THE RELATIVE SIZE OF CONSECUTIVE ODD DENOMINATORIS IN FAREY SERIES

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Abstract

Let $\mathfrak{F}_Q$ be the Farey sequence of order Q and let $\mathfrak{F}_Q(c,d)$ be the subset of those fractions whose denominators are congruent to $c \pmod d$. A fundamental property of $\mathfrak{F}_Q$ says that the sum of denominators of any pair of neighbor fractions is always greater than Q. It turns out that this property is no longer true for $d \geq 2$. We show that the set of normalized pairs $(q'/Q, q''/Q)$, where q', q'' are denominators of consecutive fractions belonging to the subset of fractions with odd denominators becomes dense, as $Q \to \infty$, in the quadrangle with vertices $(1,0)$; $(1,1)$; $(0,1)$; $(1/3,1/3)$. We also find the local densities of points in this set.

1. Introduction and Statement of Results

The distribution of Farey sequences has been studied, from various points of view, for a long time. In some questions, such as for instance those related to the connection between Farey fractions and Dirichlet $L-$functions, one is naturally lead to consider subsequences of Farey fractions defined by congruence constraints. The distribution of such subsequences is not well understood at present. One reason why it is more difficult to handle sequences of Farey fractions with congruence constraints is because these subsequences fail to have some of the nice, basic properties which the entire sequence of Farey fractions has. For example, the well known fact that $a''q' - a'q'' = 1$ for any two consecutive Farey fractions a'/q' and a''/q'', fails for subsequences as above. This phenomenon has been investigated for subsequences of Farey fractions with odd denominators in [2] and [5]. In the present paper, we are concerned with another basic property of the Farey sequence of any order Q, which says that the sum of denominators of any two consecutive Farey fractions in this finite sequence is larger than Q. We shall see that, although this property fails for subsequences of Farey fractions with odd denominators, there are meaningful things that can be proved in this case too.

For c, d integers with $d \geq 1$ and $0 \leq c < d$, let

$$\mathfrak{F}_Q(c,d) = \left\{\frac{a}{q}\colon\ 1 \leq a \leq q \leq Q,\ \gcd(a,q) = 1,\ q \equiv c \pmod{d}\right\}$$

be the set of Farey fractions of order Q with denominators congruent to c modulo d. In what follows, we always assume that the elements of $\mathfrak{F}_Q(c,d)$ are arranged in increasing order. In particular, for a given order Q, we denote by $\mathfrak{F}_Q = \mathfrak{F}_Q(0,1)$ the set of all Farey fractions, and by $\mathfrak{F}_{Q,\mathrm{odd}} = \mathfrak{F}_Q(1,2)$ the set of Farey fractions with odd denominators. We call a Farey fraction *odd* if its denominator is odd and *even* if its denominator is even, respectively.

It is well known that given the denominators of two consecutive fractions from $\mathfrak{F}_Q$, one can produce their numerators, then their neighbor fractions and afterwords, one can generate recursively the whole set $\mathfrak{F}_Q$. As was mentioned above, the classical inequality $q' + q'' > Q$, which holds for any two consecutive Farey fractions $a'/q', a''/q'' \in \mathfrak{F}_Q$, is no longer true if one replaces $\mathfrak{F}_Q$ by $\mathfrak{F}_{Q,\mathrm{odd}}$. A natural question would be to investigate how often does this property fail as $(a'/q', a''/q'')$ runs over the set of pairs of consecutive elements of $\mathfrak{F}_{Q,\mathrm{odd}}$. And, when the above inequality fails, does the pair of normalized ratios $(q'/Q, q''/Q)$ have any preference to lie in any particular subregion of the unit square? In order to find the answer to these, and similar questions, in the following we investigate, the local density of points $(q'/Q, q''/Q)$, with q', q'' denominators of consecutive Farey fractions in $\mathfrak{F}_{Q,\mathrm{odd}}$, which lie around any given point (u,v) in the unit square. We shall see that this local density approaches a certain limiting local density $g(u,v)$ as $Q \to \infty$, and we shall provide an explicit formula for $g(u,v)$, for any real numbers u, v with $0 < u, v < 1$.

Let us denote

$$\mathcal{D}_Q(c,d) := \left\{(q',q'')\colon\ q', q'' \text{ denominators of consecutive fractions in } \mathfrak{F}_Q(c,d)\right\},$$

and consider the normalized set $\mathcal{D}_Q(c,d)/Q$ and its limit $\mathcal{D}(c,d)$, as $Q \to \infty$. Precisely, we have

$$\mathcal{D}(c,d) := \left\{ (x,y) \in (0,1)^2 : \begin{array}{l} \text{there exists a sequence of pairs } (q'_n, q''_n) \in \mathcal{D}_{Q_n}(c,d), \\ \text{such that } \lim_{n\to\infty} (q'_n/Q_n, q''_n/Q_n) = (x,y) \end{array} \right\}.$$

A significant geometric interpretation of Farey fractions follows by identifying each pair $(a'/q', a''/q'')$ of consecutive fractions in $\mathfrak{F}_Q$ with the point of coordinates (q', q'') in $\mathbb{R}^2$. In this way, one can view $\mathfrak{F}_Q$ as the set of lattice points with coprime coordinates in the triangle $\mathcal{T}_Q$ with vertices $(0,Q)$; $(Q,0)$; (Q,Q). Downscaling by multiplying with $1/Q$, we get $\mathcal{T}$, the Farey triangle with vertices $(0,1)$; $(1,0)$; $(1,1)$. Then, it is not difficult to see that in the case of all Farey fractions, the sets of interior points of $\mathcal{D}(0,1)$ and $\mathcal{T}$ coincide.

