K-DISTANCE SETS, FALCONER CONJECTURE, AND DISCRETE ANALOGS

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Abstract. In this paper we prove a series of results on the size of distance sets corresponding to sets in the Euclidean space. These distances are generated by bounded convex sets and the results depend explicitly on the geometry of these sets. We also use a diophantine mechanism to convert continuous results into distance set estimates for discrete point sets.

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Introduction

Let E be a compact subset of $\mathbb{R}^d$. Let $\Delta(E) = \{|x-y| : x, y \in E\}$, where $|\cdot|$ denotes the usual Euclidean metric. The celebrated Falconer conjecture says that if the Hausdorff dimension of E is greater than $\frac{d}{2}$, then $\Delta(E)$ has positive Lebesgue measure. Falconer ([Falconer86]) obtained this conclusion if the Hausdorff dimension of E is greater than $\frac{d+1}{2}$. His result was improved by Bourgain in [Bourgain94]. The best known result in the plane is due to Tom Wolff who proved in [Wolff99] that the distance set has positive Lebesgue measure if the Hausdorff dimension of E is greater than $\frac{4}{3}$.

All these results are based on the curvature of the unit circle of the Euclidean metric. For if the Euclidean metric is replaced by the l^∞ metric, for example, the situation becomes very different. To see this, let $E = C_{2m} \times C_{2m}$, $m > 1$ an even integer, where C_{2m} is the Cantor-type subset of $[0,1]$ consisting of numbers whose base $2m$ expansions contain only even numbers. One can check that the distance set with respect to the l^∞ metric has measure 0 for any m, whereas the Hausdorff dimension of this set is $2\frac{\log(m)}{\log(2m)} \to 2$ as $m \to \infty$.

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The curvature is not the end of the story. The aforementioned results of Falconer, Bourgain and Wolff also used the smoothness of the unit circle. Falconer did so explicitly by using the formula for the Fourier transform of the characteristic function of an Euclidean annulus, whereas Bourgain and Wolff utilized it implicitly by using a reduction to circular averages which again relies on asymptotics of the Fourier transform of the Lebesgue measure on the unit circle of the distance which does not hold in the absence of smoothness. See [Mattila87] and [Sjölin93] for a background on these reductions.

The purpose of this paper to study the Falconer conjecture in the absence of smoothness using geometric features of the Fourier transforms of characteristic functions of convex sets in Euclidean space. We also develop a conversion mechanism based on diophantine approximation which allows us to obtain geometric combinatorial results from analogous facts in a continuous setting.

Continuous results: variants of the Falconer distance problem

Definition. Let K be a bounded convex set in $\mathbb{R}^d$ symmetric about the origin, and let $||\cdot||_K$ be the norm induced by K. The K-distance set of a set $E \subset \mathbb{R}^d$ is the set $\Delta_K(E) = \{||x-y||_K : x, y \in E\}$.

Our main results are the following.

Theorem 0.1. *Let $E \subset [0,1]^d$. Let K be a bounded convex set in $\mathbb{R}^d$ and let σ_K denote the Lebesgue measure on ∂K.*

i) Suppose that $|\widehat{\sigma}_K(\xi)| \leq C|\xi|^{-\gamma}$ for some $\gamma > 0$, and the Hausdorff dimension of E is greater than $d-\gamma$. Then $\Delta_K(E)$ has positive Lebesgue measure.

ii) Let $d = 1$. Assume that we have an estimate

$$\int_{S^1} \sup_{R>1} R^{\gamma} |\widehat{\sigma}_K(R\omega)| d\omega \leq C$$

and the Hausdorff dimension of E is greater than $2-\gamma$ for some $\gamma > 0$. Then $\Delta_K(\rho E)$ has positive Lebesgue measure for almost every $\rho \in S^1$, viewed (in the obvious way) as an element of $SO(2)$.

iii) Suppose that

$$|\widehat{\sigma}_K(\xi)| \leq C\gamma(|\xi|^{-1}), \tag{0.1}$$

where γ is a convex function with $\gamma(0) = 0$, and σ_K is the Lebesgue measure on ∂K. Let $E \subset [0,1]^d$. Suppose that there exists a Borel measure on E such that

$$\int |\widehat{\mu}(\xi)|^2 \gamma(|\xi|^{-1}) d\xi < \infty. \tag{0.2}$$

Then $\Delta_K(E)$ has positive Lebesgue measure.

