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DUCCI SEQUENCES IN HIGHER DIMENSIONS

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Abstract

A Ducci sequence is a sequence of n-tuples of integers obtained by iterating the map $(a_1, \dots, a_n) \mapsto (|a_1 - a_2|, |a_2 - a_3|, \dots, |a_n - a_1|)$. In this paper, we generalize the concept of Ducci sequences to sequences of d-dimensional arrays, extend some of the basic results on Ducci sequences to this case, and point out some new phenomena that occur. Our main tool is the description of Ducci sequences modulo two in terms of polynomials over the field $\mathbb{F}_2$.

1. Introduction

Ducci sequences have been studied for more than a century, and have recently enjoyed a resurgence of interest; see for example [2, 6, 7, 9, 11, 12, 16, 20]. The idea is to start with an n-tuple of numbers (in this paper we will stick to integers) $(a_1, a_2, \dots, a_n)$ and to form the n-tuple of cyclic differences $D(a_1, \dots, a_n) = (|a_1 - a_2|, |a_2 - a_3|, \dots, |a_{n-1} - a_n|, |a_n - a_1|)$. One then repeats the procedure, obtaining a sequence of n-tuples called a *Ducci sequence* in honor of E. Ducci who first studied them (see [15]).

A Ducci sequence is said to *vanish* if it stabilizes at $(0, 0, \dots, 0)$. We have the following main result, which has been rediscovered many times:

Theorem 1 *Every Ducci sequence of n-tuples of integers vanishes if and only if n is a power of two.*

Proof. The usual approach is to first notice that, since the entries of the tuples are bounded, every Ducci sequence eventually becomes periodic. Then one shows (e.g., [8] or [19]) that every tuple in the cyclic part must be a constant multiple of a *binary* tuple - a tuple whose entries lie in $\{0, 1\}$. Then we have, for binary tuples,

$$(|a_1 - a_2|, |a_2 - a_3|, \dots, |a_n - a_1|) \equiv (a_1 + a_2, a_2 + a_3, \dots, a_n + a_1) \bmod 2.$$

This means that the Ducci operator D is linear modulo two, and Theorem 1 is now easy (and fun) to prove for binary tuples; see for example Theorem 5 below. $\square$

Many other aspects of Ducci sequences have also been studied, such as the number of iterations required to reach the cyclic part (this is known as the "length of the n-number game;" see, e.g., [18]), the periods of the cyclic part (e.g., [6], [9], [13], [20] and [21]), and the relationship with cellular automata (e.g., [1], [16] and [20]). Numerous generalizations of Ducci sequences have also been studied (e.g., [5], [11] and [22]), and the purpose of this paper is to introduce yet another generalization, this time to higher dimensions.

2. Ducci Sequences Over $\mathbb{F}_2$ - A Survey

In this section, we briefly survey some known results on binary Ducci sequences, before turning to higher dimensions in the following sections. We denote by $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ the field of two elements.

Definition 2 *A Ducci sequence over $\mathbb{F}_2$ is a sequence $\underline{u}_1, \underline{u}_2, \ldots \in \mathbb{F}_2^n$ of n-tuples given by $\underline{u}_{i+1} = D\underline{u}_i$, where*

$$D(a_1, a_2, \ldots, a_n) := (a_1 + a_2, a_2 + a_3, \ldots, a_{n-1} + a_n, a_n + a_1).$$

Typical questions concern vanishing, or the period of a tuple.

Definition 3 *Let $\underline{u} \in \mathbb{F}_2^n$. We say that $\underline{u}$* vanishes *if $D^k\underline{u} = (0, 0, \ldots, 0)$ for some $k \in \mathbb{N}$. The* period *of $\underline{u}$ is the least $l \in \mathbb{N}$ such that $D^{k+l}\underline{u} = D^k\underline{u}$ for all sufficiently large k, and the smallest such k is called the* length *of $\underline{u}$. The* basic Ducci sequence *is the Ducci sequence starting with $\underline{a}_0 := (0, 0, \ldots, 0, 1)$, and its period is denoted by $P(n)$. The tuple $\underline{u}$ is said to be* in a cycle *if $D^k\underline{u} = \underline{u}$ for some $k \in \mathbb{N}$.*

Notice that the period of every tuple must divide $P(n)$, since D is linear and commutes with cyclic permutations, and every tuple is a linear combination of cyclic permutations of $\underline{a}_0$. Before we can state a theorem summarizing some of the main results concerning Ducci sequences over $\mathbb{F}_2$, we make one more definition.

