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THE STRUCTURE OF THE DISTRIBUTIVE LATTICE OF GAMES BORN BY DAY N

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Abstract

We prove structural theorems about the distributive lattice of games born by day n. For instance, the number of join-irreducible elements is exactly twice the number of elements born by day $n-1$. As an immediate corollary, all maximal chains on day n are of length exactly one plus double the number of born by day $n-1$.

1 Introduction to combinatorial game theory

In this section we give a minimal number of definitions to understand this paper, omitting any intuition or justification. For a proper introduction to combinatorial game theory, refer to [2] or [5]. In this paper we restrict our attention to finite games.

Combinatorial games, as axiomatized by Conway, form a group with a partial order. A game G is defined as a pair of sets of games written $\{\mathcal{G}^L \,|\, \mathcal{G}^R\}$. In the interests of brevity, we usually drop $\{\}$'s whenever possible, and use $||$ as a lower precedence form of $|$. For instance,

$$a \,||\, b \,|\, c, d = \{\{a\} \,|\, \{\{b\} \,|\, \{c, d\}\}\}.$$

Addition, negation and comparison are defined as follows:[1]

$$G + H \stackrel{\text{def}}{=} \{(\mathcal{G}^L + H) \cup (G + \mathcal{H}^L) \,|\, (\mathcal{G}^R + H) \cup (G + \mathcal{H}^R)\}$$

[1]Here, when we write $S + G$, for a set of games S and a game G, we mean add every element of S to G, i.e., $S + G \stackrel{\text{def}}{=} \{X + G : X \in S\}$. Similarly, $-S = \{-X : X \in S\}$.

$$-G \stackrel{\text{def}}{=} \{-\mathcal{G}^R \,|\, -\mathcal{G}^L\}$$

$$G \geq H \qquad \text{unlesss } H \geq G^R \text{ or } H^L \geq G \text{ for some } G^R \in \mathcal{G}^R \text{ or some } H^L \in \mathcal{H}^L.$$

(Analogous to "iff", the term "unlesss" means "unless and only unless".)

While these definitions are abstract, they correspond naturally to playable games. Players named Left (the positive player) and Right take turns moving, and a player wins by making the last legal move.[2] Left moves on game $G = \{\mathcal{G}^L \,|\, \mathcal{G}^R\}$ by selecting some $G^L \in \mathcal{G}^L$; play then continues from G^L. Right selects from $\mathcal{G}^R$. $G + H$ is obtained by placing G and H next to one another, and a player can move on either game. $-G$ reverses the roles of the players. The following definitions are then equivalent:

1. $G \geq H$
2. For all games X, Left wins $G + X$ (moving first, say) whenever Left wins $H + X$ (moving first).
3. Left wins moving second on $G - H$, i.e., on $G + (-H)$.

In particular, there are four possible outcomes when comparing G and H. These four outcomes can be summarized by seeing who wins on $G - H$ when Left moves first and when Right moves first:

G	$=$	H	The second player can always win $G - H$
G	$>$	H	Left wins $G - H$ moving first or second
G	$<$	H	Right wins $G - H$ moving first or second
G	$\|$	H	The first player wins $G - H$

A few especially important games have names:

$$\begin{array}{rcll} 0 & = & \{\ |\ \} & \\ n & = & \{n{-}1 \ |\ \} & n > 0 \text{ integer} \\ -n & = & \{\ |\ -n{+}1\} & \\ * & = & \{0|0\} & \end{array}$$

The reader can safely ignore other named games, all defined in [2] and [5], which will appear in the figures of this paper, for they are of no consequence to the main theorems.

Three important theorems dictate how games can be simplified. The first is intuitive:

Theorem 1 (Dominated options) *If $B \geq A$, then*

$$\{A, B, C, \ldots \,|\, D, E, F, \ldots\} = \{B, C, \ldots \,|\, D, E, F, \ldots\}.$$

[2] A *game* is an individual *position* rather than a collection of rules.

We say game A is dominated by B. (The inequality is reversed for a dominated right option, so if $E \leq D$ then game D can be removed.)