Corollary 0.2. *Let K be a bounded symmetric convex set in $\mathbb{R}^d$.*

i) Let $d=2$. Let $S_\theta = \sup_{x \in K} x \cdot \omega$, where $\omega = (\cos(\theta), \sin(\theta))$, and denote by $l(\theta, \epsilon)$ the length of the chord $C(\theta, \varepsilon) = \{x \in K : x \cdot \omega = S_\theta - \varepsilon\}$. Suppose that ∂K has everywhere non-vanishing curvature in the sense that there exists a positive uniform constant c such that

$$l(\theta, \epsilon) \leq c\sqrt{\epsilon}. \tag{0.3}$$

Let $E \subset [0,1]^2$ be a set of Hausdorff dimension $\alpha > \frac{1}{2}$. Then the Hausdorff dimension of $\Delta_K(E)$ is at least $\min(1, \alpha - \frac{1}{2})$. If furthermore $\alpha > \frac{3}{2}$, then $\Delta_K(E)$ has positive Lebesgue measure.

ii) Suppose that $E \subset [0,1]^d$ is a set of Hausdorff dimension $\alpha > \frac{d+1}{2}$, and that ∂K is smooth and has non-vanishing Gaussian curvature. Then $\Delta_K(E)$ has positive Lebesgue measure.

Corollary 0.3. *Let $E \subset [0,1]^d$.*

i) Assume that $d=2$. Let K be a symmetric convex polygon, and assume that the Hausdorff dimension of E is greater than 1. Then $\Delta_K(\rho E)$ has positive Lebesgue measure for almost every $\rho \in S^1$. Moreover, this result is sharp in the sense that for every $\alpha < 1$ there exists a set $E \subset \mathbb{R}^2$ of Hausdorff dimension α such that $\Delta_K(E)$ has Lebesgue measure 0 with respect to every convex body K.

ii) Let K be a bounded symmetric convex body in $\mathbb{R}^2$ (with no additional regularity assumptions). Suppose that the Hausdorff dimension of E is greater than $\frac{3}{2}$. Then $\Delta_K(\rho E)$ has positive Lebesgue measure for almost every $\rho \in S^1$.

iii) Suppose that E is radial in the sense that $E = \{r\omega : \omega \in S^{d-1}; r \in E_0\}$, where $E_0 \subset [0,1]$. Suppose that the Hausdorff dimension of E is greater than $\frac{d+1}{2}$. Suppose that K is any symmetric bounded convex set in the plane or a symmetric bounded convex set in higher dimensions with a finite type boundary in the sense defined above. Then $\Delta_K(E)$ has positive Lebesgue measure.

Discrete theorems and Continuous → Discrete conversion mechanism

Definition. We say that $S \subset \mathbb{R}^d$ is well-distributed if there exists a $C > 0$ such that every cube of side-length C contains at least one element of S.

Definition. We say that a set $S \subset \mathbb{R}^d$ is separated if there exists a constant $c > 0$ such that $|a - a'| \geq c$ for every $a, a' \in A$.

Definition. Let K be a bounded symmetric convex set and let $0 < \alpha_K \leq d$. We say that the (K, α_K) Falconer conjecture holds if for every compact $E \subset \mathbb{R}^d$ of dimension greater than α_K, $\Delta_K(E)$ has positive Lebesgue measure.

The essence of the conversion mechanism is captured by the following result and its proof.

Theorem 0.4. *Let S be a well-distributed and separated subset of $\mathbb{R}^d$. Let $S_q = S \cap [0,q]^d$. Suppose that (K, α_K) Falconer conjecture holds. Then there exists a constant $c > 0$ such that $\#\Delta_K(S_q) \geq cq^{\frac{d}{\alpha_k}}$.*

In view of Theorem 0.4, every result stated above in Theorem 0.1 and Corollaries 0.2–0.3 has a discrete analog. Moreover, we have the following application.

