EXTENDING A RECENT RESULT OF SANTOS ON PARTITIONS INTO ODD PARTS

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Abstract

In a recent note, Santos proved that the number of partitions of n using only odd parts equals the number of partitions of n of the form $p_1 + p_2 + p_3 + p_4 + \dots$ such that $p_1 \geq p_2 \geq p_3 \geq p_4 \geq \cdots \geq 0$ and $p_1 \geq 2p_2 + p_3 + p_4 + \dots$. Via partition analysis, we extend this result by replacing the last inequality with $p_1 \geq k_2p_2 + k_3p_3 + k_4p_4 + \dots$, where $k_2, k_3, k_4, \dots$ are nonnegative integers. Several applications of this result are mentioned in closing.

1 Background

One of the most celebrated identities in the theory of partitions is attributed to Leonhard Euler and reads as follows:

Theorem 1.1. *Let $d(n)$ be the number of partitions of n into distinct parts and let $o(n)$ be the number of partitions of n into odd parts. Then, for all $n \geq 0$, $d(n) = o(n)$.*

In a recent paper, Santos [12] proved via a bijection that $o(n)$ also equals the number of partitions of n of the form $p_1+p_2+p_3+p_4+\dots$ such that $p_1 \geq p_2 \geq p_3 \geq p_4 \geq \cdots \geq 0$ and $p_1 \geq 2p_2 + p_3 + p_4 + \dots$.

Our goal in this note is to prove Santos' result via generating functions. Actually, we will prove a much more general result using the technique of partitions analysis, introduced by Percy MacMahon [11, Vol. II, Section VIII] and heavily utilized recently by G. Andrews, P. Paule, A. Riese and others [1, 2, 3, 4, 5, 6, 7, 8, 9].

Our main theorem is as follows:

Theorem 1.2. *Let $K = (k_2, k_3, k_4, \dots)$ be an infinite vector of nonnegative integers. Define $p(n; K)$ as the number of partitions of n of the form $p_1 + p_2 + p_3 + p_4 + \dots$ with $p_1 \geq p_2 \geq p_3 \geq p_4 \cdots \geq 0$ and $p_1 \geq k_2p_2 + k_3p_3 + k_4p_4 + \dots$. Then, for all $n \geq 0$,*

$p(n;K)$ equals the number of partitions of n whose parts must be 1's or of the form $(\sum_{i=2}^{m} k_i) + (m-1)$ for some integer $m \geq 2$.

Before turning to the proof of Theorem 1.2, we briefly mention a few key items from partition analysis. First, we define the Omega operator $\underset{\geqq}{\Omega}$.

Definition 1.3. *The operator $\underset{\geqq}{\Omega}$ is given by*

$$\underset{\geqq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_j=-\infty}^{\infty} A_{s_1,\ldots,s_j} \lambda_1^{s_1} \ldots \lambda_j^{s_j} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_j=0}^{\infty} A_{s_1,\ldots,s_j},$$

where the domain of the $A_{s_1,\ldots,s_j}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to annuli of the form $1-\varepsilon < |\lambda_i| < 1+\varepsilon$.

In the work below, we will also use the symbol μ as a parameter like λ_j for some j. Finally, we need the following lemma involving the Omega operator.

Lemma 1.4.

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)} = \frac{1}{(1-x)(1-xy)}.$$

A proof of this result can be found in [3, Lemma 1.1].

2 Main Result

Now we are in position to prove Theorem 1.2 via generating function manipulations.

Proof. Note that

$$\begin{aligned} \sum_{n=0}^{\infty} p(n;K) q^n &= \sum_{\substack{p_1 \geq p_2 \geq p_3 \geq \cdots \geq 0 \\ p_1 \geq k_2 p_2 + k_3 p_3 + \ldots}} q^{p_1+p_2+p_3+\ldots} \\ &= \underset{\geqq}{\Omega} \sum_{p_1,\, p_2,\, p_3,\, \cdots \geq 0} q^{p_1+p_2+p_3+\ldots} \left(\lambda_1^{p_1-p_2} \lambda_2^{p_2-p_3} \lambda_3^{p_3-p_4} \ldots\right) \mu^{p_1-k_2p_2-k_3p_3-\ldots} \end{aligned}$$

by the definition of the Omega operator. Hence, after rewriting the above and applying

Lemma 1.4 multiple times, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p(n;K)q^n &= \underset{\geqq}{\Omega} \frac{1}{(1-q\lambda_1\mu)\left(1-\frac{q\lambda_2}{\lambda_1\mu^{k_2}}\right)\left(1-\frac{q\lambda_3}{\lambda_2\mu^{k_3}}\right)\dots} \\ &= \underset{\geqq}{\Omega} \frac{1}{(1-q\mu)\left(1-\frac{q^2\lambda_2}{\mu^{k_2-1}}\right)\left(1-\frac{q\lambda_3}{\lambda_2\mu^{k_3}}\right)\dots} \\ &= \underset{\geqq}{\Omega} \frac{1}{(1-q\mu)\left(1-\frac{q^2}{\mu^{k_2-1}}\right)\left(1-\frac{q^3\lambda_3}{\mu^{k_3+k_2-1}}\right)\dots}\end{aligned}$$

We continue to apply Lemma 1.4 to eliminate all parameters λ_j to obtain

$$\sum_{n=0}^{\infty} p(n;K)q^n = \underset{\geqq}{\Omega} \left(\frac{1}{1-q\mu}\right)\left(\frac{1}{1-\frac{q^2}{\mu^{k_2-1}}}\right)\left(\frac{1}{1-\frac{q^3}{\mu^{k_2+k_3-1}}}\right)\dots$$