Definition 4 *Let R be a ring, and $x \in R$. Then the* multiplicative order *of x in R, denoted $\mathrm{Ord}_R(x)$, is defined to be the eventual period of the sequence $x, x^2, x^3, \ldots$, if it exists. In particular, if $x \in R^\times$ then $\mathrm{Ord}_R(x)$ is the order of x in the group $R^\times$.*

Theorem 5 *Let $n = 2^k m$, where m is odd.*

1. *Every $\underline{u} \in \mathbb{F}_2^n$ vanishes if and only if $n = 2^k$.*
2. *The length of the basic Ducci sequence is 2^k, and the first tuple in the basic Ducci sequence that is in a cycle is $(0, \dots, 0, 1, \underbrace{0, \dots, 0}_{2^k - 1 \text{ zeros}}, 1)$.*
3. *$\underline{u} = (a_1, \dots, a_n)$ is in a cycle if and only if*
$$\sum_{j=0}^{m-1} a_{i+j2^k} = 0 \quad \text{for all } i = 1, \dots, 2^k.$$
4. *$P(n)$ divides $2^k(2^t - 1)$, where $t = \mathrm{Ord}_{\mathbb{Z}/m\mathbb{Z}}(2)$.*
5. *If $2^r \equiv -1 \bmod m$ for some $r \in \mathbb{Z}$, then $P(n)$ divides $n(2^r - 1)$.*

The above result is far from exhaustive, in particular, much more is known about the function $P(n)$, see for example [13] and [14]. Some of these results have been rediscovered many times. As far as the author can tell, (1) is first proved in [10], (2) appears in [14] for the special case $n = 2^r \pm 2^s$ and in [4] for general n, (3) is proved in [17], while (4) and (5) appear in [13].

A particularly useful technique for studying Ducci sequences is due to Peter Zwengrowski [23], and we will follow his approach throughout this paper. Consider the isomorphism of $\mathbb{F}_2$-vector spaces:

$$\begin{array}{rcl} \mathbb{F}_2^n & \xrightarrow{\sim} & R = \dfrac{\mathbb{F}_2[x]}{\langle x^n + 1 \rangle} \\ \underline{u} = (a_1, \dots, a_n) & \longmapsto & f_u(x) \bmod (x^n + 1), \end{array} \tag{1}$$

where $f_u(x) = a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$. Zvengrowski's idea was to exploit the ring structure of R: notice that applying the Ducci operator D corresponds to multiplication by $(x + 1)$ in R:

$$f_{Du}(x) \equiv (x + 1) f_u(x) \bmod (x^n + 1).$$

Now all questions about the Ducci sequence $\underline{u}, D\underline{u}, D^2\underline{u}, \ldots \in \mathbb{F}_2^n$ may be reformulated as questions about the sequence of polynomials $f_u(x), (x + 1)f_u(x), (x + 1)^2 f_u(x), \ldots \in R$.

Proof of Theorem 5 Let $t = \mathrm{Ord}_{\mathbb{Z}/m\mathbb{Z}}(2)$, then $(x + 1)^{2^{k+t}} = (x^{2^{k+t}} + 1) = (x^{2^k} + 1) = (x + 1)^{2^k}$ in R, and this is zero if and only if $n = 2^k$. This proves (1) and (4). It now follows that $\underline{u}$ is in a cycle if and only if $f_u(x)$ is divisible (in R) by $(x^{2^k} + 1)$, which is seen to be equivalent to (3), and this in turn completes the proof of (2). Lastly, suppose

$2^r \equiv -1 \bmod m$. Then $(x+1)^{n2^r} = (x^{2^{r+k}}+1)^m = (x^{-2^k}+1)^m = x^{-n}(1+x^{2^k})^m = (1+x)^n$, and (5) follows. $\square$

There are various expressions for the period of a given $\underline{u}$, of which we give a particularly pleasing one below. For simplicity, we will now assume that n is odd, the general case being found in [6]. Similar results are also found in [9], and in fact the first such result was attributed to D. Richman in [17], but apparently never published.

We denote by $\mu_n(\overline{\mathbb{F}}_2)$ the group of nth roots of unity in the algebraic closure of $\mathbb{F}_2$.

