,p_s}$ and for each $j \in \{0, 1, \ldots, s - 1\}$ let

$$U_{g,j} = E_{s,p_j} \cap \bigcap \{-y + E_{k,p_i+p_j} : \begin{array}{l} i \in \{0, 1, \ldots, s - 1\}, k \in \{j, j + 1, \ldots, s - 1\}, \\ \text{and } y \in Y_g \cap \bigcup_{n=0}^{s-1} E_{k,p_i+p_n}\}. \end{array}$$

For each j, $U_{g,j}$ is a finite intersection of members of p_j so is in p_j. Let $B_g = U_{g,0} \times U_{g,1} \times \ldots \times U_{g,s}$.

Induction hypothesis (a) is satisfied directly. To verify hypothesis (b) let $h \in T_{s+1}$ and for each $n \in \{0,1,\ldots,s\}$, let $h(n) = (x_{0,n}, x_{1,n}, \ldots, x_{n,n})$. Let $g = h_{|s}$ so that $B_g = U_{g,0} \times U_{g,1} \times \ldots \times U_{g,s}$. Let $\emptyset \neq F \subseteq \{0,1,\ldots,s\}$, $f : F \to \{0,1,\ldots,\min F\}$, $i = f(\min F)$, and $j = f(\max F)$. If $s \notin F$, then $\sum_{n\in F} x_{f(n),n} \in E_{\min F, p_i+p_j}$ by hypothesis. So assume that $s \in F$. If $F = \{s\}$, then $j = f(s) \in \{0,1,\ldots,s\}$ and $x_{j,s} \in U_{g,j} \subseteq E_{s,p_j}$ as required. So assume that $\emptyset \neq G = F\backslash\{s\}$ and let $m = f(\max G)$. Then $y = \sum_{n\in G} x_{f(n),n} \in E_{\min G, p_i+p_m}$ so $y \in Y_g \cap E_{\min G, p_i+p_m}$. Also $j \leq \min F \leq s-1$, $i \in \{0,1,\ldots,s-1\}$, and $m \in \{0,1,\ldots, s-1\}$ so $x_{j,s} \in U_{g,j} \subseteq -y + E_{\min F, p_i+p_j}$ and thus $\sum_{n\in F} x_{f(n),n} \in E_{\min F, p_i+p_j}$. □

Conclusions (1) and (2) of Theorem 2.8 contrast with earlier results involving finite colorings in which separate cells for different kinds of expressions were guaranteed such as [5, Theorem 1.1] and [11, Theorem 2.9]; in these results the different cells could be forced to be in different color classes.

We observe that, if one is only concerned with choosing finitely many sequences, the restriction on the range of f in conclusion (4) of Theorem 2.8 can be eliminated.

2.9 Corollary. *Let D be a central subset of $\mathbb{N}$ and let $k \in \mathbb{N}$. Then there exist a choice of $D_{i,j}$ for each $i,j \in \{0,1,\ldots,k\}$ and a tree T such that*

(1) for each $i,j \in \{0,1,\ldots,k\}$, $D_{i,j}$ is a central subset of D;

(2) if $i,j,l,m \in \{0,1,\ldots,k\}$ and $(i,j) \neq (l,m)$, then $D_{i,j} \cap D_{l,m} = \emptyset$;

(3) for each $g \in T$ there exist $U_0, U_1, \ldots, U_k$ such that $B_g = U_0 \times U_1 \times \ldots \times U_k$, each U_i is central, and each $U_i \subseteq D_{i,i}$; and

(4) if g is a path through T and for each $n \in \omega$, $g(n) = (x_{0,n}, x_{1,n}, \ldots, x_{k,n})$, then whenever $F \in \mathcal{P}_f(\omega)$, $f : F \to \{0,1,\ldots,k\}$, $i = f(\min F)$, and $j = f(\max F)$, one has $\sum_{n\in F} x_{f(n),n} \in D_{i,j}$.

Proof. Pick a tree T' as guaranteed by Theorem 2.8. Pick $g_0 \in T'$ such that $\text{domain}(g_0) = k$. Let $T'' = \{g \in T' : g_{|k} = g_0\}$. For $g \in T''$ define $\varphi(g)$ with $\text{domain}\big(\varphi(g)\big) = \text{domain}(g) - k$ (so that $\varphi(g_0) = \emptyset$) and, if $\text{domain}(g) = n > k$, $i \in \{0,1,\ldots,n-k-1\}$, and $g(k+i) = (x_0, x_1, \ldots, x_n)$, then $\varphi(g)(i) = (x_0, x_1, \ldots, x_k)$. Let $T = \{\varphi(g) : g \in T''\}$. If $g \in T''$, $\text{domain}(g) = n \geq k$, and $B_g = U_0 \times U_1 \times \ldots \times U_n$ then $B_{\varphi(g)} = U_0 \times U_1 \times \ldots \times U_k$.

Let h be a path through T, for each $n \in \omega$ let $h(n) = (x_{0,n}, x_{1,n}, \ldots, x_{k,n})$, let $F \in \mathcal{P}_f(\omega)$, let $f : F \to \{0,1,\ldots,k\}$, let $i = f(\min F)$, and let $j = f(\max F)$. Let $m = \max F$. Then $h_{|m+1} \in T$. Pick $g \in T''$ with $\text{domain}(g) = k+m+1$ such that $h_{|m+1} = \varphi(g)$. Then by Theorem 2.8(3) g extends to a path g' through T'. For $n \in \omega$ let $g'(n) = (y_{0,n}, y_{1,n}, \ldots, y_{n,n})$. Let $G = k + F$ and define $f' : G \to \{0,1,\ldots,k\} \subseteq \{0,1,\ldots,\min G\}$ by $f'(n) = f(n-k)$. Then $\sum_{n\in F} x_{f(n),n} = \sum_{n\in G} y_{f'(n),n} \in D_{i,j}$. □

3. Sums in Central Subsets of Semigroups of Variable Words – Extending the Graham-Rothschild Theorem

Throught this section A will denote a nonempty countable set (the *alphabet*). We choose a set $V = \{v_n : n \in \omega\}$ (of *variables*) such that $A \cap V = \emptyset$ and define W to be the semigroup of words over the alphabet $A \cup V$ (including the empty word), with concatenation as the semigroup operation. (Formally a *word* w is a function with domain $k \in \omega$ to the alphabet and the length $\ell(w)$ of w is k. We shall occasionally need to resort to this formal meaning, so that if $i \in \{0, 1, \ldots, \ell(w) - 1\}$, then $w(i)$ denotes the $(i+1)^{\mathrm{st}}$ letter of w.)

For each $n \in \mathbb{N}$, we define W_n to be the set of words over the alphabet $A \cup \{v_0, v_1, \ldots, v_{n-1}\})$ and we define W_0 to be the set of words over A. We note that each W_n is a subsemigroup of W.