Corollary 0.5. *Let S be a well-distributed and separated subset of $\mathbb{R}^d$.*

i) Let K be a bounded symmetric convex set. Suppose that

$$\left| \int_{\partial K} e^{-2\pi i x\cdot\xi} d\sigma_K(x) \right| \leq C\gamma(|\xi|^{-1}), \tag{0.4}$$

where γ is a convex increasing function with $\gamma(0) = 0$ and σ_K is the Lebesgue measure on ∂K. Then $\Delta_K(S)$ is not separated.

ii) Let K be any bounded symmetric convex set in $\mathbb{R}^2$. Then $\Delta_K(\rho S)$ is not separated for almost every $\rho \in SO(2)$.

This complements the following result, proved in [IoŁa2002] for $d = 2$ and in [Kol2003] for $d \geq 3$.

Theorem 0.6. *Let S be well-distributed subset of $\mathbb{R}^d$, and let $\Delta_{K,N}(S) = \Delta_K(S) \cap [0, N]$.*

(i) Assume that $d = 2$ and $\underline{\lim}_{N\to\infty} \#\Delta_{K,N}(S) \cdot N^{-3/2} = 0$. Then K is a polygon (possibly with infinitely many sides). If moreover $\#\Delta_{K,N}(S) \leq CN^{1+\alpha}$ for some $0 < \alpha < 1/2$, then the number of sides of K whose length is greater than δ is bounded by $C'\delta^{-2\alpha}$.

(iii) Let $d \geq 2$. If $\#\Delta_{K,N}(S) \leq CN$ (in particular, this holds if $\Delta_K(S)$ is separated), then K is a polytope with finitely many faces.

Stationary phase tools

In the proofs of our results, we shall make use of the following estimates on the Fourier transform of the surface carried measure, which we collect in a single theorem.

Theorem 0.7. *Let K be a bounded convex set in $\mathbb{R}^d$, and let σ_K denote the Lebesgue measure on ∂K.*

i) Suppose that ∂K is smooth and has everywhere non-vanishing Gaussian curvature. Then (see e.g. [Herz62]*)*

$$|\widehat{\sigma}_K(\xi)| \leq C|\xi|^{-\frac{d-1}{2}}. \tag{0.5}$$

ii) Suppose that $d = 2$ and ∂K has everywhere non-vanishing curvature in the sense of part i) of Corollary 0.3. Then (0.5) *holds without any additional smoothness assumptions. See, e.g.* [BRT98].

iii) (See [BHI02]*) Without any additional assumptions,*

$$\left(\int_{S^{d-1}} |\widehat{\sigma}_K(R\omega)|^2 d\omega \right)^{\frac{1}{2}} \leq CR^{-\frac{d-1}{2}},$$

$$\left(\int_{S^{d-1}} |\widehat{\chi}_K(R\omega)|^2 d\omega \right)^{\frac{1}{2}} \leq CR^{-\frac{d+1}{2}}. \tag{0.6}$$

iv) (See [BCIPT03]*). If $d = 2$ and K is any bounded convex set, then*

$$\left(\int_{S^1} \sup_{R>1} R^{\frac{1}{2}} \left| \widehat{\sigma}_K(R\omega) \right|^2 d\omega \right)^{\frac{1}{2}} \leq C. \tag{0.7}$$

v) (See [BCT97]*) Suppose that K is a polyhedron. Then*

$$\int_{S^{d-1}} |\widehat{\sigma}_K(R\omega)| d\omega \leq C \log^{d-1}(R) R^{-(d-1)}. \tag{0.8}$$

Moreover, the same proof shows that the maximal version holds, namely there is a k such that

$$\int_{S^{d-1}} \sup_{R>1} R^{d-1} |\widehat{\sigma}_K(R\omega)| d\omega \lesssim \log^k(R). \tag{0.9}$$

Proof of Theorem 0.1, Corollary 0.2, and Corollary 0.3

Let $A_{R,\delta} = \{x \in \mathbb{R}^d : R \leq ||x||_K \leq R+\delta\}$ denote an annulus of radius R and width δ, $\delta \ll R$.

Since the Hausdorff dimension of E is greater than $d - \gamma$, there is a non-zero Borel measure μ on E such that the following energy integral is finite:

$$\int |\xi|^{-\gamma} |\widehat{\mu}(\xi)|^2 d\xi < \infty. \tag{1.1}$$

For the existence of such a measure, see, for example, [Falconer85].

