A COMBINATORIAL PROOF OF A RESULT FROM NUMBER THEORY

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Abstract

Let $r_k(n)$ denote the number of representations of n as a sum of k squares and $t_k(n)$ the number of representations of n as a sum of k triangular numbers. We give an elementary, combinatorial proof of the relations

$$r_k(8n+k) = c_k t_k(n), \quad 1 \le k \le 7,$$

where $c_1 = 2$, $c_2 = 4$, $c_3 = 8$, $c_4 = 24$, $c_5 = 112$, $c_6 = 544$ and $c_7 = 2368$.

1. Introduction

Let $r_k(n)$ denote the number of solutions in integers of the equation

$$x_1^2 + x_2^2 + \cdots + x_k^2 = n,$$

and let $t_k(n)$ denote the number of solutions in non-negative integers of the equation

$$\frac{x_1(x_1+1)}{2} + \frac{x_2(x_2+1)}{2} + \cdots + \frac{x_k(x_k+1)}{2} = n.$$

For example,

$$\begin{aligned} 9 &= (\pm3)^2 + 0^2 + 0^2 = 0^2 + (\pm3)^2 + 0^2 = 0^2 + 0^2 + (\pm3)^2 \\ &= (\pm2)^2 + (\pm2)^2 + (\pm1)^2 = (\pm2)^2 + (\pm1)^2 + (\pm2)^2 = (\pm1)^2 + (\pm2)^2 + (\pm2)^2, \end{aligned}$$

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and so $r_3(9) = 30$. On the other hand, $r_3(7) = 0$. Also, $t_3(10) = 9$, because the solutions of $\frac{x_1(x_1+1)}{2} + \frac{x_2(x_2+1)}{2} + \frac{x_3(x_3+1)}{2} = 10$ in non-negative integers are $(x_1, x_2, x_3) = (4, 0, 0)$ (three possible permutations), and $(3, 2, 1)$ (six possible permutations), giving a total of nine solutions.

Geometrically, $r_k(n)$ counts the number of points with integer coordinates on the k-dimensional sphere $x_1^2 + x_2^2 + \cdots + x_k^2 = n$. Similarly, $2^k t_k(n)$ counts the number of points with integer coordinates on the k-dimensional sphere $(x_1+\frac{1}{2})^2+(x_2+\frac{1}{2})^2+\cdots+(x_k+\frac{1}{2})^2 = 2n + \frac{k}{4}$.

A great deal is known about $r_k(n)$ and $t_k(n)$. For example, generating functions which yield explicit formulas for $r_k(n)$ and $t_k(n)$ for $k = 2, 4, 6$ and 8 in terms of the divisors of n, were given by Jacobi [7, pp. 159–170]. On the other hand, explicit formulas for odd values of k are much more complicated. For both even and odd values of $k \geq 9$, explicit formulas become even more complicated. For more information, see [4], [5, Chs. 6–9], [6, Ch. 20] and [8].

In [1], a remarkable connection between $t_k(n)$ and $r_k(8n + k)$ for $1 \leq k \leq 7$ was observed. These relations were independently rediscovered in [3].

Theorem [1, Lemma 2.7], [3].
For any non-negative integer n,

$$r_k(8n+k) = 2^k \left(1 + \frac{k(k-1)(k-2)(k-3)}{48}\right) t_k(n), \quad 1 \leq k \leq 7.$$

□

Thus for $1 \leq k \leq 7$, in order to study the sequence $\{t_k(n)\}_{n\geq 0}$, it suffices to study the subsequence $\{r_k(8n+k)\}_{n\geq 0}$ of $\{r_k(n)\}_{n\geq 0}$.

The proof in [1] relies on Jacobi's explicit formula for $r_4(n)$ in terms of divisors of n. The proof in [3] uses generating functions, and depends on properties of theta functions. The purpose of this article is to give an elementary, combinatorial proof of this theorem.