3.1 Definition. Let $n \in \omega$ and let $k \in \{0, 1, \ldots, n\}$. Then $[A]\binom{n}{k}$ is the set of all words w over the alphabet $A \cup \{v_0, v_1, \ldots, v_{k-1}\}$ of length n such that

(1) for each $i \in \{0, 1, \ldots, k-1\}$, if any, v_i occurs in w and

(2) for each $i \in \{0, 1, \ldots, k-2\}$, if any, the first occurrence of v_i in w precedes the first occurrence of v_{i+1}.

3.2 Definition. Let $k \in \mathbb{N}$. Then the set of *k-variable words* is $S_k = \bigcup_{n=k}^{\infty} [A]\binom{n}{k}$. Also $S_0 = W_0$.

Given $w \in S_n$ and $u \in W$ with $\ell(u) = n$, we define $w\langle u\rangle$ to be the word with length $\ell(w)$ such that for $i \in \{0, 1, \ldots, \ell(w) - 1\}$

$$w\langle u\rangle(i) = \begin{cases} w(i) & \text{if } w(i) \in A \\ u(j) & \text{if } w(i) = v_j \,. \end{cases}$$

That is, $w\langle u\rangle$ is the result of substituting $u(j)$ for each occurrence of v_j in w. (And if u is the empty word, then $w\langle u\rangle = w$.)

The following theorem is commonly known as the Graham-Rothschild Theorem. The original theorem [8] (or see [14]) is stated in a significantly stronger fashion. However this stronger version is derivable from Theorem 3.3 in a reasonably straightforward manner. (See [4, Theorem 5.1].)

3.3 Theorem (Graham-Rothschild). *Let $m, n \in \omega$ with $m < n$, and let S_m be finitely colored. There exists $w \in S_n$ such that $\left\{w\langle u\rangle : u \in [A]\binom{n}{m}\right\}$ is monochrome.*

In Theorem 3.8 we will extend the Graham-Rothschild Theorem in a fashion similar to the way the Finite Sums Theorem is extended by Theorem 2.8.

3.4 Definition. Let $u \in W$ with length n. Then $h_u : W \to W$ is the homomorphism such that, for all $w \in A \cup V$,

$$h_u(w) = \begin{cases} w & \text{if } w \in A \\ u(j) & \text{if } w = v_j \text{ and } j < n \\ w & \text{if } w = v_j \text{ and } j \geq n\,. \end{cases}$$

Notice that if $w \in S_n$, $u \in W$, and the length of u is n, then $h_u(w) = w\langle u\rangle$. Given $u \in W$, the function h_u has a continuous extension from βW to βW. We shall also denote this extension by h_u, and observe that $h_u : \beta W \to \beta W$ is a homomorphism. (See [12, Corollary 4.22].)

The following theorem is a special case of the main algebraic result of [4]. We observe that this theorem and, indeed, all the theorems in Section 3 are valid under far more general assumptions than those stated above. They hold for the general parameter systems introduced in [4], which we shall define in Section 4. We decided to restrict Section 3 to comparatively simple parameter systems in which the transformations h_u are much easier to understand.

3.5 Theorem. *Let p be a minimal idempotent in βS_0. There is a sequence $\langle p_n\rangle_{n=0}^{\infty}$ such that*

(1) $p_0 = p$;

(2) for each $n \in \mathbb{N}$, p_n is a minimal idempotent of βS_n;

(3) for each $n \in \mathbb{N}$, $p_n \leq p_{n-1}$;

(4) for each $n \in \mathbb{N}$ and each $u \in [A]\binom{n}{n-1}$, $h_u(p_n) = p_{n-1}$.

Further, p_1 can be any minimal idempotent of βS_1 such that $p_1 \leq p_0$.

Proof. This is [4, Theorem 2.12] in the case where $D = \{e\}$ and T_e is the identity. □

We shall refer to a sequence $\langle p_n\rangle_{n=0}^{\infty}$ with the properties stated in Theorem 3.5, as a *special reductive sequence*. Note that by [4, Lemma 1.10], if $\langle p_n\rangle_{n=0}^{\infty}$ is a special reductive sequence, $0 \leq m < n$, and $u \in [A]\binom{n}{m}$, then $h_u(p_n) = p_m$.

For our Ramsey Theoretic application, Theorem 3.8, we only need $p_{i,j,n}$ for $i, j \in \omega$ as produced by the following theorem. However, we think that the additional algebraic structure described is of independent interest. We are treating cardinal numbers as ordinals, so each is the set of its predecessors. In particular, the statement $i \in 2^{\mathfrak{c}}$ says that i is an ordinal smaller than $2^{\mathfrak{c}}$.

3.6 Definition. Let S be a semigroup and let $\psi : \omega \to S$. Then ψ is called an $\mathbb{H}$-map if it is bijective and if $\psi(m+n) = \psi(m)\psi(n)$ whenever $m, n \in \mathbb{N}$ satisfy $\max\big(\mathrm{supp}(m)\big) + 1 < \min\big(\mathrm{supp}(n)\big)$.

3.7 Theorem. *Let C be a compact G_δ subset of βS_0 which has a member which is a minimal idempotent. There is a minimal idempotent $p_0 \in C$ such that, for every sequence $\langle p_n \rangle_{n=1}^{\infty}$ such that $\langle p_n \rangle_{n=0}^{\infty}$ is a special reductive sequence and every sequence $\langle C_n \rangle_{n=0}^{\infty}$ for which C_n is a compact G_δ subset of βS_n and $p_n \in C_n$ for every $n \in \omega$, there exists a choice of $p_{i,j,n} \in K(\beta S_n)$ for each $i, j \in 2^{\mathfrak{c}}$ and each $n \in \omega$ such that:*

(1) for $i, j, k, l \in 2^{\mathfrak{c}}$ and $n \in \omega$, if $(i,j) \neq (k,l)$, then $p_{i,j,n} \neq p_{k,l,n}$;

(2) for $i, j, k, l \in 2^{\mathfrak{c}}$ and $n, m \in \omega$, $p_{i,j,n} p_{k,l,m} = p_{i,l,n \vee m}$;

(3) for $i, j \in 2^{\mathfrak{c}}$ and $n, m \in \omega$, if $m < n$ and $u \in [A]\binom{n}{m}$, then $h_u(p_{i,j,n}) = p_{i,j,m}$;

(4) $p_{i,j,m} \in C_m$ for every $m \in \omega$ and every $i, j \in 2^{\mathfrak{c}}$;

(5) for each $n \in \omega$, $\{p_{i,j,n} : i, j \in \omega\}$ is discrete; and

(6) for each $n \in \omega$, $p_{0,0,n} = p_n$.