It is not as easy to find $\mathcal{D}(c,d)$ when $d \geq 2$. This is mainly due to a couple of facts. Firstly, two consecutive fractions in $\mathfrak{F}_Q(c,d)$ may be far away in $\mathfrak{F}_Q$, since there may exist many Farey fractions in $\mathfrak{F}_Q$ in between them. For example, $1/2$ has in $\mathfrak{F}_Q$ a number of $\left[\frac{Q}{4}\right] + a$ odd neighbors on each side, where $a = 0,1,1,2$ for $Q \equiv 0,1,2,3 \pmod 4$, respectively. Secondly, for a given $n \geq 3$, the number of pairs of consecutive fractions in $\mathfrak{F}_Q(c,d)$, which are the end points of an n-tuple of consecutive fractions in $\mathfrak{F}_Q$, may have a significant contribution, whence one can not neglect its influence on the local densities or even on the shape of $\mathcal{D}(c,d)$. Both these facts are tractable, but the required analysis may be quite complex.

Numerical calculations show that the closure of $\mathcal{D}(c,d)$ in $\mathbb{R}^2$ is often the same for different values of c, but the local densities are different. Also, when d gets large, the sets $\mathcal{D}(c,d)$ tend to occupy, besides $\mathcal{T}$, the South-West corner of the unit square.

More relevant information about $\mathfrak{F}_Q(c,d)$ can be deduced if one knows the local densities at points in $\mathcal{D}(c,d)$. At any $(u,v) \in (0,1)^2$, this local density is defined by

$$g(u,v) := \lim_{\mathrm{Area}(\Delta)\to 0} \frac{\lim\limits_{Q\to\infty} \frac{\#\left(\Delta \cap \mathcal{D}_Q(c,d)\right)}{\#\mathcal{D}_Q(c,d)}}{\mathrm{Area}(\Delta)}, \tag{1}$$

in which $\Delta \subset \mathbb{R}^2$ are squares centered at (u,v). We shall address the problem of finding $g(u,v)$ in the case of odd Farey fractions. The next theorem shows that the local density function on $\mathcal{D}(1,2)$ exists, and its value is calculated explicitly.

For a set of *conditions* (equalities or inequalities in variables u and v), we use the following notation for the characteristic function:

$$\varphi(\textit{conditions}) = \begin{cases} 1, & \text{if } u,v \text{ satisfy all } \textit{conditions} \\ 0, & \text{else} \end{cases}.$$

Theorem 1. *The local density in the unit square of the points $(q'/Q, q''/Q)$, where q' and q'' are denominators of neighbor fractions in $\mathfrak{F}_{Q,\mathrm{odd}}$, approaches a limiting density $g(u,v)$ as $Q \to \infty$. Moreover, for any real numbers u, v with $0 < u, v < 1$,*

$$
\begin{aligned}
g(u,v) =& \varphi\big(1 < u+2v,\ 1 < 2u+v\big) + \sum_{k=2}^{\min\left(\frac{u+v}{1-v},\,\frac{u+v}{1-u}\right)} \frac{1}{k} \\
&+ \frac{1}{2}\,\varphi\big(2u+v=1,\ \text{if } 0<u<1/3 \\
&\qquad\qquad \text{or}\ \ u+2v=1,\ \text{if } 1/3<u<1\big) \\
&+ \frac{1}{2k}\,\varphi\big(k = \tfrac{u+v}{1-v} \ge 2,\quad \text{if } \tfrac{k}{k+2} < u < 1, \\
&\qquad\qquad \text{or } k = \tfrac{u+v}{1-u} \ge 2,\quad \text{if } \tfrac{k-1}{k+1} < u < \tfrac{k}{k+2}\big) \\
&+ \frac{k+2}{4k(k+1)}\,\varphi\big(u=v=\tfrac{k}{k+2},\ k\ge 1\big)\,.
\end{aligned}
$$

(Here k is a positive integer.)

We remark that the expression of $g(u,v)$ above shows that the density is locally constant on an open subset of measure 1 of the unit square. One should compare this result with that obtained in the case of all Farey fractions. There, it is not difficult to see that the local density on $\mathcal{D}(0,1) = \mathcal{T}$ is constant $= 1$ in the interior, and on the edges it reduces to $1/2$.

Additionally, we can find how often the native property of $\mathfrak{F}_Q$, which says that the sum of neighbor denominators is $> Q$, is preserved in $\mathfrak{F}_{Q,\mathrm{odd}}$. Proposition 1 below proves that the proportion of pairs $(q', q'') \in \mathcal{D}_Q(1,2)$, which satisfy the condition $q' + q'' > Q$, tends to $5/6$, as $Q \to \infty$. The remaining points are situated on or under the line $x+y=1$, in the triangle $(0,1)$; $(1,0)$; $(1/3,1/3)$.

By definition, we know that $\mathcal{D}(1,2)$ is closed in $(0,1)^2$, and from the proof of Theorem 1 it follows that it has no isolated points. Moreover, $\mathcal{D}(1,2)$ contains exactly those points from $(0,1)^2$ where the density $g(u,v)$ does not vanish.

Corollary 1. *The set $\mathcal{D}(1,2)$ coincides with the quadrangle bounded by the lines: $y=1$, $x=1$, $2x+y=1$, and $2y+x=1$.*

Also, from Theorem 1, we see that the distribution of local densities is like a stairway ascending as the sum of an harmonic series towards the point $(1,1)$, where it blows up. This shows the preponderance of couples of neighbor denominators in $\mathfrak{F}_{Q,\mathrm{odd}}$ that are both large and almost equal in size. In Figure 2, there is a picture of $\mathcal{D}(1,2)$, in which heavier colors represent places with higher densities. More details on the limiting process that leads to $\mathcal{D}(1,2)$ are given in Section 2, and in Section 3 we complete the proof of Theorem 1.