Cover $\Delta_K(E)$ by intervals $\{[R_i, R_i + \delta_i]\}$. It follows that

$$0 < (\mu \times \mu)(E \times E) \leq \sum (\mu \times \mu)\{(x, y) : R_i \leq ||x - y||_K \leq R_i + \delta_i\}$$

$$\leq C \sum_i \int \chi_{A_{R_i,\delta_i}}(x - y) d\mu(x) d\mu(y) \leq C \sum_i \int \widehat{\chi}_{A_{R_i,\delta_i}}(\xi) |\widehat{\mu}(\xi)|^2 d\xi$$

$$\leq C' \sum_i \delta_i \int |\xi|^{-\gamma} |\widehat{\mu}(\xi)|^2 d\xi, \tag{1.2}$$

where the last line follows by the decay assumption and the definition of Lebesgue measure on a hyper-surface. By (1.1), the last expression in (1.2) is bounded by $C'' \sum_i \delta_i$.

Proof of Theorem 0.1(ii): draft.

Suppose that $|\Delta_{\rho K}(E)| = 0$ for all $\rho \in U$, where U is a subset of S^1 of positive (1-dimensional) measure. It follows from the construction of μ (or can be deduced from (1.1)) that $\mu(\{x\}) = 0$ for any x, hence from Fubini's theorem we have $\mu \times \mu\{(x,x): \ x \in E\} = 0$. Thus

$$0 < \mu \times \mu(E \times E) = \sum_k \mu \times \mu\{(x,y) \in E \times E: \ 2^{-k} \leq \|x-y\|_{\rho K} \leq 2^{-k+1}\}.$$

Denote the sets on the right by $X_{\rho,k}$. For each ρ, there are $k(\rho)$ and $n(\rho)$ such that $\mu \times \mu(X_{\rho,k(\rho)}) > 1/n(\rho)$. Finally, we choose k, n so that $|U_{k,n}| > 0$, where

$$U_{k,n} = \{\rho: \ k(\rho) = k, \ n(\rho) = n\}.$$

We now restrict our attention to $\rho \in U_{k,n}$; in fact, we will assume without loss of generality that $U = U_{k,n}$. Rescaling the set if necessary, we may also assume that $k = 0$.

Fix a small $\epsilon > 0$, to be determined later. For every $\rho \in U$ we may then choose a covering of $\Delta_{\rho K}(E) \cap [1,2]$ by intervals I_i^ρ such that $\sum_i |I_i^\rho| < \epsilon$. Let $A_i^\rho = \{x : \|x\|_{\rho K} \in I_i^\rho\}$. We may take $I_i^\rho \subset [1,2]$. Then there is a constant c_0 such that for each $\rho \in U$,

$$0 < c_0 < (\mu \times \mu)(X_{\rho,k}) \leq \sum (\mu \times \mu)\{(x,y) \in E \times E : ||x-y||_{\rho K} \in I_i^\rho\}$$

$$\leq C \sum_i \int \chi_{A_i^\rho}(x-y) d\mu(x) d\mu(y) \leq C \sum_i \int |\widehat{\chi}_{A_i^\rho}(\xi)| |\widehat{\mu}(\xi)|^2 d\xi$$

$$\leq C\epsilon \int \sup_{1 \leq R \leq 2} |\widehat{\sigma}_{R(\rho K)}(\xi)| |\widehat{\mu}(\xi)|^2 d\xi. \tag{1.3}$$

Observe that

$$|\widehat{\sigma}_{R(\rho K)}(\xi)| = |R^2 \widehat{\sigma}_{\rho K}(R\xi)| = |R^2 \widehat{\sigma}_K(\rho(R\xi))|$$

for R as above.

Integrating this over U, we get

$$0 < c_0|U| < C\epsilon \int_U d\rho \int \sup_{1 \leq R \leq 2} |\widehat{\sigma}_K(\rho(R\xi))| |\widehat{\mu}(\xi)|^2 d\xi$$

$$\leq C\epsilon \int d\xi |\widehat{\mu}(\xi)|^2 |\xi|^{-\gamma} \int_U \sup_{1 \leq R \leq 2} |\widehat{\sigma}_K(\rho(R\xi))| \, |R\xi|^\gamma d\rho. \tag{1.4}$$

We claim that the last integral (in ρ) is bounded by a constant independent of ξ. Indeed, for $|\xi| < 1$ this is trivial, since then the integrand is uniformly bounded. For $|\xi| \geq 1$, we let $t = R|\xi|$ and $\omega = \xi/|\xi|$, then the integral is bounded by

$$\int_{S^1} \sup_{t \geq 1} |\widehat{\sigma}_K(\rho(t\omega))| \, |\frac{t}{R}|^\gamma d\rho \leq C \int_{S^1} \sup_{t \geq 1} |\widehat{\sigma}_K(t\omega)| \, t^\gamma \, d\omega.$$

At the last line we changed variables and used the lower bound on R. But this is bounded by our assumption on K.