Proof. We regard S_0 as embedded in the free group G generated by A and βS_0 as embedded in βG. We claim that G can be embedded algebraically in a compact metrizable topological group. To see this, for each $g \in G \setminus \{\emptyset\}$ pick by [12, Theorem 1.23] a finite group F_g and a homomorphism $\varphi_g : G \to F_g$ such that $\varphi_g(g)$ is not the identity of F_g. Let each F_g have the discrete topology and let $H = \times_{g \in G \setminus \{\emptyset\}} F_g$. Since G is countable, H is metrizable. Define $\tau : G \to H$ by $\tau(h)(g) = \varphi_g(h)$.

By [12, Theorem 7.28], there is an $\mathbb{H}$-map $\psi : \omega \to G$ such that the continuous extension from $\beta\omega$ to βG (also denoted by ψ) is an isomorphism on $\mathbb{H}$. Furthermore, $\psi[\mathbb{H}]$ contains all the idempotents of G^*.

Let r be a minimal idempotent of βS_0 for which $r \in C$. Choose V, q, q' as guaranteed by Lemma 2.5 with $p = \psi^{-1}(r)$ and $D = \psi^{-1}[C]$. Let $p_{-1} = \psi(q')$ and $p_0 = \psi(q)$. We observe that p_0 is minimal in $\psi[V]$ and hence $p_0 \in K(\beta S_0)$, because $r \in \psi[V]$ and so $K(\beta S_0) \cap \psi[V] \neq \emptyset$. Note that $p_0 < p_{-1}$.

Let $\langle p_n \rangle_{n=1}^{\infty}$ and $\langle C_n \rangle_{n=0}^{\infty}$ be sequences such that $\langle p_n \rangle_{n=0}^{\infty}$ is a special reductive sequence and for each $n \in \omega$, C_n is a compact G_δ subset of βS_n with $p_n \in C_n$. Choose a compact G_δ subset C_{-1} of βS_0 with $p_{-1} \in C_{-1}$ e.g. C_{-1} could be βS_0.

We shall define a subsemigroup Y of $\bigcup_{n=-1}^{\infty} C_n$ such that $p_n \in Y$ for every $n \in \{-1, 0, 1, \ldots\}$ and $Y \cap \beta S_n$ is a compact G_δ for every $n \in \omega$.

Given $n \in \omega$ pick a countable subfamily $\mathcal{F}_{n,0}$ of $\mathcal{P}(S_n)$ such that $C_n = \bigcap\{\overline{B} : B \in \mathcal{F}_{n,0}\}$. (We have that βS_n is extremally disconnected by [7, Exercise 6M] or [12, Theorem 3.18(f)] and so C_n is the intersection of countably many clopen sets, each of which is of the form $\overline{B}$ for some $B \subseteq S_n$ by [12, Theorem 3.18(b)].) Similarly pick a countable subfamily $\mathcal{F}_{-1,0}$ of $\mathcal{P}(S_0)$ such that $C_{-1} = \bigcap\{\overline{B} : B \in \mathcal{F}_{-1,0}\}$. Notice that for each $n \in \{-1,0,1,\ldots\}$, $\mathcal{F}_{n,0} \subseteq p_n$.

Inductively, let $k \in \omega$ and assume that we have chosen for each $n \in \{-1,0,1,\ldots\}$ a countable subfamily $\mathcal{F}_{n,k}$ of p_n. For each $n \in \{-1,0,1,\ldots\}$ let

$$\mathcal{F}_{n,k+1} = \mathcal{F}_{n,k} \cup \big\{\{w \in W : w^{-1}B \in p_m\} : m \in \{-1,0,1,\ldots\} \text{ and } B \in \mathcal{F}_{m\vee n,k}\big\} \cup \{w^{-1}B : B \in \mathcal{F}_{m,k} \text{ for some } m \geq n,\ w \in W, \text{ and } w^{-1}B \in p_n\}\,.$$

Given $B \in \mathcal{F}_{m\vee n,k}$ and $m \in \{-1,0,1,\ldots\}$, $\{w : w^{-1}B \in p_m\} \in p_n$ because $B \in p_{m\vee n} = p_n p_m$.

For each $n \in \{-1,0,1,\ldots\}$ let $\mathcal{F}_n = \bigcup_{k=0}^{\infty} \mathcal{F}_{n,k}$. Let

$$Y = \bigcup_{n=-1}^{\infty}\{s \in \beta W : \mathcal{F}_n \subseteq s\}\,.$$

We have immediately that $Y \subseteq \bigcup_{n=-1}^{\infty} C_n$, $p_n \in Y$ for every $n \in \{-1,0,1,\ldots\}$, and $Y \cap \beta S_n$ is a compact G_δ for every $n \in \omega$. It remains to show that Y is a subsemigroup. To this end, let $m,n \in \{-1,0,1,\ldots\}$ and let $s,t \in \beta W$ with $\mathcal{F}_n \subseteq s$ and $\mathcal{F}_m \subseteq t$. We show that $\mathcal{F}_{m\vee n} \subseteq st$. So let $B \in \mathcal{F}_{m\vee n}$ and pick $k \in \omega$ such that $B \in \mathcal{F}_{m\vee n,k}$. We need to show that $\{w \in W : w^{-1}B \in t\} \in s$. Now $\{w \in W : w^{-1}B \in p_m\} \in \mathcal{F}_{n,k+1} \subseteq s$. Given $w \in W$, if $w^{-1}B \in p_m$, then $w^{-1}B \in \mathcal{F}_{m,k+1} \subseteq t$.

Now $L = \psi^{-1}[Y \cap \beta S_0] \cap V$ is a compact G_δ subsemigroup of V which contains q and q'. By Lemma 2.5, there is a compact G_δ subsemigroup M of L which contains q and q' and there are compact subsets B and D of $\beta\omega$, with $|B| = |D| = 2^{\mathfrak{c}}$, for which there is a continuous surjective homomorphism $f : M \to C_{B,D}$. Furthermore, $q' \in K(f^{-1}[B \times D])$ and $q \in K(f^{-1}[D])$. Let $Z = \psi[M]$ and $f_1 = f \circ \psi^{-1}$. Then $f_1 : Z \to C_{B,D}$ is a continuous surjective homomorphism. Furthermore, $p_0 \in K({f_1}^{-1}[B \times D])$ and $p_{-1} \in K({f_1}^{-1}[D])$, because $p_0 = \psi(q)$ and $p_{-1} = \psi(q')$.

Pick by Lemma 2.3 a homomorphism $g : C_{B,D} \to Z$ such that $f_1 \circ g$ is the identity on $C_{B,D}$, $g[D] \subseteq K({f_1}^{-1}[B])$, $g[B \times D] \subseteq K({f_1}^{-1}[B \times D])$, $g(y_0) = p_{-1}$ and $g(x_0,y_0) = p_0$ for some $x_0 \in B$ and $y_0 \in D$.