Theorem 6 *Suppose n is odd, and let $\underline{u} \in \mathbb{F}_2^n$. Then the period of $\underline{u}$ is given by*

$$\mathrm{Per}(\underline{u}) = \mathrm{lcm}\{\mathrm{Ord}_{\overline{\mathbb{F}}_2}(\zeta+1) \mid \zeta \in \mu_n(\overline{\mathbb{F}}_2), f_u(\zeta) \neq 0\}.$$

Proof. We start by factorizing x^n+1 into distinct irreducible factors in $\mathbb{F}_2[x]$:

$$x^n+1 = \prod_{i=1}^{N} \varphi_i(x).$$

Now, by the Chinese Remainder Theorem, we have

$$R = \frac{\mathbb{F}_2[x]}{\langle x^n+1\rangle} \cong \prod_{i=1}^{N} \frac{\mathbb{F}_2[x]}{\langle \varphi_i(x)\rangle} = \prod_{i=1}^{N} F_i,$$

where each $F_i = \frac{\mathbb{F}_2[x]}{\langle \varphi_i(x)\rangle}$ is a finite field, in fact it is the extension of $\mathbb{F}_2$ obtained by adjoining a root $\zeta_i \in \mu(\overline{\mathbb{F}}_2)$ of $\varphi_i(x)$, with the isomorphism $F_i \xrightarrow{\sim} \mathbb{F}_2(\zeta_i)$ induced by $x \mapsto \zeta_i$.

Now $\mathrm{Per}(\underline{u})$ is the eventual period of the sequence

$$f_u(x),\ (x+1)f_u(x),\ (x+1)^2 f_u(x),\ \ldots \quad \text{in } R,$$

which in turn is the lowest common multiple over all i of the eventual periods of

$$f_u(\zeta_i),\ (\zeta_i+1)f_u(\zeta_i),\ (\zeta_i+1)^2 f_u(\zeta_i),\ \ldots \quad \text{in } F_i.$$

If $f_u(\zeta_i) = 0$, then this period is 1, otherwise the period equals $\mathrm{Ord}_{F_i}(\zeta_i+1)$, since $f_u(\zeta_i)$ is a unit. Lastly, it is clear that one may take the lowest common multiple over all nth roots of unity ζ, and not just the chosen ζ_i's, since all the roots of $\varphi_i(x)$ are conjugate over $\mathbb{F}_2$. The theorem follows. $\square$

3. On to Higher Dimensions

Definition 7 *Let R be a ring, and $n_1, n_2, \ldots, n_d$ natural numbers. Denote by $M_{n_1\times\cdots\times n_d}(R)$ the R-module of $n_1 \times \cdots \times n_d$ arrays with entries in R. Elements of $M_{n_1\times\cdots\times n_d}(R)$ are*

$$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix} \xrightarrow{D} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&1\\0&0&1&1 \end{bmatrix} \xrightarrow{D} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&1\\0&0&0&0\\0&1&0&1 \end{bmatrix} \xrightarrow{D} \begin{bmatrix} 0&0&0&0\\1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{bmatrix} \xrightarrow{D} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

Figure 1: The basic 5×4 Ducci sequence vanishes.

denoted by $(a_{i_1,\dots,i_d})$*, where each index* i_j *ranges over the residue classes modulo* n_j*, for* $j = 1, 2, \dots, d$*. For ease of notation, we choose* $\{1, 2, \dots, n_j\}$ *as a set of representatives for these residue classes.*

We now introduce Ducci sequences of dimension $d \geq 1$, and we start with those over $\mathbb{F}_2$.

Definition 8 *An* $n_1 \times \cdots \times n_d$ Ducci sequence *over* $\mathbb{F}_2$*, also called a* D-sequence*, is a sequence* $U_1, U_2, \ldots \in M_{n_1 \times \cdots \times n_d}(\mathbb{F}_2)$ *defined by the recurrence relation* $U_{i+1} = D(U_i)$*, where the* Ducci operator D *is defined by*

$$D(a_{i_1,\dots,i_d}) = \left(\sum_{j_1,\dots,j_d \in \{0,1\}} a_{i_1+j_1, i_2+j_2, \dots, i_d+j_d} \right).$$

A basic *Ducci sequence is one starting with* A_0*, which has* 1 *in its* $(n_1, n_2, \dots, n_d)$*th entry, and zeros everywhere else.*

For example, $n_1 \times n_2$ Ducci sequences are sequences of matrices. This is illustrated in Figure 1, which shows the first few elements (matrices) of the basic 5×4 Ducci sequence, which vanishes. In Figure 2 we show the first 31 elements of the basic 32×32 Ducci sequence stacked beneath each other, where each cube represents a one and a gap represents a zero. The combined effect gives us a discrete approximation to the Sierpinski pyramid.