Returning to (1.4), we conclude that

$$0 < c_0|U| < C\epsilon \int |\widehat{\mu}(\xi)|^2 |\xi|^{-\gamma}\, d\xi \le C'\epsilon,$$

a contradiction if ϵ was chosen small enough.

The third part follows by an identical argument.

Proof of Corollary 0.2. The second part of Corollary 0.2 follows from the first part of Theorem 0.1 and the first part of Theorem 0.7. We now prove the first part. We shall need the following classical result. See, for example, [BRT98] for a simple proof.

Lemma 1.1. *Let $K \subset \mathbb{R}^2$ be a convex body. Let $\omega = (\cos(\theta), \sin(\theta))$. As before, let $S_\theta = \sup_{x\in K} x \cdot \omega$, and denote by $l(\theta, \epsilon)$ the length of the chord $C(\theta, \varepsilon) = \{x \in K : x \cdot \omega = S_\theta - \varepsilon\}$. Let σ_K denote the Lebesgue measure on ∂K. Then, for a constant C independent of smoothness and curvature, we have*

$$|\widehat{\chi}_K(t\omega)| \leqslant \frac{C}{t}\left(l\left(\theta, \frac{1}{2t}\right) + l\left(-\theta, \frac{1}{2t}\right)\right). \tag{1.6}$$

Lemma 1.2. *Let K be a convex bounded symmetric set in the plane. Assume, in addition, that ∂K has non-vanishing curvature in the sense of Corollary 0.2 (i). Let $A_{R,\delta} = \{x : R \le ||x||_K \le R + \delta\}$. Then for $R, |\xi| > 1$, $\delta \ll 1$ we have*

$$\left|\widehat{\chi}_{A_{R,\delta}}(\xi)\right| \le CR^{\frac{1}{2}}|\xi|^{-\frac{1}{2}} \min\{|\xi|^{-1}, \delta\}, \tag{1.7}$$

where C is a constant that depends only on K.

We shall prove Lemma 1.2 in a moment. We first complete the proof of the first part of Corollary 0.2.

Fix β so that $0 \le \beta \le 1$ and $\alpha > \frac{1}{2} + \beta$, where α is the Hausdorff dimension of E. Then there is a non-zero Borel measure μ supported on E such that the following energy integral is finite:

$$\int |\xi|^{-\frac{3}{2}+\beta} |\widehat{\mu}(\xi)|^2 d\xi < \infty \tag{1.8}$$

(cf. the proof of Theorem 0.1).

Let K be a convex bounded symmetric planar set satisfying the assumptions of Corollary 0.3 (i), and let $A_{R,\delta}$ be as above. We have by Lemma 1.2,

$$\int\int \chi_{A_{R,\delta}}(x-y) d\mu(x) d\mu(y) = \int \widehat{\chi}_{A_{R,\delta}}(\xi) |\widehat{\mu}(\xi)|^2 d\xi$$

$$\leq CR^{\frac{1}{2}} \left(\int_{|\xi|>\delta^{-1}} |\xi|^{-\frac{3}{2}} |\widehat{\mu}(\xi)|^2 d\xi + \delta \int_{|\xi|\leq\delta^{-1}} |\xi|^{-\frac{1}{2}} |\widehat{\mu}(\xi)|^2 d\xi \right)$$

$$\leq CR^{\frac{1}{2}} \left(\delta^\beta \int_{|\xi|>\delta^{-1}} |\xi|^{-\frac{3}{2}+\beta} |\widehat{\mu}(\xi)|^2 d\xi + \delta \cdot \delta^{\beta-1} \int_{|\xi|\leq\delta^{-1}} |\xi|^{-\frac{3}{2}+\beta} |\widehat{\mu}(\xi)|^2 d\xi \right)$$

$$\leq CR^{\frac{1}{2}} \delta^\beta. \tag{1.9}$$

It follows that

$$(\mu \times \mu)\{(x,y) : R \leq ||x-y||_K \leq R+\delta\} \leq CR^{\frac{1}{2}}\delta^\beta. \tag{1.10}$$