Note that for any $a \in A$,

$$(*)\qquad \begin{aligned} p_0 g(a,y_0) &= g(x_0,y_0)g(a,y_0) \\ &= g\big((x_0,y_0)(a,y_0)\big) \\ &= g(x_0,y_0) \\ &= p_0\,. \end{aligned}$$

Pick injective sequences $\langle a_j\rangle_{j=0}^{\infty}$ in B and $\langle b_j\rangle_{j=0}^{\infty}$ in D such that $a_0 = x_0$, $b_0 = y_0$, and $\{a_j : j \in \omega\}$ and $\{b_j : j \in \omega\}$ are discrete. (One may do this because $\beta\omega$ contains no convergent sequences that are not eventually constant. See [7, Corollary 9.12] or [12, Theorem 3.59].) Extend these sequences to enumerations $\langle a_j\rangle_{j\in 2^{\mathfrak{c}}}$ and $\langle b_j\rangle_{j\in 2^{\mathfrak{c}}}$ of B and D respectively.

For $i, j \in 2^{\mathfrak{c}}$ and $n \in \omega$, let $p_{i,j,n} = g(a_i, y_0)p_n g(b_j)$.

To verify conclusion (1), let $i, j, k, l \in 2^{\mathfrak{c}}$, let $n \in \omega$, and assume that $p_{i,j,n} = p_{k,l,n}$. Then $g(a_i, y_0)p_n g(b_j) = g(a_k, y_0)p_n g(b_l)$. Pick any $u \in [A]\binom{n}{0}$. Then

$$g(a_i, b_j) = g(a_i, y_0)g(x_0, y_0)g(b_j) = g(a_i, y_0)p_0 g(b_j) = h_u\big(g(a_i, y_0)p_n g(b_j)\big)$$

and $g(a_k, b_l) = h_u\big(g(a_k, y_0)p_n g(b_l)\big)$ so $(i, j) = (k, l)$ since g is injective.

To verify conclusion (2), let $i, j, k, l \in 2^{\mathfrak{c}}$ and let $n, m \in \omega$. Then

$$\begin{aligned} p_{i,j,n}p_{k,l,m} &= g(a_i, y_0)p_n g(b_j)g(a_k, y_0)p_m g(b_l) \\ &= g(a_i, y_0)p_n g\big(b_j(a_k, y_0)\big)p_m g(b_l) \\ &= g(a_i, y_0)p_n p_0 g(a_k, y_0)p_m g(b_l) \\ &= g(a_i, y_0)p_n p_0 p_m g(b_l) \text{ by } (*) \\ &= g(a_i, y_0)p_{m\vee n} g(b_l) \\ &= p_{i,l,m\vee n}\,. \end{aligned}$$

To verify conclusion (3) let $i, j \in 2^{\mathfrak{c}}$, let $m, n \in \omega$ with $m < n$ and let $u \in [A]\binom{n}{m}$. Note that since $g(a_i, y_0) \in \beta S_0$ and $g(b_j) \in \beta S_0$, $h_u\big(g(a_i, y_0)\big) = g(a_i, y_0)$ and $h_u\big(g(b_j)\big) = g(b_j)$. Thus

$$\begin{aligned} h_u(p_{i,j,n}) &= h_u\big(g(a_i, y_0)\big)h_u(p_n)h_u\big(g(b_j)\big) \\ &= g(a_i, y_0)p_m g(b_j) \\ &= p_{i,j,m}\,. \end{aligned}$$

Conclusion (4) holds directly.

Conclusion (5) holds because $\{a_j : j \in \omega\}$ and $\{b_j : j \in \omega\}$ are discrete and f_1 is continuous. This implies that $\{p_{i,j,0} : i, j \in \omega\}$ is discrete. Since, for any $u \in [A]\binom{n}{0}$ $h_u(p_{i,j,n}) = p_{i,j,0}$, $\{p_{i,j,n} : i, j \in \omega\}$ is discrete for every $n \in \omega$.

To verify conclusion (6), let $n \in \omega$. Then

$$p_{0,0,n} = g(a_0, y_0)p_n g(b_0) = g(x_0, y_0)p_n g(y_0) = p_0 p_n p_{-1} = p_n \qquad \square$$

The following theorem is the main result of this section. However, its statement may be a bit intimidating, so we shall attempt to describe loosely what it says. One starts with a finite coloring of S_n for each $n \in \omega$ and one chooses a monochrome central subset $D_{i,j,m}$ of S_m for each $i, j, m \in \omega$. Conclusion (5) tells us that for each j and m in ω one may choose a sequence $\langle w_{j,m,n}\rangle_{n=j\vee m}^{\infty}$ in S_m so that all suitably restricted products of the form $\prod_{n\in F} w_{j_n,m_n,n}\langle u_n\rangle$ lie in one of the chosen cells; the cell in which the product lies is determined solely by $j_{\min F}$, $j_{\max F}$, and the number of variables in the result. Conclusion (4) tells us that, having chosen the terms $\langle w_{j,m,n}\rangle_{n=j\vee m}^{k}$, one has a large number of choices for terms of the form $w_{j,m,k+1}$; specifically, there is a central set from which each such term may be chosen. (The notation $\prod_{n\in F} x_n$ represents the product taken in increasing order of indices.)

In conclusion (4) of the following theorem, we write "$B_g = \times_{j=0}^{n} \times_{m=0}^{n} U_{g,j,m}$". Formally a member of $\times_{j=0}^{n} \times_{m=0}^{n} U_{g,j,m}$ is a function with domain $\{0, 1, \ldots, n\}$ taking values in $\times_{m=0}^{n} U_{g,j,m}$. We shall pretend however that

$$\times_{j=0}^{n} \times_{m=0}^{n} U_{g,j,m} = \times_{(j,m)\in\{0,1,\ldots,n\}\times\{0,1,\ldots,n\}} U_{g,j,m}$$

so that one has $x(j, m) \in U_{g,j,m}$ if $x \in \times_{j=0}^{n} \times_{m=0}^{n} U_{g,j,m}$ and $j, m \in \{0, 1, \ldots, n\}$.