The cycle formed by the basic $7 \times 7 \times 7$ Ducci sequence is shown in Figure 3.

As before, every D-sequence eventually forms a cycle, and we may ask the same questions about which elements are in a cycle, what the period might be, and which Ducci sequences vanish (i.e., stabilize at the zero element). The period of the basic $n_1 \times \cdots \times n_d$ Ducci sequence over $\mathbb{F}_2$ is denoted by $P(n_1, \dots, n_d)$. Before we can state our generalization of Theorem 5, we need one more definition.

Definition 9 *Let* $U = (a_{i_1,\dots,i_d}) \in M_{n_1 \times \cdots \times n_d}(\mathbb{F}_2)$ *and* $1 \leq j \leq d$*. Then a* j-row *of* U *is the* n_j*-vector consisting of those entries* $a_{i_1,\dots,i_d}$ *all of whose coordinates are fixed at some constant values, except for the* j*th coordinate, which ranges from* 1 *to* n_j *to make up the vector.*

Figure 2: The basic 32×32 Ducci sequence

Theorem 10 *Let $n_1, \ldots, n_d$ be natural numbers, and write $n_i = 2^{k_i} m_i$ with m_i odd, for $i = 1, \ldots, d$. Let $k = \max\{k_1, \ldots, k_d\}$ and $t = \operatorname{lcm}_i\{\operatorname{Ord}_{\mathbb{Z}/m_i\mathbb{Z}}(2)\}$. Then*

1. *Every $U \in M_{n_1 \times \cdots \times n_d}(\mathbb{F}_2)$ vanishes if and only if at least one of the n_i's is a power of two.*

2. *$U \in M_{n_1 \times \cdots \times n_d}(\mathbb{F}_2)$ is in a cycle if and only if every j-row $(b_1, \ldots, b_{n_j})$ of U, for every $j = 1, \ldots, d$, satisfies*

$$\sum_{l=0}^{m_j-1} b_{i+l2^{k_j}} = 0 \quad \text{for all } i = 1, \ldots, 2^{k_j}.$$

3. *The first element of the basic $n_1 \times \cdots \times n_d$ Ducci sequence which is in a cycle is $D^{2^k}(A_0)$.*

4. *$P(n_1, \ldots, n_d)$ divides $2^k(2^t - 1)$.*

5. *$P(n_1, \ldots, n_d) = \operatorname{lcm}\{P(n_1), \ldots, P(n_d)\}$.*

As in the one-dimensional case, we consider the isomorphism of $\mathbb{F}_2$-vector spaces

$$\begin{array}{rcl} M_{n_1 \times \cdots \times n_d}(\mathbb{F}_2) & \xrightarrow{\sim} & R := \dfrac{\mathbb{F}_2[x_1, \ldots, x_d]}{\langle x_1^{n_1} + 1, \ldots, x_d^{n_d} + 1 \rangle} \\ U = (u_{i_1, \ldots, i_d}) & \longmapsto & f_U(x_1, \ldots, x_d) \bmod \langle x_1^{n_1} + 1, \ldots, x_d^{n_d} + 1 \rangle, \end{array} \tag{2}$$

where

$$f_U(x_1, \ldots, x_d) = \sum_{i_1, \ldots, i_d} u_{i_1, \ldots, i_d} x_1^{n_1 - i_1} \cdots x_d^{n_d - i_d} \in \mathbb{F}_2[x_1, \ldots, x_d].$$