2. Prerequisites and Geometric Aspects of Farey Fractions

We begin by stating some fundamental properties of Farey fractions, which will be used in the sequel. For a proof of them, we refer to Hardy and Wright [4] and Hall [3]. The first one says that if $a'/q' < a''/q''$ are neighbor fractions in $\mathfrak{F}_Q$, then

$$a''q' - a'q'' = 1\,. \tag{2}$$

A second statement, called *the mediant property*, reveals the relation among three consecutive fractions of $\mathfrak{F}_Q$. Thus, if $a'/q' < a''/q'' < a'''/q'''$ are consecutive elements of $\mathfrak{F}_Q$ then

$$\frac{a''}{q''} = \frac{a' + a'''}{q' + q'''}\,. \tag{3}$$

This implies that the integer k that reduces the mediant fraction (called the *index* of the Farey fraction a'/q') satisfies the relations:

$$k = \frac{a' + a'''}{a''} = \frac{q' + q'''}{q''} = a''q' - a'q''' = \left[\frac{Q + q'}{q''}\right]. \tag{4}$$

In the proof of (2)–(4) one needs the next lemma.

Lemma 1. *The positive integers q', q'' are denominators of neighbor fractions in $\mathfrak{F}_Q$ if and only if $(q', q'') \in \mathcal{T}_Q$ and $\gcd(q', q'') = 1$. Also, the pair (q', q'') appears exactly once as a pair of denominators of consecutive Farey fractions.*

By Lemma 1 and relation (4) it follows that any h-tuple $(q', q'', q''', \ldots, q^{(h)})$ of denominators of neighbor fractions in $\mathfrak{F}_Q$ is uniquely determined by q' and q''. Also, one should notice that although any pair (q', q'') with coprime components $\leq Q$ does appear exactly once as a pair of neighbor denominators of Farey fractions, the components of longer tuples must satisfy supplementary conditions in order to appear as neighbor denominators of fractions in $\mathfrak{F}_Q$.

For any positive integer k, we consider the convex polygon defined by

$$\mathcal{T}_{Q,k} := \left\{(x, y) \colon\ 0 < x, y \leq Q,\ x + y > Q,\ ky \leq Q + x < (k+1)y\right\}.$$

For $k = 1$, one can see that $\mathcal{T}_{Q,1}$ is the triangle with vertices $(0, Q)$, $\left(\frac{Q}{3}, \frac{2Q}{3}\right)$, (Q, Q), and for any $k \geq 2$, $\mathcal{T}_{Q,k}$ is the quadrilateral with vertices $\left(Q, \frac{2Q}{k}\right)$; $\left(\frac{Q(k-1)}{k+1}, \frac{2Q}{k+1}\right)$; $\left(\frac{Qk}{k+2}, \frac{2Q}{k+2}\right)$;

$\left(Q, \frac{2Q}{k+1}\right)$. Downscaling by a factor of Q, for any $k \geq 1$ we get $\mathcal{T}_k = \mathcal{T}_{Q,k}/Q$, a polygon which is independent of Q, and might be thought as the limiting domain as $Q \to \infty$. In Figure 1, there is a representation of $\mathcal{T}_k$, for $k \geq 1$.

The polygons $\mathcal{T}_k$ play an important role because the index function, defined by $(x, y) \mapsto \left[\frac{1+x}{y}\right]$ is locally constant on each $\mathcal{T}_k$. This result is summarized in the following lemma, which characterizes the triplets of neighbor denominators of Farey fractions.

Lemma 2. *The positive integers q', q'', q''' are denominators of consecutive fractions in $\mathfrak{F}_Q$ and $k = \frac{q'+q'''}{q''}$ if and only if $(q', q'') \in \mathcal{T}_{Q,k}$ and* $\gcd(q', q'') = 1$.

We remark that the sets $\mathcal{T}_k$, with $k \geq 1$, are disjoint and their union equals $\mathcal{T}$, that is, they form a partition of $\mathcal{T}$.

Usually, finding the number of Farey fractions with a certain property can be done by counting the number of lattice points in a certain domain. This may be achieved in a general context, as is presented in Lemma 1 below, which is a variation of Lemma 2 from [1]. For any domain $\Omega \subset \mathbb{R}^2$ we denote:

$$N_{\text{odd,odd}} := \#\left\{(x, y) \in \Omega \cap \mathbb{Z}^2 \colon\ x \text{ odd},\ y \text{ odd},\ \gcd(x, y) = 1\right\},$$
$$N_{\text{odd,even}} := \#\left\{(x, y) \in \Omega \cap \mathbb{Z}^2 \colon\ x \text{ odd},\ y \text{ even},\ \gcd(x, y) = 1\right\}.$$

Lemma 3. [2, Corollary 3.2] *Let R_1, $R_2 > 0$, and $R \geq \min(R_1, R_2)$. Then, for any region $\Omega \subseteq [0, R_1] \times [0, R_2]$ with rectifiable boundary, we have:*

$$\begin{aligned} N_{odd,odd}(\Omega) &= 2\text{Area}(\Omega)/\pi^2 + O(C_{R,\Omega}), \\ N_{odd,even}(\Omega) &= 2\text{Area}(\Omega)/\pi^2 + O(C_{R,\Omega}), \end{aligned}$$

where $C_{R,\Omega} = \text{Area}(\Omega)/R + R + \text{length}(\partial\Omega)\log R$.