3.8 Theorem. *Assume that the alphabet A is finite. Let C be a central subset of S_0. For each $m \in \omega$, let S_m be finitely colored. There exist a choice of $D_{i,j,m}$ for $i, j, m \in \omega$ and a tree T such that*

(1) for each $i, j \in \omega$, $D_{i,j,0} \subseteq C$;

(2) for each $i, j, m \in \omega$, $D_{i,j,m}$ is a central subset of S_m and for each $m \in \omega$, $\bigcup\{D_{i,j,m} : i, j \in \omega\}$ is monochrome;

(3) if $i, j, k, l, m \in \omega$ and $(i, j) \neq (k, l)$, then $D_{i,j,m} \cap D_{k,l,m} = \emptyset$;

(4) for each $g \in T$, if $\mathrm{domain}(g) = n$, then for each $j, m \in \{0, 1, \ldots, n\}$ there exists a subset $U_{g,j,m}$ of $D_{j,j,m}$ which is central in S_m such that $B_g = \times_{j=0}^{n} \times_{m=0}^{n} U_{g,j,m}$; and

(5) if g is a path through T and for each $n \in \omega$ and each $j, m \in \{0, 1, \ldots, n\}$, $w_{j,m,n} = g(n)(j, m)$, then given any $F \in \mathcal{P}_f(\omega)$, any $f : F \to \{0, 1, \ldots, \min F\}$, any $\alpha : F \to \{0, 1, \ldots, \max F\}$ with $\alpha(n) \leq n$ for $n \in F$, and any $\delta \in \times_{n\in F} \bigcup_{b=0}^{\min F} [A]\binom{\alpha(n)}{b}$, if $j = f(\max F)$, $i = f(\min F)$, and

$$r = \max\left\{b : \textit{ there exists } n \in F \textit{ such that } \delta(n) \in [A]\binom{\alpha(n)}{b}\right\},$$

then $\prod_{n\in F} w_{f(n),\alpha(n),n}\langle\delta(n)\rangle \in D_{i,j,r}$.

Proof. Choose $p_0 \in \overline{C}$ as guaranteed by Theorem 3.7 and choose by Theorem 3.5 $\langle p_n\rangle_{n=1}^{\infty}$ such that $\langle p_n\rangle_{n=0}^{\infty}$ is a special reductive sequence. For each $n \in \omega$, let C_n be a monochrome subset of S_n in p_n. We may assume that $C_0 \subseteq C$. Choose $p_{i,j,n}$ for each $i, j, n \in \omega$ as guaranteed by Theorem 3.7. For each $i,j,n \in \omega$ pick $M_{p_{i,j,n}} \in p_{i,j,n}$ such that $M_{p_{i,j,n}} \subseteq S_n$, $M_{p_{i,j,n}} \subseteq C_n$, and $M_{p_{i,j,n}} \cap M_{p_{k,l,n}} = \emptyset$ if $(i,j) \neq (k,l)$. Let $D_{i,j,n} = M_{p_{i,j,n}}$.

For each $k \in \omega$, let $H_k = \big\{p_{i,j,n} : i,j,n \in \{0,1,\ldots,k\}\big\}$. Then H_k is a finite subsemigroup of βW. For $p \in H_k$, let $E_{k,p} = \{w \in M_p : \text{for all } q \in H_k\,,\ w^{-1}M_{pq} \in q\}$. If $p, q \in H_k$ and $y \in E_{k,p}$ we have by Lemma 2.2 that $E_{k,p} \in p$ and $y^{-1}E_{k,pq} \in q$.

We define the tree T by defining $T_n = \{g \in T : \text{domain}(g) = n\}$ inductively. Let $T_0 = \{\emptyset\}$ (of course). Let $U_{\emptyset,0,0} = E_{0,p_{0,0,0}}$, let $B_\emptyset = \times_{j=0}^{0} \times_{m=0}^{0} U_{\emptyset,j,m}$, and let $T_1 = \big\{\{(0,x)\} : x \in B_\emptyset\big\}$.

Now let $k \in \mathbb{N}$ and assume that we have defined T_l for $l \in \{0,1,\ldots,k\}$ such that

(a) if $l \in \{0,1,\ldots,k-1\}$ and $g \in T_l$, then $B_g = \times_{j=0}^{l} \times_{m=0}^{l} U_{g,j,m}$ where for each $j,m \in \{0,1,\ldots,l\}$, $U_{g,j,m} \in p_{j,j,m}$ and $U_{g,j,m} \subseteq D_{j,j,m}$ and

(b) if $l \in \{1,2,\ldots,k\}$, $g \in T_l$, and for each $n \in \{0,1,\ldots,l-1\}$ and each $j,m \in \{0,1,\ldots,n\}$, $w_{j,m,n} = g(n)(j,m)$, then given any F with $\emptyset \neq F \subseteq \{0,1,\ldots,l-1\}$, any $\alpha : F \to \{0,1,\ldots,\max F\}$ with $\alpha(n) \leq n$ for $n \in F$, any $f : F \to \{0,1,\ldots,\min F\}$, and any $\delta \in \times_{n\in F}\Big(\bigcup_{b=0}^{\min F}[A]\binom{\alpha(n)}{b}\Big)$, if $i = f(\min F)$, $j = f(\max F)$, and $r = \max\Big\{b :\ \text{there exists } n \in F \text{ with } \delta(n) \in [A]\binom{\alpha(n)}{b}\Big\}$, then
$$\textstyle\prod_{n\in F}\, w_{f(n),\alpha(n),n}\langle\delta(n)\rangle \in E_{\min F,p_{i,j,r}}\,.$$

One sees directly that hypothesis (a) holds at $k = 1$. Hypothesis (b) says that if $w_{0,0,0} \in U_{\emptyset,0,0}$ and $\delta(0) \in [A]\binom{0}{0}$, Then $w_{0,0,0}\langle\delta(0)\rangle \in E_{0,p_{0,0,0}}$. Since then $\delta(0)$ is the empty word, this is true.

We define T_{k+1} by defining B_g for each $g \in T_k$ and then letting
$$T_{k+1} = \{g^\frown x : g \in T_k \text{ and } x \in B_g\}\,.$$

So let $g \in T_k$. For each $n \in \{0,1,\ldots,k-1\}$ and each $j,m \in \{0,1,\ldots,n\}$, let $w_{j,m,n} = g(n)(j,m)$. Let
$$\begin{aligned} Y_g = \Big\{ &\textstyle\prod_{n\in F}\, w_{f(n),\alpha(n),n}\langle\delta(n)\rangle : \emptyset \neq F \subseteq \{0,1,\ldots,k-1\}\,,\\ &\alpha : F \to \{0,1,\ldots,\max F\}\,,\ \alpha(n) \leq n \text{ for } n \in F\,,\\ &f : F \to \{0,1,\ldots,\min F\}, \text{ and } \delta \in \times_{n\in F}\Big(\textstyle\bigcup_{b=0}^{\min F}[A]\binom{\alpha(n)}{b}\Big)\Big\}\,. \end{aligned}$$

Note that Y_g is finite.

Now let $j,m \in \{0,1,\ldots,k\}$ and let
$$L_1 = \bigcap\big\{{h_u}^{-1}[E_{k,p_{j,j,r}}] : r \in \{0,1,\ldots,m\} \text{ and } u \in [A]\tbinom{m}{r}\big\}\,.$$

Notice that for any $r \in \{0,1,\ldots,m\}$ and any $u \in [A]\binom{m}{r}$, $h_u(p_{j,j,m}) = p_{j,j,r}$ so ${h_u}^{-1}[E_{k,p_{j,j,r}}] \in p_{j,j,m}$. (If $r < m$, then $h_u(p_{j,j,m}) = p_{j,j,r}$ by Theorem 3.7(3) while if $r = m$, h_u is the identity on W and ${h_u}^{-1}[E_{k,p_{j,j,r}}] = E_{k,p_{j,j,m}}$.) Thus $L_1 \in p_{j,j,m}$ and $L_1 \subseteq E_{k,p_{j,j,m}}$. If $j = k$, let $U_{g,j,m} = L_1$. Then $U_{g,j,m} \in p_{j,j,m}$.