Theorem 2 (Reversible moves) *Let game G be of the form:*

$$G = \{A, B, C, \ldots | W, X, Y, \ldots\}$$

and suppose A has some right option $A^R \leq G$. Then A can be replaced by all left options from A^R. I.e., if

$$A^R = \{a, b, c, \ldots | w, x, y, \ldots\},$$

then

$$G = \{a, b, c, \ldots, B, C, \ldots | W, X, Y, \ldots\}$$

We say that G has a reversible option A and that the move to A reverses through A^R to $a, b, c, \ldots$.

Theorem 3 (Canonical form) *If game $G = H$ and G and H have no reversible or dominated options, then G and H are* identical: *Each left (or right) option of G is equal to some left (or right) option of H.*

The theorem admits a unique minimal game equal to finite game G called G's *canonical form.*

2 The day n lattice

Define $\mathcal{G}_n$, the *games born by day n*, recursively as follows:

$$\begin{aligned} \mathcal{G}_0 &\stackrel{\text{def}}{=} \{0\} \\ \mathcal{G}_n &\stackrel{\text{def}}{=} \{\{\mathcal{G}^L | \mathcal{G}^R\} : \mathcal{G}^L, \mathcal{G}^R \subseteq \mathcal{G}_{n-1}\}. \end{aligned}$$

The games born by days 1 and 2 are shown in Figures 1 and Figures 2. Due to Theorem 1, dominated options can always be removed from a game leaving an equivalent game. Thus, while the left and right options of games born by day n are subsets of $\mathcal{G}_{n-1}$, we can assume without loss of generality that no two elements are comparable. Hence the figures show the left and right options of a game born by day n as anti-chains from the partial order of $\mathcal{G}_{n-1}$.

Calistrate, Paulhus and Wolfe [4] show that for fixed n, $\mathcal{G}_n$ forms a *distributive lattice.* In a *lattice* any pair of elements a and b has a unique least upper bound or *join* denoted $a \vee b$ and a greatest lower bound or *meet* denoted $a \wedge b$. A lattice is *distributive* if join distributes over meet (or, equivalently, meet distributes over join), i.e., $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

<table>
<tr><td></td><td></td><td colspan="2">Right</td></tr>
<tr><td></td><td></td><td>0</td><td>$\emptyset$</td></tr>
<tr><td rowspan="2">Left</td><td>0</td><td>$*$</td><td>1</td></tr>
<tr><td>$\emptyset$</td><td>-1</td><td>0</td></tr>
</table>

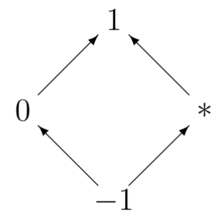

Figure 1: The four games born by day 1. Left's and Right's options are either $\{0\}$ or $\emptyset$. Shown on the right is the partial order of the four games.

<table>
<tr><td></td><td></td><td colspan="6">Right</td></tr>
<tr><td></td><td></td><td>-1</td><td>$0,*$</td><td>0</td><td>$*$</td><td>1</td><td>$\emptyset$</td></tr>
<tr><td rowspan="6">Left</td><td>1</td><td>± 1</td><td>$1\|0,*$</td><td>$1\|0$</td><td>$1\|*$</td><td>$1*$</td><td>2</td></tr>
<tr><td>$0,*$</td><td>$0,*\|-1$</td><td>$*2$</td><td>$\uparrow *$</td><td rowspan="2">$\uparrow$</td><td rowspan="2">$\frac{1}{2}$</td><td rowspan="2">1</td></tr>
<tr><td>0</td><td>$0\|-1$</td><td>$\downarrow *$</td><td>$*$</td></tr>
<tr><td>$*$</td><td>$*\|-1$</td><td colspan="2">$\downarrow$</td><td colspan="3" rowspan="3">0</td></tr>
<tr><td>-1</td><td>$-1*$</td><td colspan="2">$-\frac{1}{2}$</td></tr>
<tr><td>$\emptyset$</td><td>-2</td><td colspan="2">-1</td></tr>
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Figure 2: The 22 games born by day 2. Left and right options are selected from the six anti-chains from the partial order of games born by day 1.