Cover $\Delta_K(E)$ by intervals $\{[R_i, R_i+\delta_i]\}$. Suppose, without loss of generality, that $R_i \leq 10$. It follows that

$$0 < (\mu\times\mu)(E\times E) \leq \sum (\mu\times\mu)\{(x,y) : R_i \leq ||x-y||_K \leq R_i+\delta_i\} \leq C \sum_i \delta_i^\beta. \tag{1.11}$$

This shows that the β-dimensional Hausdorff measure of $\Delta_K(E)$ is non-zero for any $\beta < \min(1, \alpha - \frac{1}{2})$, so that the Hausdorff dimension of $\Delta_K(E)$ is at least $\min(1, \alpha - \frac{1}{2})$. If furthermore $\alpha > \frac{3}{2}$, we may take $\beta = 1$ and thus deduce that $\Delta_K(E)$ has positive Lebesgue measure.

We now prove Lemma 1.2. For a fixed ξ, let

$$F(s) = s^2 \widehat{\chi}_K(s\xi). \tag{1.12}$$

We have

$$\widehat{\chi}_{A_{R,\delta}}(\xi) = F(R+\delta) - F(R). \tag{1.13}$$

By the mean value theorem,

$$|F(R+\delta) - F(R)| \leq \delta \sup_{s\in(R,R+\delta)} |F'(s)|. \tag{1.14}$$

Now,

$$F'(s) = 2s\widehat{\chi}_K(s\xi) + s^2 \int_K e^{-2\pi i s\xi\cdot x}(-2\pi i \xi\cdot x)dx = I + II. \tag{1.15}$$

By Lemma 1.1 and the non-vanishing curvature assumption,

$$I \leq Cs|s\xi|^{-\frac{3}{2}} \leq C|\xi|^{-\frac{3}{2}}. \tag{1.16}$$

On the other hand, following word for word the proof of Lemma 1.1 given in [BRT98] we obtain that

$$II \leq Cs^2|\xi||s\xi|^{-\frac{3}{2}} \leq CR^{\frac{1}{2}}|\xi|^{-\frac{1}{2}}. \tag{1.17}$$

This proves the second estimate in (1.7). The first estimate follows from the inequality

$$|F(R+\delta) - F(R)| \leq |F(R+\delta)| + |F(R)| \leq CR^2|R\xi|^{-\frac{3}{2}} \leq CR^{\frac{1}{2}}|\xi|^{-\frac{3}{2}}, \tag{1.18}$$

where we again used Lemma 1.1 and the non-vanishing curvature assumption.

Proof of Corollary 0.3. Part i) follows from part ii) of Theorem 0.1 and part v) of Theorem 0.7. The sharpness result can be obtained as follows. Let $0 < s \leq d$. Let $q_1, q_2, \dots, q_i \dots$ be a sequence of positive integers such that $q_{i+1} \geq q_i^i$. Let $E_i = \{x \in \mathbb{R}^d : 0 \leq x_j \leq 1, |x_j - p_j/q_i| \leq q_i^{-\frac{d}{s}}$ for some integers $p_j,\ j = 1, 2\}$. It is not hard to see (see e.g. [Falconer85], Chapter 8, or [Wolff02]) that the Hausdorff dimension of $E = \cap_{i=1}^{\infty} E_i$ is s. Also, $\Delta(E) \subset \bigcap_{i=1}^{\infty} \Delta(E_i)$.

Let K be a bounded symmetric convex set in $\mathbb{R}^d$, and let $P_i = \{p = (p_1, p_2, \dots, p_d) : 0 \leq p_j \leq q_i\}$. Then $\{\|p - p'\|_K :\ p, p' \in P_i\} \subset \{\|p\|_K : p \in P_i\}$, by translational invariance. The cardinality of the latter set can be estimated trivially by $\# P_i \leq (q_i + 1)^d$. We conclude that $\Delta(E_i)$ is contained in at most $C(q_i + 1)^d$ intervals of length bounded by $C' q_i^{-\frac{d}{s}}$. It follows that the Hausdorff dimension of $\Delta(E)$ is at most s. Thus if $s < 1$, $\Delta(E)$ has Lebesgue measure 0.