Assume now that $j \in \{0,1,\ldots,k-1\}$. Let

$$\begin{aligned} L_2 = \bigcap \big\{ {h_u}^{-1}[y^{-1}E_{t,p_{i,j,s\vee b}}] : {}& \text{for some } b \in \{0,1,\ldots,m \wedge (k-1)\}, \text{ some} \\ & t \in \{b, b+1,\ldots,k-1\}, \text{ some } i,l,s,j \in \{0,1,\ldots,t\}, \\ & \text{some } y \in Y_g \cap E_{t,p_{i,l,s}}, \text{ and some } u \in [A]\tbinom{m}{b} \big\} \end{aligned}$$

Let $U_{g,j,m} = L_1 \cap L_2$. To see that $L_2 \in p_{j,j,m}$ let $b \in \{0,1,\ldots,m \wedge (k-1)\}$, let $t \in \{b, b+1,\ldots,k-1\}$, let $i,l,s,j \in \{0,1,\ldots,t\}$, let $y \in Y_g \cap E_{t,p_{i,l,s}}$, and let $u \in [A]\binom{m}{b}\}$. Then $p_{i,l,s}$ and $p_{j,j,b}$ are in H_t and $p_{i,l,s}p_{j,j,b} = p_{i,j,s\vee b}$ so by Lemma 2.2 $y^{-1}E_{t,p_{i,j,s\vee b}} \in p_{j,j,b}$. Since $u \in [A]\binom{m}{b}$, $h_u(p_{j,j,m}) = p_{j,j,b}$ and so ${h_u}^{-1}[y^{-1}E_{t,p_{i,j,s\vee b}}] \in p_{j,j,m}$. Since Y_g is finite, $L_2 \in p_{j,j,m}$.

Let $B_g = \times_{j=0}^{k} \times_{m=0}^{k} U_{g,j,m}$. Then hypothesis (a) is satisfied.

To verify hypothesis (b), let $g' \in T_{k+1}$ and for each $n \in \{0,1,\ldots,k\}$ and each $j,m \in \{0,1,\ldots,n\}$, let $w_{j,m,n} = g'(n)(j,m)$. Let $g = {g'}_{|k}$ and note that for all $j,m \in \{0,1,\ldots,k\}$, $w_{j,m,k} \in U_{g,j,m}$. Let $\emptyset \neq F \subseteq \{0,1,\ldots,k\}$, let $\alpha : F \to \{0,1,\ldots,\max F\}$ with $\alpha(n) \leq n$ for $n \in F$, let $f : F \to \{0,1,\ldots,\min F\}$, let $\delta \in \times_{n\in F}\left(\bigcup_{b=0}^{\min F}[A]\binom{\alpha(n)}{b}\right)$, let $i = f(\min F)$, let $j = f(\max F)$, and let

$$r = \max\left\{b : \text{ there exists } n \in F \text{ with } \delta(n) \in [A]\tbinom{\alpha(n)}{b}\right\} .$$

Let $z = \prod_{n\in F} w_{f(n),\alpha(n),n}\langle\delta(n)\rangle$. We must show that $z \in E_{\min F, p_{i,j,r}}$.

We may assume that $k \in F$. Let $m = \alpha(k)$ and $u = \delta(k)$. Pick $b \in \{0,1,\ldots,m\}$ such that $u \in [A]\binom{m}{b}$. Notice that $b \leq \min F$.

Assume first that $F = \{k\}$. Then $i = j = f(k)$, $r = b$, and $w_{f(k),\alpha(k),k} = w_{j,m,k} \in U_{g,j,m} \subseteq {h_u}^{-1}[E_{k,p_{j,j,r}}]$ so $z = w_{j,m,k}\langle u\rangle \in E_{k,p_{i,j,r}}$ as required.

Now assume that $\{k\} \subsetneq F$ and let $G = F \setminus \{k\}$. Let $t = \min F = \min G$, let $l = f(\max G)$, let $s = \max\left\{a : \text{ there exists } n \in G \text{ with } \delta(n) \in [A]\binom{\alpha(n)}{a}\right\}$, and let $y = \prod_{n\in G} w_{f(n),\alpha(n),n}\langle\delta(n)\rangle$. Then $z = yw_{j,m,k}\langle u\rangle$ and $y \in Y_g \cap E_{t,p_{i,l,s}}$. Notice that $r = s \vee b$. Since $w_{j,m,k} \in L_2$ we have $w_{j,m,k}\langle u\rangle = h_u(w_{j,m,k}) \in y^{-1}E_{t,p_{i,j,r}}$ so $z \in E_{t,p_{i,j,r}}$. $\square$

4. Image Partition Regular Matrices

In this section we give an application of [4, Theorem 2.12] to the theory of image partition regular matrices. We recall that a finite $u \times v$ matrix A with entries in ω is said to be *image partition regular* over $\mathbb{N}$ if, given any finite coloring of $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ for which the set of entries of $A\vec{x}$ is monochrome. Many of the classical theorems of Ramsey Theory can be expressed in terms of image partition regularity. For example, van der Waerden's Theorem which states that, given any finite coloring of $\mathbb{N}$, there is an arbitrarily long monochrome arithmetic progression, is equivalent to the claim that all matrices of the form $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ \vdots & \vdots \\ 1 & l \end{pmatrix}$ are image partition regular over $\mathbb{N}$. Image partition regularity over $\mathbb{N}$ has been extensively studied. We shall see that some of the important theorems on this subject are valid for matrices whose entries lie in more general sets.

We first need to introduce parameter systems more general than those defined in Section 3. In the process we will be redefining W_n, S_n, $[A]\binom{n}{k}$, and h_u.

Let A be a nonempty set and let D be a set with a binary operation mapping $(f,g) \in D \times D$ to $fg \in D$. We assume that D has a nonempty set E of right identities for this operation. We also assume that, for each $f \in D$, we have defined a mapping $T_f : A \to A$. We shall call $(A, D, E, \langle T_f \rangle_{f\in D})$ a *parameter system.*

Given a parameter system $(A, D, E, \langle T_f \rangle_{f\in D})$, we choose a set $V = \{\nu_n : n \in \omega\}$ such that $A \cap (D \times V) = \emptyset$ and define W to be the semigroup of words over $A \cup (D \times V)$, with concatenation as the semigroup operation. For each $n \in \mathbb{N}$, we define W_n to be the set of words over $A \cup (D \times \{\nu_0, \nu_1, \cdots, \nu_{n-1}\})$, and we define W_0 to be the set of words over A.