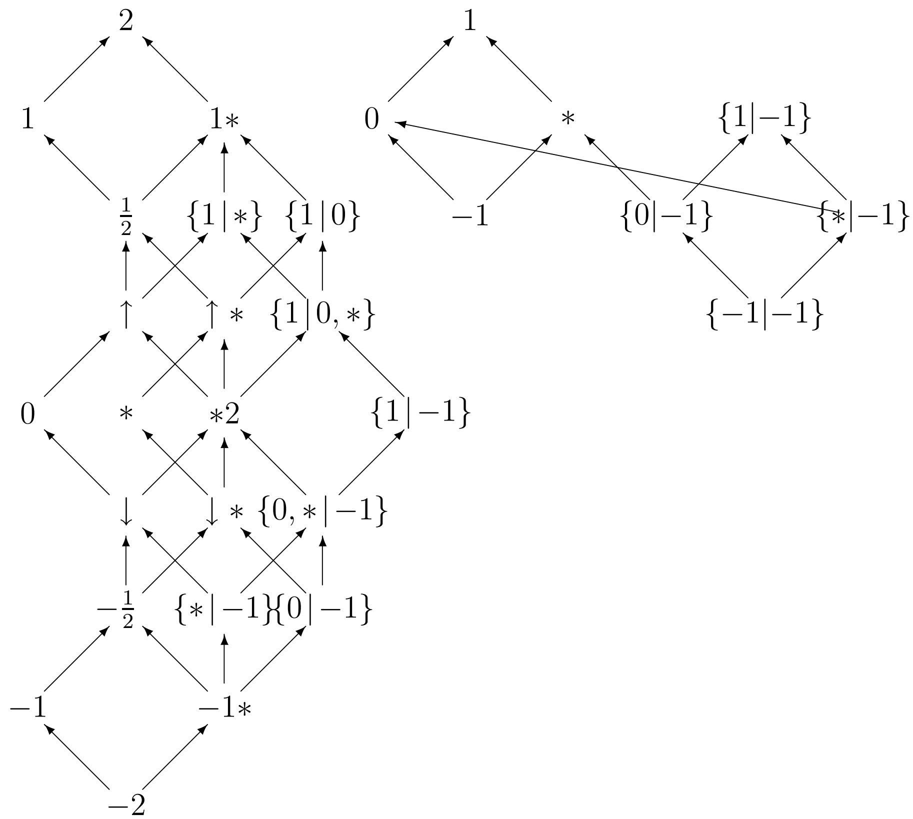


Figure 3: The day 2 lattice is shown on the right. The join irreducibles from day 2 are shown on the left.

For the lattice of games born by day n, the join and meet operations are given by:

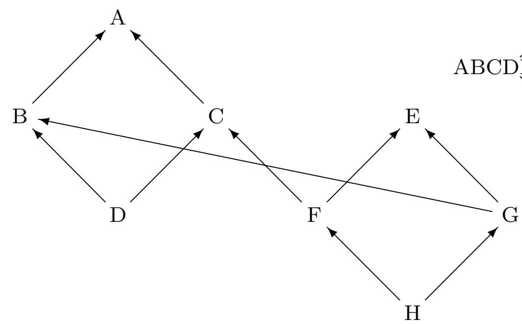
$$\begin{aligned} G_1 \vee G_2 &\stackrel{\text{def}}{=} \left\{ \mathcal{G}_1^L \cup \mathcal{G}_2^L \mid \lceil G_1 \rceil \cap \lceil G_2 \rceil \right\}, \text{ and} \\ G_1 \wedge G_2 &\stackrel{\text{def}}{=} \left\{ \lfloor G_1 \rfloor \cap \lfloor G_2 \rfloor \mid \mathcal{G}_1^R \cup \mathcal{G}_2^R \right\}. \end{aligned}$$