It is worthwhile to view $\mathfrak{F}_{Q,\text{odd}}$ as being produced through a sieving process that removes from $\mathfrak{F}_Q$ the even fractions. Then we see that a pair (q', q'') of neighbor denominators in $\mathfrak{F}_{Q,\text{odd}}$ comes up in one of the following two ways: either q', q'' are denominators of consecutive fractions in $\mathfrak{F}_Q$ or there exists an even positive integer $q \leq Q$ such that q', q, q'' are denominators of consecutive fractions in $\mathfrak{F}_Q$. We call them of `Type I` and `Type II`, respectively.

We now turn to the problem of finding the set $\mathcal{D}(1, 2)$. Let us first remark that if we cross out in $\mathcal{T}_Q$ the points situated on the vertical and horizontal lines with even abscissa and ordinate respectively, we are left with all the pairs of `Type I`. In the limit, when $Q \to \infty$, these points produce a subset of $\mathcal{D}(1, 2)$ that is dense in $\mathcal{T}$. We used here the fact that when $n \to \infty$, the set of points m/n with $1 \le m < n$ and $\gcd(m,n) = 1$ becomes dense in $[0,1]$.

Next, let us look at the triplets that produce the pairs of `Type II` . By (4), we know that they have the form $(q', q'', kq'' - q')$, with $k = \left[\frac{Q+q'}{q''}\right]$ and q' odd, q'' even. This means that, for any fixed $k \ge 1$, we need to retain the lattice points in the domain

$$\mathcal{V}_{Q,k} = \left\{(q', kq'' - q') \colon \ (q', q'') \in \mathcal{T}_{Q,k}\right\}$$

with q' odd, q'' even, and $\gcd(q', q'') = 1$. In the limit, when $Q \to \infty$, these points give a subset of $\mathcal{D}(1,2)$, which is dense in

$$\mathcal{V}_k = \left\{(x, ky - x) \colon \ (x, y) \in \mathcal{T}_k\right\}.$$

A straightforward calculation shows that $\mathcal{V}_1$ is the triangle with vertices $(0,1)$; $\left(\frac{1}{3}, \frac{1}{3}\right)$; $(1,0)$, and for $k \ge 2$ the set $\mathcal{V}_k$ is the quadrilateral with vertices $\left(\frac{k-1}{k+1}, 1\right)$; $\left(\frac{k}{k+2}, \frac{k}{k+2}\right)$; $\left(1, \frac{k-1}{k+1}\right)$; $(1,1)$.

Putting together the above facts, we have shown that the closure of $\mathcal{D}(1,2)$ in $(0,1)^2$, which coincides with $\mathcal{D}(1,2)$ (since by definition $\mathcal{D}(1,2)$ is closed) is

$$\mathcal{D}(1,2) = \left(\mathcal{T} \cup \bigcup_{k=1}^{\infty} \mathcal{V}_k\right) \cap (0,1)^2 . \tag{5}$$

Notice that $\mathcal{V}_2 \supset \mathcal{V}_3 \supset \mathcal{V}_4 \supset \ldots$, whence $\mathcal{D}(1,2) = (\mathcal{V}_1 \cup \mathcal{T}) \cap (0,1)^2$ is the quadrilateral with vertices $(0,1)$; $\left(\frac{1}{3}, \frac{1}{3}\right)$; $(1,0)$; $(1,1)$. In Figure 2, one can see a representation of $\mathcal{D}(1,2)$ covered by $\mathcal{T}$, $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, $\ldots$. In addition, the union from the right hand side of (5) gives a first hint on the local densities on $\mathcal{D}(1,2)$. Complete calculations are postponed to Section 3.

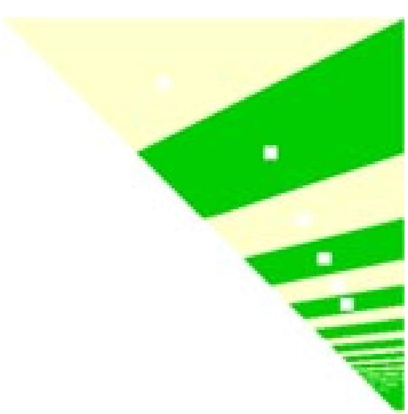

Figure 1
The tessellation of the Farey triangle with the polygons $\mathcal{T}_k$.

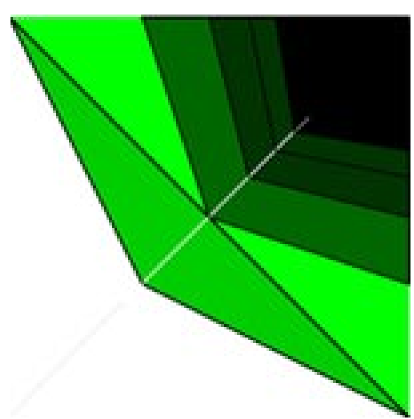

Figure 2
The covering of $\mathcal{D}(0,1)$ with $\mathcal{T}$, $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$,$\ldots$, from light to dark.

To conclude this section, we answer a question mentioned earlier, which asks to find the probability that two consecutive fractions $\frac{a'}{q'}, \frac{a''}{q''} \in \mathfrak{F}_{Q,\mathrm{odd}}$ satisfy the condition $q'+q'' > Q$, as neighbor denominators in $\mathfrak{F}_Q$ do. Firstly, the pairs (q', q'') of `Type I` satisfy the condition $q' + q'' > Q$, and by Lemma 1 we know that when $Q \to \infty$, the proportion of such pairs is $1/2$ of all the pairs of consecutive denominators of fractions in $\mathfrak{F}_{Q,\mathrm{odd}}$.