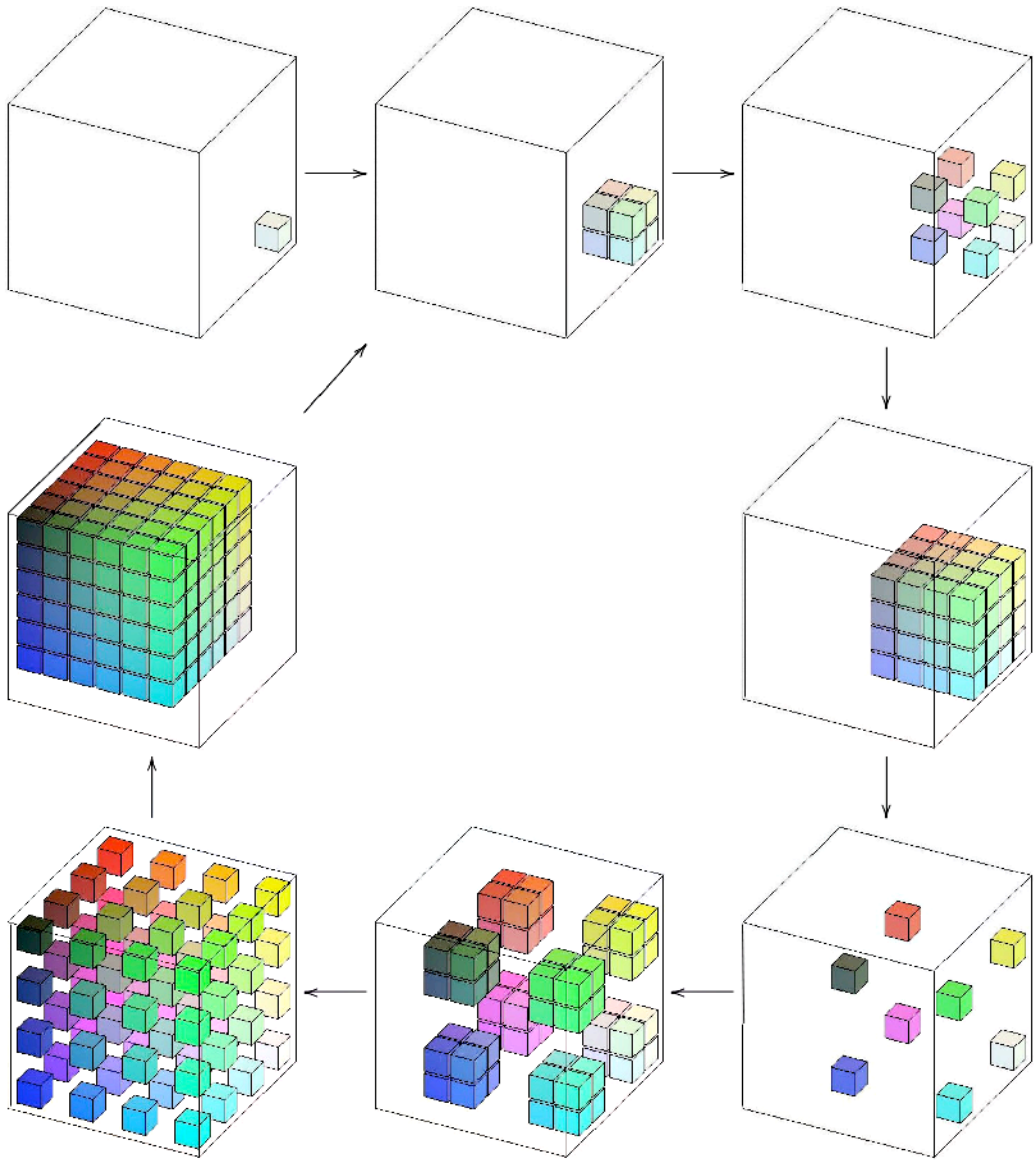

Figure 3: The basic $7 \times 7 \times 7$ Ducci sequence

We see that under this isomorphism, the Ducci operator D corresponds to multiplication by $(x_1+1)\cdots(x_d+1)$ in the ring R.

Proof of Theorem 10 We first point out that k and t as defined in the Theorem are the least integers $k \geq 0$, $t \geq 1$ satisfying $2^{k+t} \equiv 2^k \bmod n_i$ for all $i = 1\ldots, d$, and hence satisfying

$$\big((x_1+1)\cdots(x_d+1)\big)^{2^{k+t}} = \big((x_1+1)\cdots(x_d+1)\big)^{2^k} \quad \text{in } R.$$

(3) and (4) now follow immediately, as does (1) once we point out that $\big((x_1+1)\cdots(x_d+1)\big)^{2^k}$ is non-zero in R if and only if none of the n_i's are powers of 2. It also follows that U is in a cycle if and only if $f_U(x_1,\ldots,x_d)$ is divisible by $\big((x_1+1)\cdots(x_d+1)\big)^{2^k}$, from which (2) may be deduced by examining what this means for the coefficients of f_U. Lastly, (5) follows from the fact that the multiplicative order of (x_i+1) in R equals its multiplicative order in $\frac{\mathbb{F}_2[x_i]}{\langle x^{n_i}+1\rangle}$. □

Next, we concern ourselves with computing periods of Ducci sequences. We have the following generalization of Theorem 6.