4.1 Definition. Let $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$. Then $[A]\binom{n}{k}$ is the set of all words $w \in W_k$ of length n such that:

(1) for each $i \in \{0, 1, \cdots, k-1\}$, if any, some member of $E \times \{\nu_i\}$ occurs in w;

(2) for each $i \in \{0, 1, \cdots, k-1\}$, if any, the first member of $D \times \{\nu_i\}$ which occurs in w is in $E \times \{\nu_i\}$; and

(3) for each $i \in \{0, 1, \cdots, k-2\}$, if any, the first occurrence of a member of $D \times \{\nu_i\}$ in w precedes the first occurrence of a member of $D \times \{\nu_{i+1}\}$.

4.2 Definition. Let $k \in \mathbb{N}$. Then $S_k = \bigcup_{n=k}^{\infty}[A]\binom{n}{k}$.

For each $u \in [A]\binom{n}{k}$, we define $h_u : S_n \to S_k$ by stating that, for each $w \in S_n$ of length

l, $h_u(w)$ is the word of length l defined as follows for $i \in \{0, 1, 2, \cdots, l-1\}$:

$$h_u(w)(i) = \begin{cases} w(i) \text{ if } w(i) \in A \\ T_s\big(u(j)\big) \text{ if } w(i) = (s, \nu_j) \text{ and } u(j) \in A \\ (st, \nu_m) \text{ if } w(i) = (s, \nu_j) \text{ and } u(j) = (t, \nu_m) \end{cases} .$$

Note that the mappings h_u of Definition 3.4 coincide with those defined in the preceding paragraph in the special case in which $D = \{e\}$, a singleton, $v_j = (e, \nu_j)$ for each j, and T_e is the identity mapping.

Theorem 3.5 is valid for any parameter system $(A, D, E, \langle T_f \rangle_{f \in D})$ by [4, Theorem 2.12]. In this section, we shall apply Theorem 3.5 to a parameter system which we now proceed to define.

Let $(M, +)$ be a commutative semigroup with an identity 0_M and let R be a set with two distinguished elements, 0_R and 1_R. Suppose that there is a mapping $(r, m) \mapsto rm$ from $R \times M$ to M with the following properties:

For every $r \in R$ and every $m, n \in M$,

$$r(m+n) = rm + rn\,,$$

$$0_R m = 0_M, \text{ and}$$

$$1_R m = m\,.$$

We give some examples of algebraic structures of this kind to show that they occur very widely.

(a) R could be a ring and M could be an R-module.

(b) R could be ω and M could be an arbitrary commutative semigroup with an identity.

(c) If S is an arbitrary set, we could have $M = (\mathcal{P}(S), \cup)$, $R = \mathcal{P}(S)$, and $rm = r \cap m$.

To define our parameter system, we choose D to be the union of disjoint copies of R and M. We put $R' = R \times \{0\}$, $M' = M \times \{1\}$ and $D = R' \cup M'$. We define a binary operation on D by putting $d_1 d_2 = d_1$ unless $d_1 \in M'$ and $d_2 \in R'$. If $m \in M$ and $r \in R$, we put $(m, 1)(r, 0) = (rm, 1)$. We put $E = \{(1_R, 0)\}$. The alphabet A can be an arbitrary non-empty set and the mappings T_d can be arbitrary.

4.3 Definition. Let B be a finite matrix over R. We shall say that B is a *first entries matrix* provided no row of B has all its entries equal to 0_R and the first nonzero entries of any two rows are equal if they occur in the same column.

The first nonzero entry of any row will be called a *first entry.*

4.4 Theorem. *Let M and R satisfy the conditions stated in the preceding paragraphs. Let*

B be a finite $s \times t$ first entries matrix over R whose first entries are all equal to 1_R. For any central subset C of $(M,+)$, there is a vector $\vec{x} \in M^t$ such that $B\vec{x} \in C^s$.

Proof. For each $k \in \omega$, we define $f_k : W \to M$ as follows. For $w \in W$ let $\ell(w)$ be the length of w and let $I_k = \big\{t \in \{0,1,\ldots,\ell(w)-1\} : w(t) \in M' \times \{\nu_k\}\big\}$. Then let $f_k(w) = \sum_{t \in I_k} \pi\big(w(t)\big)$, where $\pi\big((m,1),\nu_k\big) = m$ and $\sum \emptyset = 0_M$. Notice that f_k is a homomorphism. We shall also use f_k to denote the continuous extension of f_k mapping βW to βM. Notice that this extension is also a homomorphism by [12, Corollary 4.22].

Let q be a minimal idempotent of $(\beta M,+)$ for which $C \in q$ and pick a minimal idempotent p_0 of βW_0. We observe that $f_0[S_1] = M$, because if $w = \big((1_R,0),\nu_0\big)\big((m,1)\nu_0\big)$, then $f_0(w) = m$. So by [12, Exercise 1.7.3], ${f_0}^{-1}[\{q\}]$ meets $K(\beta S_1)$ and consequently $K({f_0}^{-1}[\{q\}]) \subseteq K(\beta S_1)$. We observe that ${f_0}^{-1}[\{q\}]p_0 \subseteq {f_0}^{-1}[\{q\}]$, because $f_0(p_0) = 0_M$ and f_0 is a homomorphism. Note also that ${f_0}^{-1}[\{q\}]p_0$ is a left ideal of ${f_0}^{-1}[\{q\}]$. Similarly, $p_0{f_0}^{-1}[\{q\}]$ is a right ideal of ${f_0}^{-1}[\{q\}]$. So we can choose a minimal idempotent p_1 of ${f_0}^{-1}[\{q\}]$ in ${f_0}^{-1}[\{q\}]p_0 \cap p_0{f_0}^{-1}[\{q\}]$. Then $p_1 \leq p_0$ and $p_1 \in K(\beta S_1)$.

By Theorem 3.5, pick a sequence $\langle p_k\rangle_{k=2}^{\infty}$ such that $\langle p_k\rangle_{k=0}^{\infty}$ is a special reductive sequence. Denote the entry in row i and column j of B by $b_{i,j}$.