Secondly, the required pairs of `Type II` are "children" of triplets of neighbor denominators in (q', q'', q''') from $\mathfrak{F}_Q$ of parity (odd, even, odd) and satisfying $q' + q''' > Q$. Using (4), this last condition can be written as $q' + \left[\frac{Q+q'}{q''}\right] q'' - q' > Q$, or $q'' > \frac{Q}{k}$ for any $(q', q'') \in \mathcal{T}_k$ and $k \geq 1$. One can easily see that this condition is satisfied by all pairs $(q', q'') \in \mathcal{T}_k$ when $k \geq 2$. Letting $Q \to \infty$, we get the region $\mathcal{T} \setminus \mathcal{T}_1$ whose area counts the required proportions of pairs of `Type II` . This proportion is $\mathrm{Area}(T \setminus \mathcal{T}_1) = 1/3$. (Notice that we have ignored the parity and the coprimality conditions, but they do not influence the final result, as follows from Lemma 1.)

Superimposing both contributions, we get the probability that the sum of denominators of two consecutive fractions from $\mathfrak{F}_{Q,\mathrm{odd}}$ is larger than Q. This equals $1/2+1/3 = 5/6$.

Proposition 1. *The probability that the sum of neighbor denominators of fractions from $\mathfrak{F}_{Q,\mathrm{odd}}$ is $> Q$ equals $5/6$ as $Q \to \infty$.*

3. Proof of Theorem 1

Let a'/q' and a''/q'' be two consecutive Farey fractions from $\mathfrak{F}_{Q,\mathrm{odd}}$. Then they are either neighbor fractions in $\mathfrak{F}_Q$ or there is an even Farey fraction between them. In the language introduced in Section 2, the pair (q', q'') is either of `Type I` or of `Type II` . In the following, we shall also say that the scaled pairs $(q'/Q, q''/Q)$ are of `Type I` or of `Type II` , as the pair (q', q'') is. Let $g(x, y)$ be the function that gives the local densities of the points $(q'/Q, q''/Q)$ in the unit square as $Q \to \infty$, and denote by $g_1(x, y)$ and $g_2(x, y)$ the local densities in the unit square of the points $(q'/Q, q''/Q)$ of `Type I` and `Type II` , respectively, as $Q \to \infty$. These objects are defined as limits similar to that in (1). Then

$$g(u, v) = g_1(u, v) + g_2(u, v)\,, \tag{6}$$

provided we show that both local densities $g_1(u, v)$ and $g_2(u, v)$ exist.

3.1. The density $g_1(u, v)$

Let (x_0, y_0) be a fixed point in the unit square $(0, 1)^2$. We first assume that $x_0+y_0 > 1$. In this case, any neighborhood of (x_0, y_0) contains points $(q'/Q, q''/Q)$ of `Type I` if Q is large enough. This is due to the fact that points of Type I satisfy the property $q' + q'' > Q$. We choose $\eta > 0$ such that $(x_0 - \eta) + (y_0 - \eta) \geq 1$, and denote by $\mathcal{A}_Q$ the set of the points $(q'/Q, q''/Q)$ of Type I that fall in the square $(x_0 - \eta, x_0 + \eta) \times (y_0 - \eta, y_0 + \eta)$, that is,

$$\mathcal{A}_Q = \left\{ (q', q'') \in \mathbb{N}^2 \colon \begin{array}{l} 1 \leq q', q'' \leq Q,\ q' + q'' > Q,\ \gcd(q', q'') = 1,\ q', q'' \text{ odd;} \\ \dfrac{q'}{Q} \in (x_0 - \eta, x_0 + \eta),\ \dfrac{q''}{Q} \in (y_0 - \eta, y_0 + \eta) \end{array} \right\}.$$

Then the cardinality of $\mathcal{A}_Q$ is $N_{\text{odd,odd}}(\Omega_Q)$, where $\Omega_Q = \Omega_Q(x_0, y_0, \eta)$ is given by

$$\Omega_Q = \left\{ (x, y) \in \mathbb{R}^2 \colon \begin{array}{l} 1 \leq x, y \leq Q,\ x + y > Q, \\ Q(x_0 - \eta) < x < Q(x_0 + \eta),\ Q(y_0 - \eta) < y < Q(y_0 + \eta) \end{array} \right\}.$$

By Lemma 1, we get:

$$\#\mathcal{A}_Q = 8\, Q^2 \eta^2 / \pi^2 + O(Q \log Q).$$

On the other hand, it is well known that $\#\mathfrak{F}_{Q,\text{odd}} = 2Q^2/\pi^2 + O(Q \log Q)$, and because the number of points $(q'/Q, q''/Q)$ from $(0,1)^2$, where q' and q'' are the denominators of two consecutive elements from $\mathfrak{F}_{Q,\text{odd}}$ is $\#\mathfrak{F}_{Q,\text{odd}} - 1$, we have:

$$\int\limits_{x_0-\eta}^{x_0+\eta} \int\limits_{y_0-\eta}^{y_0+\eta} g_1(x, y)\, dx dy = \lim_{Q \to \infty} \frac{\#\mathcal{A}_Q}{\#\mathfrak{F}_{Q,\text{odd}} - 1} = 4\eta^2\,.$$

By the Lesbegue differentiation theorem, we obtain:

$$g_1(x_0, y_0) = \lim_{\eta \to 0} \frac{\int\limits_{x_0-\eta}^{x_0+\eta} \int\limits_{y_0-\eta}^{y_0+\eta} g_1(x, y)\, dx dy}{4\eta^2} = 1\,,$$

in other words, the measure associated to the distribution of points of Type I from $\mathcal{T}$ approaches the Lesbegue measure, as $Q \to \infty$.