Theorem 11 *Suppose that $n_1,\ldots,n_d$ are all odd. Then the period of $U \in M_{n_1\times\cdots\times n_d}(\mathbb{F}_2)$ is given by*

$$\mathrm{Per}(U) = \mathrm{lcm}\{\mathrm{Ord}_{\overline{\mathbb{F}}_2}\big((\zeta_1+1)\cdots(\zeta_d+1)\big) \mid \zeta_i \in \mu_{n_i}(\overline{\mathbb{F}}_2),\ i=1,\ldots,d,\ f_U(\zeta_1,\ldots,\zeta_d) \neq 0\}.$$

Proof. For ease of notation, we will assume that $d = 2$, and it will be clear how to extend the proof to arbitrary dimension d. As in Theorem 6, we wish to factorize our ring R into a product of finite fields.

$$\begin{aligned} R = \frac{\mathbb{F}_2[x_1,x_2]}{\langle x_1^{n_1}+1, x_2^{n_2}+1\rangle} &\cong \frac{\frac{\mathbb{F}_2[x_1]}{\langle x_1^{n_1}+1\rangle}[x_2]}{\langle x_2^{n_2}+1\rangle} \\ &\cong \prod_{i=1}^{N} \frac{F_i[x_2]}{\langle x_2^{n_2}+1\rangle}, \end{aligned}$$

where $F_i = \frac{\mathbb{F}_2[x_1]}{\langle \varphi_i(x_1)\rangle} \cong \mathbb{F}_2(\zeta_i)$ as in the proof of Theorem 6. This is a finite field, so we may similarly factor each $\frac{F_i[x_2]}{\langle x_2^{n_2}+1\rangle}$ into a product of finite fields F_{ij} to obtain

$$R \cong \prod_{i=1}^{N}\prod_{j=1}^{M_i} F_{ij}, \qquad \text{where} \quad F_{ij} \cong \mathbb{F}_2(\zeta_i,\zeta_j),$$

and now $\zeta_i \in \mu_{n_1}(\overline{\mathbb{F}}_2)$ and $\zeta_j \in \mu_{n_2}(\overline{\mathbb{F}}_2)$. The rest of the proof is now identical to that of Theorem 6. □

4. Preimages of D

Under the isomorphism (2) the image and kernel of the Ducci operator are given by

$$\mathrm{Im}(D) \cong \frac{(x_1+1)(x_2+1)\cdots(x_d+1)\mathbb{F}_2[x_1,\ldots,x_d]}{\langle x_1^{n_1}+1,\ldots,x_d^{n_d}+1\rangle}$$

and

$$\ker(D) \cong \frac{\langle x_1^{n_1-1}+x_1^{n_1-2}+\cdots+1,\ldots,x_d^{n_d-1}+x_d^{n_d-2}+\cdots+1\rangle\mathbb{F}_2[x_1,\ldots,x_d]}{\langle x_1^{n_1}+1,\ldots,x_d^{n_d}+1\rangle}.$$

Intuitively, this means that the array U is in the image of D if every j-row, for every $j=1,\ldots,d$, contains an even number of ones. In this case, U is uniquely determined by any of its $(n_1-1)\times(n_2-1)\times\cdots\times(n_d-1)$-subarrays. Thus we have

$$\begin{aligned}\dim_{\mathbb{F}_2}\mathrm{Im}(D) &= (n_1-1)(n_2-1)\cdots(n_d-1)\\ \dim_{\mathbb{F}_2}\ker(D) &= n_1n_2\cdots n_d-(n_1-1)(n_2-1)\cdots(n_d-1).\end{aligned}$$

Definition 12 *A* row flip *on U is the action of inverting all the elements in one j-row of U. In terms of polynomials, this means adding $x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}(x_j^{n_j-1}+x_j^{n_j-2}+\cdots+x_j+1)$ to the polynomial representing U.*

We see that if U has a preimage under U, then it actually has $2^{n_1n_2\cdots n_d-(n_1-1)(n_2-1)\cdots(n_d-1)}$ preimages, and each preimage can be obtained from any other by performing a sequence of row flips upon it. Finding one preimage of U is straightforward, but finding one which has a minimal number of ones is much harder, as the following result shows.

Theorem 13 *The following decision problem is NP-complete: Given $U\in M_{n_1\times\cdots\times n_d}(\mathbb{F}_2)$ and $M\geq 0$, does U have a preimage under D containing at most M ones?*

Proof. We start by assuming $U\in\mathrm{Im}(D)$, otherwise there is nothing to prove. From the above discussion follows that our decision problem is equivalent to the following problem:

Row flipping(d): Given $U\in M_{n_1\times\cdots\times n_d}(\mathbb{F}_2)$ and $M\geq 0$, does there exist a sequence of row flips reducing U to an array containing at most M ones?