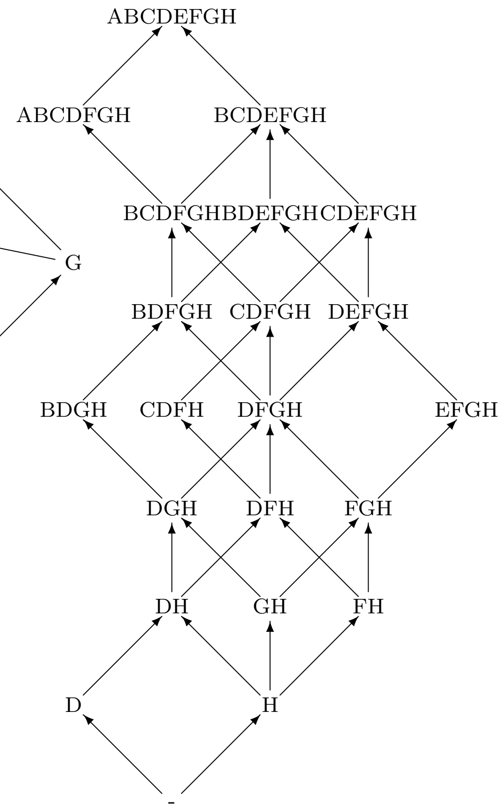
where

$$\begin{aligned} \lceil G \rceil &\stackrel{\text{def}}{=} \{H \in \mathcal{G}_{n-1} : H \not\leq G\}, \text{ and} \\ \lfloor G \rfloor &\stackrel{\text{def}}{=} \{H \in \mathcal{G}_{n-1} : H \not\geq G\}. \end{aligned}$$

Note that $G_1 \vee G_2$ and $G_1 \wedge G_2$ are in $\mathcal{G}_n$ since their left and right options are chosen from $\mathcal{G}_{n-1}$.

In a lattice, the *join irreducible* elements are those elements that cannot be formed by the join of other elements. Looking at the Hasse diagram of the lattice, a join irreducible element has exactly one element immediately below it in the lattice. (The single element at the bottom is not considered a join irreducible for it's the join of the empty set.) Figure 3 shows the Hasse diagram of the day 2 lattice along with the partial order of the day 2 join irreducibles.





ABCDEFGH	DFGH
ABCDFGH	EFGH
BCDEFGH	DFH
BCDFGH	DGH
BDEFGH	FGH
CDEFGH	DH
BDFGH	FH
CDFGH	GH
DEFGH	D
BDGH	H
CDFH	-

Figure 4: The upper left shows the partial order of the join-irreducible elements of the day 2 distributive lattice. The lower-left lists all 22 downsets of the partial order. A downset is a subsets, S, of elements such that if $x \leq y$ and $y \in S$ then $x \in S$. On the right is the original lattice as reconstructed from the downsets. Downset $S_1 \leq S_2$ if $S_1 \subset S_2$.

A few things are worth observing about the partial ordering of the day 2 join-irreducibles: It consists of two copies of the day 1 lattice (the diamonds) connected by a few cross edges. The upper left copy consists of the four day 1 games, while the lower right copy consists of games $G\,|\,{-1}$ for $G \in \mathcal{G}_1$. We'll generalize and prove this in the next section.

Birkhoff [3] showed (amazingly) that partial orders are in 1-1 correspondence with distributive lattices. In particular, a distributive lattice can be constructed from the partial order of its join irreducibles. Since the construction is informative, the construction is explained by example in Figure 4.[3]

For comparison, we also provide the join irreducibles of the 1474 game lattice from day 3 in Figure 5. The number 1474 was found by hand by Dean Hickerson and Robert Li in

[3]Note that the join operation in the lattice of Figure 3 is simply set union. Also, when comparing Figures 3 and 4, observe that the closure under $\leq$ of an element, say G, of the partial order in Figure 4, yields its representation, GH in the lattice of the same Figure. Both correspond to the $\{*|-1\}$ in respective diagrams of Figure 3.

1974, and confirmed more recently by computer by Marc Paulhus and others [6].