In the case $x_0 + y_0 < 1$, we clearly have $g_1(x_0, y_0) = 0$. Finally, on the boundary, that is, when $x_0 + y_0 = 1$, we get $g_1(x_0, y_0) = 1/2$, using the same argument as in the first case, the only change being that now Ω_Q is a right isosceles triangle of area $2\eta^2$. These results can be written in closed form as

$$g_1(u, v) = \varphi(u + v > 1) + \frac{1}{2}\varphi(u + v = 1)\,, \quad \text{for } (u, v) \in (0,1)^2. \tag{7}$$

3.2. The density $g_2(u, v)$

Let (x_0, y_0) be a fixed point in the unit square $(0,1)^2$, and fix a small $\eta > 0$. We know that any pair (q', q''') of Type II has in $\mathfrak{F}_Q$ a “parent” (q', q'', q''') with q', q''' odd, q'' even, and $q' + q''' = \left[\frac{Q+q'}{q''}\right] q''$. Next, we consider the set $\mathcal{B}_Q$ of points (q', q''') of `Type II` for which $(q'/Q, q'''/Q)$ falls in the square $(x_0 - \eta, x_0 + \eta) \times (y_0 - \eta, y_0 + \eta)$, that is,

$$\mathcal{B}_Q = \left\{ (q', q'') \in \mathbb{N}^2 \colon \begin{array}{l} 1 \le q', q'' \le Q,\ \gcd(q', q'') = 1,\ q' \text{ odd},\ q'' \text{ even}; \\ \dfrac{q'}{Q} \in (x_0 - \eta, x_0 + \eta),\ \left[\dfrac{Q+q'}{q''}\right] \dfrac{q''}{Q} - \dfrac{q'}{Q} \in (y_0 - \eta, y_0 + \eta) \end{array} \right\}.$$

Then the cardinality of $\mathcal{B}_Q$ is $\#\mathcal{B}_Q = N_{\text{odd,even}}(\Omega_Q)$, where $\Omega_Q = \Omega_Q(x_0, y_0, \eta)$ is given by

$$\Omega_Q = \left\{ (x, y) \in \mathbb{R}^2 \colon \begin{array}{l} 1 \le x, y \le Q,\ x + y > Q,\ k = \left[\frac{Q+x}{y}\right]; \\ Q(x_0 - \eta) < x < Q(x_0 + \eta),\ Q(y_0 - \eta) < ky - x < Q(y_0 + \eta) \end{array} \right\}.$$

Downscaling by multiplication with $1/Q$, we get the bounded set

$$\Omega = \left\{ (x, y) \in (0,1)^2 \colon \begin{array}{l} x + y > 1,\ k = \left[\frac{1+x}{y}\right]; \\ x_0 - \eta < x < x_0 + \eta,\ y_0 - \eta < ky - x < y_0 + \eta \end{array} \right\}.$$

and $Q \cdot \Omega = \Omega_Q$. By Lemma 1, it follows that

$$\#\mathcal{B}_Q = \frac{2Q^2 \operatorname{Area}(\Omega)}{\pi^2} + O(Q \log Q)\,. \tag{8}$$

In order to calculate its area, we split Ω using the angular domains

$$\mathcal{U}_k := \left\{ (x, y) \in (0,1)^2 \colon\ k = \left[\frac{1+x}{y}\right] \right\}, \quad \text{for } k = 1, 2, 3 \dots.$$

Notice that $\mathcal{U}_k \cap \mathcal{T} = \mathcal{T}_k$, the polygons depicted in Figure 1. Then

$$\begin{aligned} \Omega \cap \mathcal{U}_k &= \{(x, y) \in \mathcal{T}_k \colon\ x_0 - \eta < x < x_0 + \eta,\ y_0 - \eta < ky - x < y_0 + \eta\} \\ &= \mathcal{T}_k \cap \mathcal{P}_k\,, \end{aligned}$$

where $\mathcal{P}_k = \mathcal{P}_k(x_0, y_0, \eta)$ is the parallelogram

$$\mathcal{P}_k = \left\{ (x, y) \in \mathbb{R}^2 \colon\ x_0 - \eta < x < x_0 + \eta,\ y_0 - \eta < ky - x < y_0 + \eta \right\}$$

and

$$\operatorname{Area}(\Omega) = \sum_{k=1}^{\infty} \operatorname{Area}(\mathcal{T}_k \cap \mathcal{P}_k)\,. \tag{9}$$

The above series is convergent. In fact, one can easily see by a compactness argument that only a finite number of terms in the sum are non-zero. In order to get a concrete expression for the density, one requires an explicit form of the series in (9).

The center of $\mathcal{P}_k$ has coordinates $C_k = \big(x_0, (x_0 + y_0)/k\big)$, the length of the vertical edges is $2\eta/k$ and the height is 2η. Therefore

$$\operatorname{Area}(\mathcal{P}_k) = 2\eta \cdot 2\eta/k = 4\eta^2/k. \tag{10}$$

We need to find out under what conditions $\mathcal{P}_k$ intersects $\mathcal{T}_k$. Since η can be chosen as small as we please, and eventually it tends to 0, while x_0, y_0 are kept fixed, we only need to see when C_k lies either in the interior or on the boundary of $\mathcal{T}_k$. In the following, we shall assume that η is small enough. We may also assume that k is bounded, since all the parallelograms $\mathcal{P}_k$ are contained in the vertical strip given by the inequalities $x_0 - \eta < x < x_0 + \eta$, and since only finitely many polygons $\mathcal{T}_k$ intersect this strip.