It is shown in [3] that *Row flipping(2)* is NP-complete. (In fact, there is an interesting connection between $n_1\times n_2$ Ducci sequences over $\mathbb{F}_2$ and the game of "Squares" (see [3]): The Ducci operator D gives precisely the transition from the *solution board* to the *game board*).

It is now easy to show that a polynomial-time solution to *Row flipping(d)* would also yield a polynomial-time solution to *Row flipping(d-1)*, for $d \geq 3$. Indeed, suppose given $U \in M_{n_1 \times \cdots \times n_{d-1}}(\mathbb{F}_2)$ and $0 \leq M \leq n_1 n_2 \cdots n_{d-1}$, we let $n_d := n_1 n_2 \ldots n_{d-1} + 2$, and construct $W \in M_{n_1 \times \cdots \times n_d}(\mathbb{F}_2)$, which contains U as the subarray with dth coordinate equal to 1, and all whose other entries are zero. Then any sequence of row flips reducing the number of ones in W to at most M cannot include any d-row flips (this requires a few moments' thought), hence this same sequence of row flips will reduce U to at most M ones. Thus *Row flipping(d-1)* reduces to *Row flipping(d)*, and the result follows by induction on d. $\square$

5. Ducci Sequences Over $\mathbb{Z}$

Now that we have made some progress on $n_1 \times \cdots \times n_d$ Ducci sequences over $\mathbb{F}_2$, what about Ducci sequences in $M_{n_1 \times \cdots \times n_d}(\mathbb{Z})$ defined somehow in terms of absolute differences? One may propose several definitions for the Ducci operator $\mathcal{D}$, and it is not clear which will lead to the richest theory. Let's restrict ourselves, for now, to the following class of operators:

Definition 14 *A map $\mathcal{D} : M_{n_1 \times \cdots \times n_d}(\mathbb{Z}) \to M_{n_1 \times \cdots \times n_d}(\mathbb{Z})$ is called a* Ducci operator *if it satisfies $D(aU) = aD(U)$ for every scalar $a \in \mathbb{Z}$, and if it reduces modulo 2 to the usual Ducci operator D of Definition 8.*

The Ducci operator $\mathcal{D}$ is called bounded *if for every $U \in M_{n_1 \times \cdots \times n_d}(\mathbb{Z})$ there exists some $M_U > 0$ such that the entries of $\mathcal{D}^k(U)$ are bounded by M_U for all k.*

An $n_1 \times \cdots \times n_d$ Ducci sequence over $\mathbb{Z}$, also called a $\mathcal{D}$-sequence, is a sequence $U_1, U_2, \ldots \in M_{n_1 \times \cdots \times n_d}(\mathbb{Z})$ given by $U_{i+1} = \mathcal{D}(U_i)$, where $\mathcal{D}$ is a Ducci operator.

Then we have the following result.

Theorem 15 *Suppose that $\mathcal{D} : M_{n_1 \times \cdots \times n_d}(\mathbb{Z}) \to M_{n_1 \times \cdots \times n_d}(\mathbb{Z})$ is a bounded Ducci operator. Then every $\mathcal{D}$-sequence vanishes if and only if at least one of the n_i's is a power of two.*

Proof. Suppose that some n_i is a power of two. Since $\mathcal{D}$ is bounded, every $\mathcal{D}$-sequence eventually forms a cycle. Suppose that $U \in M_{n_1 \times \cdots \times n_d}(\mathbb{Z})$ is in such a cycle, and is not the zero element. Since we may divide out by common factors, we may assume that at least one entry of U is odd. But $U \bmod 2$ must also be in a cycle of the corresponding Ducci sequence over $\mathbb{F}_2$, which vanishes by Theorem 10, hence $U \bmod 2$ is the zero element. This is a contradicton.

On the other hand, if none of the n_i's are powers of 2, then there are $\mathcal{D}$-sequences which do not vanish modulo 2, hence do not vanish. $\square$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{\mathcal{D}_1} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{\mathcal{D}_1} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\mathcal{D}_1} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{\mathcal{D}_1} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{\mathcal{D}_1} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Figure 4: A 3×3 $\mathcal{D}_1$-sequence which becomes cyclic with period 3

$$\begin{bmatrix} 2 & 3 & 5 \\ 5 & 2 & 3 \\ 3 & 5 & 2 \end{bmatrix} \xrightarrow{\mathcal{D}_2} \begin{bmatrix} 4 & 1 & 5 \\ 5 & 4 & 1 \\ 1 & 5 & 4 \end{bmatrix} \xrightarrow{\mathcal{D}_2} \begin{bmatrix} 2 & 7 & 5 \\ 5 & 2 & 7 \\ 7 & 5 & 2 \end{bmatrix} \xrightarrow{\mathcal{D}_2} \begin{bmatrix} 8 & 7 & 1 \\ 1 & 8 & 7 \\ 7 & 1 & 8 \end{bmatrix} \xrightarrow{\mathcal{D}_2} \begin{bmatrix} 8 & 5 & 13 \\ 13 & 8 & 5 \\ 5 & 13 & 8 \end{bmatrix} \xrightarrow{\mathcal{D}_2} \cdots$$