2. Proofs

Lemma. Let

$$\begin{aligned} A_n &= \big\{(i,j,k,l) \in \mathbb{Z}^4 : i+j+k+l \equiv 0 \pmod 2, \\ &\qquad (2i+1)^2 + (2j+1)^2 + (2k+1)^2 + (2l+1)^2 = 8n+4\big\}, \end{aligned}$$

$$\begin{aligned} B_n &= \big\{(i,j,k,l) \in \mathbb{Z}^4 : i+j+k+l \equiv 1 \pmod 2, \\ &\qquad (2i+1)^2+(2j+1)^2+(2k+1)^2+(2l+1)^2 = 8n+4\big\}, \\ C_n &= \big\{(i,j,k,l) \in \mathbb{Z}^4 : (2i)^2+(2j)^2+(2k)^2+(2l)^2 = 8n+4\big\}. \end{aligned}$$

Then the sets A_n, B_n and C_n are equinumerous. Note that for the set C_n, the condition $i+j+k+l \equiv 1 \pmod 2$ also holds.

Proof. Define $f : A_n \to B_n$ by

$$f(i,j,k,l) = (i,j,k,-l-1).$$

Then f is readily seen to be a bijection, and so A_n and B_n are equinumerous. Similarly, define $g : B_n \to C_n$ by

$$g(i,j,k,l) = \frac{1}{2}(i+j+k-l+1, i+j-k+l+1, i-j+k+l+1, -i+j+k+l+1).$$

Then it may be easily verified that

$$g^{-1}(i,j,k,l) = \frac{1}{2}(i+j+k-l-1, i+j-k+l-1, i-j+k+l-1, -i+j+k+l-1),$$

and g is a bijection. Thus B_n and C_n are equinumerous. □

Corollary. The number of representations of $8n+4$ as a sum of four odd squares equals twice the number of representations of $8n+4$ as a sum of four even squares.

Proof of the Theorem. We will show that each representation of n as a sum of k triangular numbers gives rise to $2^k\left(1+\frac{k(k-1)(k-2)(k-3)}{48}\right)$ representations of $8n+k$ as a sum of k squares, and that every representation of $8n+k$ as a sum of k squares arises once and only once in this way.

Suppose

$$n = \frac{x_1(x_1+1)}{2} + \cdots + \frac{x_k(x_k+1)}{2} \tag{1}$$

is a representation of n as a sum of k triangular numbers. Then multiplying by 8 and completing the square gives

$$8n+k = (\pm(2x_1+1))^2 + \cdots + (\pm(2x_k+1))^2. \tag{2}$$

This gives rise to 2^k representations of $8n+k$ as a sum of k odd squares, because there are 2^k possibilities for the signs. Conversely, each of the 2^k representations in (2) arises only from the corresponding representation (1).

If $1 \le k \le 3$, then the only way $8n+k$ may be expressed as a sum of k squares is if all the squares are odd, and so we have $r_k(8n+k) = 2^k t_k(n)$ in this case.

If $4 \le k \le 7$ and $8n+k$ is a sum of k squares, then parity considerations show that either all k squares are odd, or $k-4$ are odd and 4 are even. In the first case, equation

(2) gives 2^k representations of $8n+k$ as a sum of k odd squares for each instance of (1), and this accounts for all representations of $8n+k$ as a sum of k odd squares. In the latter case, there are $\binom{k}{4}$ orderings of $x_1, \cdots, x_k$ by parity, in which four of the squares are even and the others odd. Consider the equation

$$x_1^2 + \cdots + x_k^2 = 8n + k \tag{3}$$

where $x_1,\ x_2,\ x_3$ and x_4 are even and the other x_is are odd. The number of such representations is half the number of representations of $8n+k$ as a sum of k odd squares. To see this, rewrite (3) in the form

$$x_1^2 + \cdots + x_4^2 = 8n + k - \sum_{j=5}^{k} x_j^2,$$

and apply the corollary. It follows that the number of representations of $8n+k$ as a sum of k squares, 4 of which are even, arising from the single representation (1) is $\frac{1}{2}\binom{k}{4}2^k$.

Combining the two cases we complete the proof of the Theorem. □

Remark. It is clear from this proof of the Theorem that extra complications will arise if $k \geq 8$. In fact, using modular forms it was shown in [2] that for each value of $k \geq 8$, $r_k(8n+k)/t_k(n)$ is not a constant function of n. Therefore the Theorem does not hold if $k \geq 8$.

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