Fix now a positive integer k. Firstly, if C_k belongs to the interior $\overset{\circ}{\mathcal{T}_k}$ of $\mathcal{T}_k$, it follows that $\mathcal{P}_k \subset \mathcal{T}_k$ for η small enough, so $\operatorname{Area}(\mathcal{T}_k \cap \mathcal{P}_k) = \operatorname{Area}(\mathcal{P}_k)$.

Secondly, if $C_k \in \partial\mathcal{T}_k$, but is not a vertex of $\mathcal{T}_k$, using the fact that any line that crosses a parallelogram through its center cuts the parallelogram into two pieces of equal area, it follows that in this case $\operatorname{Area}(\mathcal{T}_k \cap \mathcal{P}_k) = \operatorname{Area}(\mathcal{P}_k)/2$.

Thirdly, we need to know when C_k coincides with a vertex of $\mathcal{T}_k$. We use the fact that, by definition, it follows that the edges of $\mathcal{T}_k$ have equations $y = (x+1)/k$ (top), $y = (x+1)/(k+1)$ (bottom), $x + y = 1$ (left), and $x = 1$ (right), for $k \geq 1$, except for $k = 1$, where the top edge is $y = 1$ and the other two edges have equations $x + y = 1$ (bottom left) and $y = (x+1)/2$ (bottom right). Considering all the vertices of $\mathcal{T}_k$ and using the hypothesis that (x_0, y_0) is in the open unit square, one finds that C_k may only coincide with the South-West vertex of $\mathcal{T}_k$, for any $k \geq 1$. This gives

$$C_k \text{ vertex of } \mathcal{T}_k \quad \Longleftrightarrow \quad x_0 = \frac{k}{k+2}, \;\; y_0 = \frac{k}{k+2}, \quad \text{for } k \geq 1.$$

Suppose C_k coincides with such a corner of $\mathcal{T}_k$, that is, $x_0 = y_0 = k/(k+2)$. Then, as η is small, $\mathcal{P}_k \cap \mathcal{T}_k$ is a quadrilateral with edges $x = x_0 + \eta$, $x + y = 1$, $y = (x+1)/(k+1)$ and $y = (x + y_0 + \eta)/k$ and vertices $\big(\frac{k}{k+2}, \frac{2}{k+2}\big)$, $\big(\frac{k}{k+2} + \eta, \frac{2}{k+2} + \frac{\eta}{k+1}\big)$, $\big(\frac{k}{k+2} + \eta, \frac{2}{k+2} + \frac{2\eta}{k}\big)$, and $\big(\frac{k}{k+2} - \frac{\eta}{k+1}, \frac{2}{k+2} + \frac{\eta}{k+1}\big)$. Its area is $\operatorname{Area}(\mathcal{T}_k \cap \mathcal{P}_k) = \eta^2(k+2)/\big(k(k+1)\big)$.

Let $V(\mathcal{T}_k)$ be the set of vertices of the polygon $\mathcal{T}_k$. Using the last remarks together with (10) in (9), we obtain:

$$\operatorname{Area}(\Omega) = 4\eta^2 \sum_{C_k \in \overset{\circ}{\mathcal{T}_k}} \frac{1}{k} + 2\eta^2 \sum_{C_k \in \partial\mathcal{T}_k \setminus V(\mathcal{T}_k)} \frac{1}{k} + \eta^2 \sum_{C_k \in V(\mathcal{T}_k)} \frac{k+2}{k(k+1)}\,. \tag{11}$$

To finish the proof of the theorem, we only need to translate the condition of summation in terms of x_0 and y_0. Suppose first that $k \geq 2$. Then $C_k \in \overset{\circ}{\mathcal{T}_k}$ if and only if the following conditions hold simultaneously:

$$\begin{cases} \frac{x_0+1}{k+1} < \frac{x_0+y_0}{k} < \frac{x_0+1}{k}, \\ x_0 + \frac{x_0+y_0}{k} > 1, \\ x_0 < 1. \end{cases}$$

Here, the last condition is trivial, and writing jointly the first two, we get

$$C_k \in \overset{\circ}{\mathcal{T}_k} \quad \Longleftrightarrow \quad k < \min\left(\frac{x_0+y_0}{1-y_0}, \frac{x_0+y_0}{1-x_0}\right), \quad \text{for } k \geq 2. \tag{12}$$

Proceeding similarly in the case $k = 1$, we have

$$C_1 \in \overset{\circ}{\mathcal{T}_1} \quad \Longleftrightarrow \quad 1 < x_0 + 2y_0, \quad x_0 + y_0 < 1, \quad 1 < 2x_0 + y_0\,. \tag{13}$$

Next, we find necessary and sufficient conditions for C_k to be on the open edges of $\mathcal{T}_k$. Suppose $k \geq 2$. Then the coordinates of C_k should satisfy one of the following conditions:

$$\frac{x_0+1}{k+1} = \frac{x_0+y_0}{k}, \quad \text{for } \frac{k}{k+2} < x_0 < 1$$

or

$$\frac{x_0+1}{k} = \frac{x_0+y_0}{k}, \quad \text{for } \frac{k-1}{k+1} < x_0 < 1$$

or

$$x_0 + \frac{x_0+y_0}{k} = 1, \quad \text{for } \frac{k-1}{k+1} < x_0 < \frac{k}{k+2}$$

or

$$x_0 = 1.$$

The second and the last equality can not hold for any (x_0, y_0) in the open unit square. It remains:

$$C_k \in \partial\mathcal{T}_k \setminus V(\mathcal{T}_k) \quad \Longleftrightarrow \quad \begin{cases} k = \frac{x_0+y_0}{1-y_0}, & \text{for } \frac{k}{k+2} < x_0 < 1, \\ \quad \text{or} & \\ k = \frac{x_0+y_0}{1-x_0}, & \text{for } \frac{k-1}{k+1} < x_0 < \frac{k}{k+2}, \end{cases} \quad \text{for } k \geq 2. \tag{14}$$