Figure 5: A divergent 3×3 $\mathcal{D}_2$-sequence

The above argument also gives an alternative proof to Theorem 1.

We give two examples of Ducci operators.

Example 1. The operator $\mathcal{D}_1 : M_{n_1 \times n_2}(\mathbb{Z}) \to M_{n_1 \times n_2}(\mathbb{Z})$ defined by

$$\mathcal{D}_1(a_{i,j}) = \big(\big||a_{i,j} - a_{i+1,j+1}| - |a_{i+1,j} - a_{i,j+1}|\big|\big)$$

is a bounded Ducci operator, but as Figure 4 shows, it is no longer true that the elements in a cycle are binary, as was the case in one dimension.

Example 2. The operator $\mathcal{D}_2 : M_{n_1 \times n_2}(\mathbb{Z}) \to M_{n_1 \times n_2}(\mathbb{Z})$ defined by

$$\mathcal{D}_2(a_{i,j}) = (|a_{i,j} + a_{i+1,j+1} - a_{i+1,j} - a_{i,j+1}|)$$

is also a Ducci operator. If $A = (a_{i,j}) \in M_{n \times n}(\mathbb{Z})$ is such that $a_{i,j} = a_{i+1,j+1}$ for all i and j, then $\mathcal{D}_2(A)$ has the same symmetry, and the sequence of n-tuples formed by the first rows of the matrices in the $\mathcal{D}_2$-sequence starting with A is what Marc Chamberland [11] calls a Ducci sequence with $(-1, \underline{2}, -1)$-weighting. In particular, such sequences can diverge (e.g., Figure 5), so $\mathcal{D}_2$ is not a bounded Ducci operator.

Interestingly, divergent $\mathcal{D}_2$-sequences only seem to occur for certain values of n_1 and n_2: It is easy to show that all $\mathcal{D}_2$-sequences vanish if $\min(n_1, n_2) \leq 2$, and it follows from [11] and the above argument that there exist diverging 3×3, 8×8 and 16×16 $\mathcal{D}_2$-sequences. Computations suggest that all $\mathcal{D}_2$-sequences vanish if n_1 or n_2 equals 4, and that some 3×5 and 5×5 $\mathcal{D}_2$-sequences diverge, whereas all 3×7 sequences seem to be bounded. What about other values of n_1 and n_2?

At this stage we can say very little about periods of Ducci sequences over $\mathbb{Z}$, other than the obvious fact that the period of some U is divisible by the period of (U mod 2). The

$n_1 \times n_2$	D	$\mathcal{D}_1$	$\mathcal{D}_2$
3×3	$1, 3$	$1, 3, 6$	$1, 3, \infty$
3×5	$1, 5, 15$	$1, 5, 10, 15, 30$	$1, 10, 15, 30, 120, 390, 660, 1920, (\infty)$
3×6	$1, 3, 6$	$1, 3, 6, 12$	$1, 2, 3, 6, 12, 36, 66, 72, 78, 114, 174, 282, \infty$
3×7	$1, 21$	$1, 21$	$1, 21, 42, 84, 147, 168, 210, 273, 420, 546, 588, 672$
5×5	$1, 15$	$1, 15, 30, 45$	$1, 60, 90, 3720, 16335, (\infty)$

Table 1: Some periods of D, $\mathcal{D}_1$ and $\mathcal{D}_2$-sequences. The entries for $\mathcal{D}_1$ and $\mathcal{D}_2$ are likely incomplete. The symbols ∞ and (∞) denote sequences known and suspected to diverge, respectively.

above two examples show a wider range of periods for given values of n_1 and n_2 than do Ducci sequences over $\mathbb{F}_2$. Table 1 gives a few results, intended not as an exhaustive list, but merely to demonstrate the wealth of periods of higher-dimensional Ducci sequences over $\mathbb{Z}$.

Clearly, much remains to be discovered here.

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