At this point, the only parameter to eliminate is μ. We now rewrite the generating function above in terms of geometric series and annilihate μ based on the definition of the Omega operator. Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} p(n;K)q^n &= \underset{\geqq}{\Omega} \sum_{a_1\geq 0}(q\mu)^{a_1}\sum_{a_2\geq 0}(q^2\mu^{-k_2+1})^{a_2}\sum_{a_3\geq 0}(q^3\mu^{-k_3-k_2+1})^{a_3}\dots \\ &= \underset{\geqq}{\Omega} \sum_{a_1,a_2,a_3,\dots\geq 0} q^{a_1+2a_2+3a_3+\dots}\mu^{a_1+(-k_2+1)a_2+(-k_3-k_2+1)a_3+\dots} \\ &= \underset{\geqq}{\Omega} \sum_{\substack{a_2,\,a_3,\,\dots\,\geq\,0 \\ a_1\,\geq\,(k_2-1)a_2+(k_3+k_2-1)a_3+\dots}} q^{a_1+2a_2+3a_3+\dots}\mu^{a_1-[(k_2-1)a_2+(k_3+k_2-1)a_3+\dots]} \\ &= \sum_{a_2\geq 0} q^{2a_2}\times\sum_{a_3\geq 0} q^{3a_3}\times\dots\times \sum_{a_1\geq(k_2-1)a_2+(k_3+k_2-1)a_3+\dots} q^{a_1} \\ &= \sum_{a_2\geq 0} q^{2a_2}\times\sum_{a_3\geq 0} q^{3a_3}\times\dots\times \frac{q^{(k_2-1)a_2+(k_3+k_2-1)a_3+\dots}}{1-q} \\ &= \frac{1}{(1-q)(1-q^{k_2+1})(1-q^{k_3+k_2+2})(1-q^{k_4+k_3+k_2+3})\dots}.\end{aligned}$$

The result follows. □

3 Applications

We close with several comments related to Theorem 1.2. First off, Santos' result is clearly proven via Theorem 1.2 using the vector $K = (2, 1, 1, 1, \dots)$. Next, note that the vector

$K = (1, 0, 0, 0, \dots)$ also yields an obvious result. Namely, the number of partitions of n of the form $p_1 + p_2 + p_3 + \dots$ with $p_1 \geq p_2 \geq p_3 \geq \cdots \geq 0$ and $p_1 \geq p_2$ is simply $p(n)$, whose generating function is

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\dots},$$

which is what we obtain in Theorem 1.2 with $K = (1, 0, 0, 0, \dots)$.

A third example of Theorem 1.2 arises in connection with the vector $K = (1, 1, 1, 1, \dots)$. From Theorem 1.2 we find that the number of partitions of n with $p_1 \geq p_2 + p_3 + p_4 + \dots$ equals the number of partitions of n using 1's and even integers as parts. This means

$$\sum_{n=0}^{\infty} p(n; (1, 1, 1, 1, \dots))q^n = \frac{1}{(1-q)(1-q^2)(1-q^4)(1-q^6)\dots}.$$

Note that, by generating function dissection, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} p(2n; (1, 1, 1, 1, \dots))q^{2n} \\
&= \frac{1}{2}\left[\sum_{n=0}^{\infty} p(n; (1, 1, 1, 1, \dots))q^n + \sum_{n=0}^{\infty} p(n; (1, 1, 1, 1, \dots))(-q)^n\right] \\
&= \frac{1}{2}\left(\frac{1}{(1-q^2)(1-q^4)(1-q^6)\dots}\right)\left(\frac{1}{1-q} + \frac{1}{1+q}\right) \\
&= \frac{1}{2}\left(\frac{1}{(1-q^2)(1-q^4)(1-q^6)\dots}\right)\left(\frac{2}{(1-q)(1+q)}\right) \\
&= \frac{1}{(1-q^2)^2(1-q^4)(1-q^6)\dots}.
\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} p(2n; (1, 1, 1, 1, \dots))q^n = \frac{1}{(1-q)^2(1-q^2)(1-q^3)\dots}.$$

Similar analysis shows that $p(2n + 1; (1, 1, 1, 1, \dots))$ has the same generating function. A variant of this generating function recently arose in the context of graphical forest partitions [10]. Namely, let $gf(2k)$ be the number of partitions of $2k$ such that each partition, when viewed as the degree sequence of a graph, has a graphical representation which is a tree or union of trees (forest). Since the generating function for $gf(2n)$, as shown in [10], is

$$\frac{q}{(1-q)^2(1-q^2)(1-q^3)\dots},$$

we now know that

$$p(2n - 2; (1, 1, 1, 1, \dots)) = gf(2n)$$

for all $n \geq 1$.

We close with one last well-known partition function which is related to the Rogers-Ramanujan identities. Namely, let $p_5^*(n)$ be the number of partitions of n into parts congruent to $\pm 1 \pmod 5$. Then it is clear that $p_5^*(n) = p(n;(3,1,2,1,2,1,2,\dots))$ for all n. By way of generalization, let $p_m^*(n)$ be the number of partitions of n into parts congruent to $\pm 1 \pmod m$ (for $m \geq 3$). Then, for all $n \geq 0$,

$$p_m^*(n) = p(n;(m-2,1,m-3,1,m-3,1,m-3,\dots)).$$

Of course, the case $m = 4$ returns us to Santos' result, the original motivation for this note.

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