3 The structure of the day n lattice

Define $\mathcal{G}_{|n} = \{\{G\,|\,-n\} : \ G \in \mathcal{G}_n\}$. Observe that $\mathcal{G}_{|n} \subset \mathcal{G}_{n+1}$. We'll first prove, through a series of lemmas that, the join-irreducible elements from day $n+1$ are exactly $\mathcal{G}_n \cup \mathcal{G}_{|n}$. We'll proceed to verify the entire structure of the partial order on these $2 \cdot |\mathcal{G}_n|$ elements.

Lemma 1 *If $G \in \mathcal{G}_n$, then $n \geq G$. (Also, if $G \in \mathcal{G}_{n-1}$, then $n > G$.)*

Proof. As in most proofs in combinatorial game theory, use induction on the birthday of G. For the first assertion, $n \geq 0$ has no right options, so $n \geq G$ unlesss $G^L \geq n$. But $G^L \in G_{n-1}$ so $n-1 \geq G^L$. ■

Lemma 2 *If G is born by day n, then $G\,|\,-n$ is in canonical form.*

Proof. Followers are singletons, so there are no dominated options. The only opportunity for a reversible option is if some $G^R \leq \{G\,|\,-n\}$.[4] However, $G^R > -n$ by Lemma 1. ■

Corollary 2.1 *All of the games $G\,|\,-n$ for $G \in \mathcal{G}_n$ are distinct from each other and from any game in $\mathcal{G}_n$.*

Proof. This is a direct consequence of the Lemma and uniqueness of canonical form (Theorem 3). ■

Lemma 3 *If $G \in \mathcal{G}_n$, then $\{G\,|\,-n\}$ is join-irreducible on day $n+1$.*

Proof. Let J be the join of elements $H < \{G\,|\,-n\}$, where the join is formally constructed as per the definition on page 5. We see that $-n$ is the only right option of J, since $-n \not\leq H$ and $-n$ dominates all other options. After removing any J^L which are dominated, we note that $J^{LR} > -n$ (by Lemma 1), so $J^{LR} \not\leq J$ and J has no reversible options and J is in canonical form.

[4] When we write "some G^R" we mean "some right option, G^R, of G," i.e., "some $G^R \in \mathcal{G}^R$ where $G = \{\mathcal{G}^L\,|\,\mathcal{G}^R\}$."