Given $r \in \{0,1,\ldots,s-1\}$, define $u_r \in [A]\binom{n}{1}$ as follows. Pick $a \in A$. If the first nonzero entry of row r of B is in column i and $k \in \{0,1,\ldots,t-1\}$, then

$$u_r(k) = \begin{cases} a & \text{if } k < i \\ \big((b_{r,k},0),\nu_0\big) & \text{if } k \geq i\,. \end{cases}$$

We claim that for any $w \in W_t$,

$$f_0\big(h_{u_r}(w)\big) = f_i(w) + b_{r,i+1}f_{i+1}(w) + b_{r,i+2}f_{i+2}(w) + \ldots + b_{r,t-1}f_{t-1}(w)\,.$$

To see this, let the length of w be l and let $j \in \{0,1,\ldots,l-1\}$. If $w(j) \in A$ or $w(j) \in D \times \{\nu_k\}$ for some $k < i$, then $h_{u_r}(w)(j) \in A$ and so adds nothing to $f_0\big(h_{u_r}(w)\big)$. If $w(j) = \big((x,0),\nu_k\big)$ for some $k \in \{i,i+1,\ldots,t-1\}$ and some $x \in R$, then $h_{u_r}(w)(j) = \big((x,0),\nu_0\big)$, which again adds nothing to $f_0\left(h_{u_r}(w)\right)$. If $w(j) = \big((m,1),\nu_k\big)$ for some $k \in \{i,i+1,\cdots,t-1\}$ and some $m \in M$, then $h_{u_r}(w)(j) = \big((b_{r,k}m,1),\nu_0\big)$. So $b_{r,k}m$ is added to $f_0\big(h_{u_r}(w)\big)$ whenever m is added to $f_k(w)$.

Now ${f_0}^{-1}[C] \in p_1$ and for each $r \in \{0,1,\ldots,s-1\}$, $h_{u_r}(p_t) = p_1$. Thus

$$\textstyle\bigcap_{r=0}^{s-1} {h_{u_r}}^{-1}\big[{f_0}^{-1}[C]\big] \in p_t\,.$$

Pick $w \in S_t \cap \bigcap_{r=0}^{s-1} {h_{u_r}}^{-1}\big[{f_0}^{-1}[C]\big]$. For $i \in \{0,1,\ldots,t-1\}$, let $x_i = f_i(w)$. Then, given $r \in \{0,1,\ldots,m-1\}$, $\sum_{i=0}^{t-1} b_{r,i}x_i = f_0\big(h_{u_r}(w)\big) \in C$. $\square$

4.5 Corollary. *Let B and C be as stated in Theorem 4.3. There is a vector $\vec{x} \in C^t$ such that all the entries of $B\vec{x}$ are in C.*

Proof. We can apply Theorem 4.3 to the first entries matrix $\begin{pmatrix} I \\ B \end{pmatrix}$, where I denotes the identity $t \times t$ matrix. $\square$

4.6 Corollary. *Suppose that R is a ring with multiplicative identity, that M is an infinite R-module and that B is a finite first entries $s \times t$ matrix over R whose first entries have inverses in R. Then, for any central subset C of $(M, +)$, there exists $\vec{x} \in (M \setminus \{0_M\})^t$ such that the entries of $B\vec{x}$ are all in C.*

Proof. We may assume that $0_M \notin C$, because $C \setminus \{0_M\}$ is a central set in $(M, +)$. (By [12, Theorem 4.36] $K(\beta M) \subseteq \beta M \setminus M$ so 0_M is not a minimal idempotent.) Assume that the first entries of B occur in the ${j_1}^{\text{th}}$, ${j_2}^{\text{th}}$, . . ., ${j_k}^{\text{th}}$ columns, and that the first entries in these columns are $c_1, c_2, \cdots, c_k$ respectively. Let P denote the diagonal $t \times t$ matrix whose ${j_i}^{\text{th}}$ diagonal entry is c_i^{-1} for each $i \in \{1, 2, \cdots, k\}$ and whose other diagonal entries are 1_R. Then BP is a first entries matrix over R whose first entries are all 1_R. Our claim follows by applying Corollary 4.4 to this matrix in place of B, observing that $P\vec{x} \in (M \setminus \{0_M\})^s$ if $\vec{x} \in (M \setminus \{0_M\})^t$. $\square$

4.7 Corollary. *Suppose that M is infinite and that M, R and B satisfy either the hypotheses of Theorem 4.4 or those of Corollary 4.6. Then, for any finite coloring of $M \setminus \{0_M\}$, there exists $\vec{x} \in (M \setminus \{0_M\})^t$ such that the entries of $B\vec{x}$ are monochrome.*

Proof. There is a monochrome central subset of $M \setminus \{0_M\}$. $\square$

A concept closely related to image partition regularity is that of kernel partition regularity. We shall say that a finite $s \times t$ matrix B with entries in R is *kernel partition regular* over a subset N of M if, for any finite coloring of $N \setminus \{0_M\}$, there exists $\vec{x} \in (N \setminus \{0_M\})^t$, with monochrome entries, such that all the entries of $B\vec{x}$ are equal to 0_M.

In the case in which R is a field, this concept is related to a computable condition, called the *columns condition*, introduced by R. Rado.

4.8 Definition. Let R be a field and let B be an $s \times t$ matrix with entries from R. Then B satisfies the columns condition over R if and only if there exist m and a $t \times m$ first entries matrix H with entries from R such that $BH = O$, where O is the $s \times m$ matrix all of whose entries are 0_R.

It is easy to verify that this definition is equivalent to the original. (See [12, Definition 15.28] for the original definition.) A celebrated theorem, due to Rado [15], states that a finite matrix with entries in $\mathbb{Z}$ is kernel partition regular over $\mathbb{N}$ if and only if it satisfies the columns condition over $\mathbb{Q}$.

The following theorem is closely related to [12, Theorem 15.30] which was in turn based on a result in [1].

4.9 Theorem. *Assume that R is a field and that M is a vector space over R. Let B be a finite $s \times t$ matrix over R.*

(i) If B satisfies the columns condition and if M is infinite, then B is kernel partition regular over M.

(ii) If R is finite and B is kernel partition regular over M, then B satisfies the columns condition.

Proof. (i) If B satisfies the columns condition, $BH = O$ for some first entries matrix H with entries in R. Our claim follows by applying Corollary 4.7 to H in place of B.

(ii) Choose a Hamel basis X for M, and assign X a well-ordering. Every element $m \in M \setminus \{0\}$ can be expressed uniquely as $m = \sum_{x \in F_m} \gamma(m,x)x$ for some $F_m \in \mathcal{P}_f(X)$, where each $\gamma(m,x) \in R \setminus \{0_R\}$. We color $M \setminus \{0\}$ according to the value of $\gamma\big(m, \min(F_m)\big)$. Since B is kernel partition regular over M, there exist $m_1, m_2, \cdots, m_t \in M \setminus \{0_M\}$ and $a \in R \setminus \{0_R\}$ such that $\gamma\big(m_i, \min(F_{m_i})\big) = a$ for every $i \in \{1, 2, \cdots, t\}$ and all entries of $B \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_t \end{pmatrix}$ are equal to 0_M. Let $F = \bigcup_{i=1}^{t} F_{m_i}$. Write F as $\{x_1, x_2, \ldots, x_n\}$, with this sequence arranged in increasing order. Define a $t \times n$ matrix H over R as the matrix of coefficients of the elements $m_1, m_2, \ldots, m_t$ relative to the basis $\{x_1, x_2, \ldots, x_n\}$ of $\sum_{i=1}^{n} Rx_i$. Then H is a first entries matrix for which $BH = O$. □

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