For the three edges of $\mathcal{T}_1$, we get:

$$C_1 \in \partial\mathcal{T}_1 \setminus V(\mathcal{T}_1) \quad \Longleftrightarrow \quad \begin{cases} x_0 + y_0 = 1, & \\ \quad \text{or} & \\ 2x_0 + y_0 = 1, & \text{for } 0 < x_0 < \frac{1}{3}, \\ \quad \text{or} & \\ x_0 + 2y_0 = 1, & \text{for } \frac{1}{3} < x_0 < 1, \end{cases} \tag{15}$$

Finally, the last condition, that is, $C_k \in V(\mathcal{T}_k)$ translates into:

$$C_k \in V(\mathcal{T}_k) \quad \Longleftrightarrow \quad x_0 = y_0 = \frac{k}{k+2}, \quad \text{for } k \geq 1. \tag{16}$$

We see, from (14) and (15), that the second sum on the right hand side of (11) consists of at most one term. More precisely, this sum vanishes unless $x_0 + y_0 = 1$ or one of the fractions $(x_0 + y_0)/(1 - x_0)$ or $(x_0 + y_0)/(1 - y_0)$ is an integer. Geometrically, this means that the second sum on the right hand side of (11) has a non-zero contribution only if our point (x_0, y_0) lies on one of the edges of one of the quadrilaterals from Figure 2, or if it lies on the diagonal $x + y = 1$. Similarly, the last sum on the right hand side of (11) consists of at most one term, and this happens if and only if (x_0, y_0) coincides with a vertex of one of the quadrilaterals from Figure 2 that lies on the diagonal $x = y$. In other words, the last sum on the right hand side of (11) has a non-zero contribution if and only if (x_0, y_0) is one of the points $\left(\frac{k}{k+2}, \frac{k}{k+2}\right)$, with $k \geq 1$.

Replacing the condition of summation from the right-hand side of (11) by their equivalents from (12) – (16), we get:

$$
\begin{aligned}
\operatorname{Area}(\Omega) =& 4\eta^2 \varphi\big(1 < x_0 + 2y_0,\ x_0 + y_0 < 1,\ 1 < 2x_0 + y_0\big) + 4\eta^2 \sum_{k=2}^{\min\left(\frac{x_0+y_0}{1-y_0}, \frac{x_0+y_0}{1-x_0}\right)} \frac{1}{k} \\
&+ 2\eta^2 \varphi\big(x_0 + y_0 = 1 \ \text{ or } \ 2x_0 + y_0 = 1, \text{ if } \ 0 < x_0 < 1/3 \\
&\qquad\qquad \text{or } \ x_0 + 2y_0 = 1, \text{ if } \ 1/3 < x_0 < 1\big) \\
&+ \frac{2\eta^2}{k} \varphi\big(k = \tfrac{x_0+y_0}{1-y_0} \geq 2, \ \text{ if } \ \tfrac{k}{k+2} < x_0 < 1, \\
&\qquad\qquad \text{or } k = \tfrac{x_0+y_0}{1-x_0} \geq 2, \ \text{ if } \ \tfrac{k-1}{k+1} < x_0 < \tfrac{k}{k+2}\big) \\
&+ \frac{\eta^2(k+2)}{k(k+1)} \varphi\big(x_0 = y_0 = \tfrac{k}{k+2},\ k \geq 1\big) .
\end{aligned}
\tag{17}
$$

Next we proceed as in the last part of Section 3.1 with the expression of $\#\mathcal{B}_Q$, obtained by replacing (17) in (8), instead of $\#\mathcal{A}_Q$. This produces the formula for the density corresponding to the points of `Type II` :

$$
\begin{aligned}
g_2(u, v) =& \varphi\big(1 < u + 2v,\ u + v < 1,\ 1 < 2u + v\big) + \sum_{k=2}^{\min\left(\frac{u+v}{1-v}, \frac{u+v}{1-u}\right)} \frac{1}{k} \\
&+ \frac{1}{2} \varphi\big(u + v = 1 \ \text{ or } \ 2u + v = 1, \text{ if } \ 0 < u < 1/3 \\
&\qquad\qquad \text{or } \ u + 2v = 1, \text{ if } \ 1/3 < u < 1\big) \\
&+ \frac{1}{2k} \varphi\big(k = \tfrac{u+v}{1-v} \geq 2, \ \text{ if } \ \tfrac{k}{k+2} < u < 1, \\
&\qquad\qquad \text{or } k = \tfrac{u+v}{1-u} \geq 2, \ \text{ if } \ \tfrac{k-1}{k+1} < u < \tfrac{k}{k+2}\big) \\
&+ \frac{k+2}{4k(k+1)} \varphi\big(u = v = \tfrac{k}{k+2},\ k \geq 1\big) .
\end{aligned}
\tag{18}
$$

Now the theorem follows by replacing the expression of $g_1(u, v)$ and $g_2(u, v)$ from (7) and (18) into (6).

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