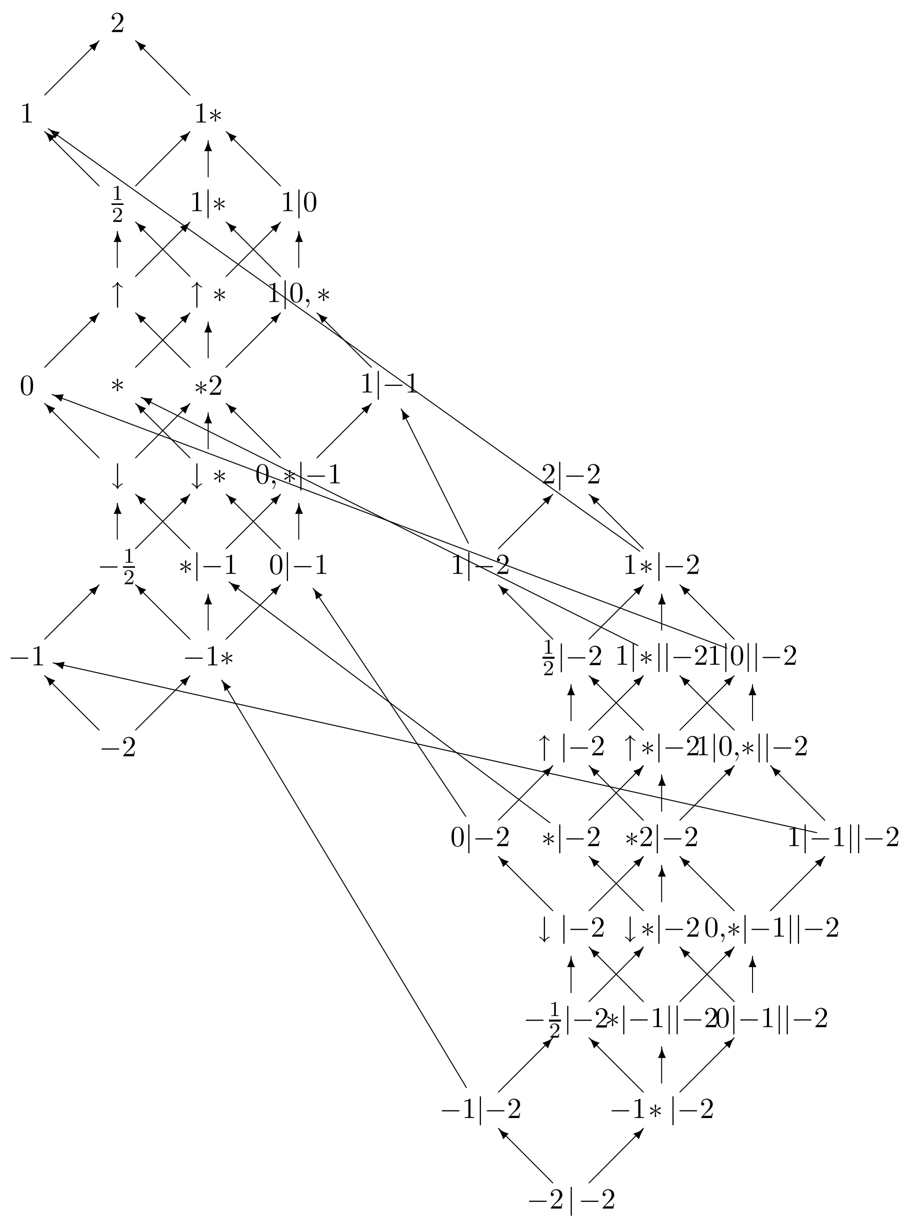


Figure 5: The partial order of the 44 join-irreducible elements of the day 3 lattice, consisting of two copies of the day 2 lattice. The two copies have games of the form G and $G \mid -2$ respectively for $G \in \mathcal{G}_2$.

If $G \in \mathcal{H}^L$ for some H, then $H \geq \{G \mid -n\}$, so $G \notin \mathcal{J}^L$. Therefore, $J \neq G$ by the uniqueness of canonical form. ■

Lemma 4 *If $G \in \mathcal{G}_{n+1}$ is an irreducible join of $\mathcal{G}_{n+1}$, then $G \in \mathcal{G}_n \cup \mathcal{G}_{|n}$.*

Proof. It suffices to prove that $G \in \mathcal{G}_{n+1}$ is equal to the day $n+1$ join of all elements $H \leq G$ such that $H \in \mathcal{G}_n$ or $H \in \mathcal{G}_{|n}$.

Suppose J is the join of all such H. Since G is greater than all of the Hs, and $\mathcal{G}_n$ is a lattice, $J \leq G$.

It remains to prove $J \geq G$, i.e., Left wins $J - G$. It's always the case that $G^L \ngeq G$ for right has a winning move on $G^L - G$ to $G^L - G^L$. Hence, if Right plays to $J - G^L$ then $\{G^L \mid -n\} \leq G$, so $G^L \in \mathcal{J}^L$ and Left can play to $G^L - G^L$. If Right plays to $J^R - G$, we observe that $J^R \in \mathcal{G}_n$ and $J^R \nleq H$ (in particular $J^R \neq H$) for any $H \in \mathcal{G}_n$ such that $H \leq G$. Thus $J^R \nleq G$ and Left can win $J^R - G$. ■

Lemma 5 *On day $n+1$, $G \in \mathcal{G}_n$ is an irreducible join.*

Proof. Let J be the join of all elements $K < G$ in $\mathcal{G}_{n+1}$. By Lemma 4, J is the join of all elements $H < G$ in $\mathcal{G}_n$ or $\mathcal{G}_{|n}$. As above, we see that $J \leq G$. Consider the game $J - G$. Since $H \ngeq G$ for all H, Right can play to $G - G = 0$, $J \ngeq G$. Thus $J < G$ and G is join-irreducible. ■