Part ii) follows from Theorem 0.1 (ii) and Theorem 0.7 (iv).

Part iii) of Corollary 0.3 requires a bit of work. Let μ be a probability measure supported on E with the following energy integral finite:

$$\int |\xi|^{-\frac{d-1}{2}} |\widehat{\mu}(\xi)|^2 d\xi < \infty.$$

Averaging over rotations if necessary, we may assume that μ is rotation-invariant. Thus $\widehat{\mu}$ is also rotation-invariant. Let $F(s) = |\widehat{\mu}(s\omega)|$, $\omega \in S^{d-1}$, and define $A_{R,\delta}$ as in Lemma 1.2. We first claim that

$$\int_{S^{d-1}} |\widehat{\chi}_{A_{R,\delta}}(r\omega)| d\omega < C\delta r^{-\frac{d-1}{2}}. \tag{1.19}$$

Indeed, by (0.6) and Cauchy-Schwarz we have

$$\int_{S^{d-1}} |\widehat{\chi}_K(r\omega)| d\omega < C r^{-\frac{d+1}{2}}.$$

Now the claim follows by the same calculation as in the proof of Lemma 1.2, with the above inequality substituted for Lemma 1.1; the term analogous to II in the proof of Lemma 1.2 is estimated by following the proof of (0.6) in [BHI02] and using Cauchy-Schwarz again.

We now have

$$\begin{aligned}
\int\int \chi_{A_{R,\delta}}(x-y) d\mu(x) d\mu(y) &= \int \widehat{\chi}_{A_{R,\delta}}(\xi) |\widehat{\mu}(\xi)|^2 d\xi \\
&= \int \int_{S^{d-1}} \widehat{\chi}_{A_{R,\delta}}(r\omega) |\widehat{\mu}(r\omega)|^2 d\omega r^{d-1} dr \\
&\leq \int \int |\widehat{\chi}_{A_{R,\delta}}(r\omega)| d\omega F^2(r) r^{d-1} dr \\
&\leq C\delta \int r^{-\frac{d-1}{2}} F^2(r) r^{d-1} dr
\end{aligned}$$

$$(1.20) \qquad = C\delta \int |\xi|^{-\frac{d-1}{2}} |\widehat{\mu}(\xi)|^2 d\xi \leq C'\delta$$

if $\alpha > \frac{d+1}{2}$ where α is the Hausdorff dimension of E. The fourth line in (1.20) follows by (1.19). The rest of the proof is exactly like that of Theorem 0.1.

Proof of Theorem 0.4 and Corollary 0.5.

Let $p = (p_1, \dots, p_d)$. Let q_i denote a sequence of positive integers such that $q_{i+1} \geq q_i^i$. This sequence will be specified more precisely below. Let $0 < s \leq d$. Let

$$(2.1) \qquad E_i = \{x \in [0,1]^d : |x_j - p_j/q_i| \leq q_i^{-\frac{d}{s}} \text{ for some } \ p \in S\}.$$

Let $E = \bigcap_i E_i$. A standard calculation (see e.g. [Falconer85], Ch. 8) shows that the Hausdorff dimension of E is s. Also, $\Delta(E) \subset \bigcap_i \Delta(E_i)$.

Let S_q be defined as in the statement of Theorem 0.2. Suppose that $\#\Delta_K(S_{q_i}) \leq Cq_i^{\beta}$ for a sequence q_i going to infinity. By refining this sequence we can make sure that it satisfies the growth condition above. It follows that $\Delta(E_i)$ can be covered by at most Cq_i^{β} intervals of length $C'q_i^{-\frac{d}{s}}$. It follows that the Hausdorff dimension of $\Delta_K(E)$ is at most $\frac{s\beta}{d}$.

On the other hand, by assumption, (K, α_K) Falconer conjecture holds. Letting $s = \alpha_K$, we see that $\frac{\alpha_K \beta}{d} \geq 1$, so that $\beta \geq \frac{d}{\alpha_k}$ and we are done.

To prove Corollary 0.5 observe that the proof of Theorem 0.4 shows that if (0.4) holds, then $\frac{\Delta_K(S \cap [0,R]^d)}{R^d} \to \infty$ as $R \to \infty$. This shows that that $\Delta_K(S)$ cannot be separated.

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