Theorem 4 *The irreducible joins of $\mathcal{G}_{n+1}$ are $\mathcal{G}_n \cup \mathcal{G}_{|n}$.*

Proof. Lemma 3 and Lemmas 4 and 5 show inclusion in both directions. ■

Lemma 6 *If $G \in \mathcal{G}_n$, then $G \geq \{n \mid G \| -(n+1)\}$.*

Proof. Play $G - \{n \mid G \| -(n+1)\}$. If Right plays to $G^R - \{n \mid G \| -(n+1)\}$, Left plays to $n + 1 + G^R$ and wins. Thus Right must play to $G - \{n \mid G\}$, but Left can play to $G - G$ and win. ■

Lemma 7 *If $G \in \mathcal{G}_n$, then $\{G \mid -n\} \geq \{G \mid -(n+1)\}$.*

Proof. $-(n+1)$ dominates $-n$. ■

Lemma 8 *If $G \in \mathcal{G}_{n+1}$, then $G \geq \{\mathcal{G}^L \,|\, -n\}$ and $G \leq \{n \,|\, \mathcal{G}^R\}$.*

Proof. n dominates any follower of G. ■

Corollary 8.1 $G \leq \{n \,|\, \mathcal{G}^R\}$.

Theorem 5 *The partial order between the elements of $\mathcal{G}_{n+1}$ and $\mathcal{G}_{|n+1}$, is completely described by the induced partial order plus the inequalities given in Lemmas 6 and 7.*

Proof. Suppose $G, K \in \mathcal{G}_{n+1}$. First, we note that Right can win in $\{K \,|\, -(n+1)\} - G$ by playing to $-(n+1) - G$, so $G \not\leq \{K \,|\, -(n+1)\}$. Now suppose $G \geq \{K \,|\, -(n+1)\}$ and play $G - \{K \,|\, -(n+1)\}$. Left must have some winning response to $G - K$.

If Left's winning move is to $G^L - K$, then $G^L \geq K$. Hence $G \geq \{G^L \mid -n\} \geq \{G^L \mid -(n+1)\} \geq \{K \mid -(n+1)\}$ shows that the inequality is implied by transitivity with Lemma 7.

If, on the other hand, Left's move is to $G - K^R$, then $G \geq K^R$. Hence $G \geq K^R \geq \{n \,|\, K^R \,||\, -(n+1)\} \geq \{K \,|\, -(n+1)\}$ shows that the inequality is implied by transitivity and Lemma 6. ■

4 Conclusions and open questions

Because the join irreducible elements in $\mathcal{G}_n$ are exactly $\mathcal{G}_{n-1} \cup \mathcal{G}_{|n-1}$, we arrive at the following result:

Theorem 6 *All maximal chains on day n are of length exactly one plus double the number of games born by day $n-1$.*

Proof. This is an immediate consequence of Birkhoff's construction of the distributive lattice from the join irreducibles in Figure 4. ■

A number of questions still elude us.

Clearly, the lattice is symmetric about the middle level. Is it the case that the middle level is the largest level? Does the structure yield information about the number of games born by day n? Are there interesting substructures induced by infinitesimals born by day n or games which are all-small? Can any single longest chain in the day n lattice be completely characterized for all n?

Berlekamp [1] suggests other possible definitions for games born by day n, $\mathcal{G}_n$, depending on how one defines $\mathcal{G}_0$. Our definition is 0-based, as $\mathcal{G}_0 = \{0\}$. Other natural definitions are integer-based (where $\mathcal{G}_0$ are integers) or number-based (see [2] for definition of number). By an argument identical to that given in Theorem 3 of [4], these two alternatives do not form a lattice, for if G_1 and G_2 are born by day k,

$$H_n \stackrel{\text{def}}{=} \{G_1, G_2 \mid\mid G_1, G_2 \mid -n\}$$

form a decreasing sequence of games born by day $k+2$ exceeding any $G \geq G_1, G_2$, and the day $k+2$ join of G_1 and G_2 cannot exist. What is the structure of the partial order given by one of these alternative definitions of birthday?

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