THE NUMBER OF REPRESENTATIONS BY SUMS OF SQUARES AND TRIANGULAR NUMBERS

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Abstract

In this paper, we present eighteen interesting infinite products and their Lambert series expansions. From these, we deduce formulae for the number of representations of an integer n by eighteen quadratic forms in terms of divisor sums.

–Dedicated to the memory of my grandmother Yuet Kwai Mah.

1. Introduction and Statement of Results

Let τ be a fixed complex number satisfying Im $(\tau) > 0$ and let $q = e^{i\pi\tau}$, so that $|q| < 1$. Let

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2},$$

and

$$\psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}.$$

The purpose of this paper is to study and give proofs of eighteen theorems in the area of the number of representations by sums of squares and triangular numbers. Most of these results appear to be new. The results in this paper can be found in the author's thesis [7]. Section 2 contains preliminary results and will be used as a basis for Section 3. In Section 3, we will prove the following results.

Theorem 1

$$\begin{aligned}
\varphi(q)\,\varphi\left(q^4\right) &= 1+2\sum_{j=1}^{\infty}\frac{(-1)^j\,q^{2j}}{1+q^{4j}}-2\sum_{j=1}^{\infty}\frac{(-1)^j\,q^{2j-1}}{1-q^{2j-1}}, &(1)\\
\varphi(q)\,\psi\left(q^8\right) &= -\sum_{j=1}^{\infty}\frac{(-1)^j\,q^{2j-2}}{1-q^{2j-1}}-\sum_{j=1}^{\infty}\frac{(-1)^j\,q^{2j-1}}{1+q^{4j}}, &(2)\\
\varphi\left(q^4\right)\psi\left(q^2\right) &= -\frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j\,q^{\frac{j-1}{2}}}{1-q^{\frac{2j-1}{2}}}-\frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j\,(-q)^{\frac{j-1}{2}}}{1-(-q)^{\frac{2j-1}{2}}}. &(3)
\end{aligned}$$

Theorem 2

$$\begin{aligned}
\varphi^3(q)\,\psi\left(q^8\right) &= 2\sum_{j=1}^{\infty}\frac{jq^{j-1}}{1+(-q)^j}-2\sum_{j=1}^{\infty}\frac{(-1)^j\,jq^{4j-1}}{1+q^{4j}}\\
&\quad+\sum_{j=1}^{\infty}\frac{(-1)^j\,(2j-1)\,q^{2j-2}}{1+q^{4j-2}}, &(4)\\
\varphi^2(q)\,\psi^2(q) &= \frac{1}{2}\sum_{j=1}^{\infty}\frac{(2j-1)\,q^{\frac{j-1}{2}}}{1-q^{\frac{2j-1}{2}}}-\frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j\,(2j-1)\,q^{\frac{j-1}{2}}}{1+q^{\frac{2j-1}{2}}}, &(5)\\
\varphi^2(q)\,\psi^2\left(q^4\right) &= \sum_{j=1}^{\infty}\frac{jq^{j-1}}{1+(-q)^j}-\sum_{j=1}^{\infty}\frac{(-1)^j\,jq^{2j-1}}{1+q^{2j}}, &(6)\\
\varphi^2(q)\,\psi^2\left(q^8\right) &= \frac{1}{2}\sum_{j=1}^{\infty}\frac{jq^{j-2}}{1+(-q)^j}-\sum_{j=1}^{\infty}\frac{(-1)^j\,jq^{4j-2}}{1+q^{4j}}\\
&\quad+\frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j\,(2j-1)\,q^{2j-3}}{1+q^{4j-2}}+\frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j\,jq^{2j-2}}{1+q^{2j}}, &(7)\\
\varphi(q)\,\psi^3\left(q^8\right) &= \frac{1}{8}\sum_{j=1}^{\infty}\frac{jq^{j-3}}{1+(-q)^j}+\frac{3}{8}\sum_{j=1}^{\infty}\frac{(-1)^j\,jq^{2j-3}}{1+q^{2j}}\\
&\quad-\frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j\,jq^{4j-3}}{1+q^{4j}}+\frac{1}{8}\sum_{j=1}^{\infty}\frac{(-1)^j\,(2j-1)\,q^{2j-4}}{1+q^{4j-2}}, &(8)\\
\varphi\left(q^4\right)\psi^3\left(q^2\right) &= \frac{1}{16}\sum_{j=1}^{\infty}\frac{(2j-1)q^{\frac{j-2}{2}}}{1-q^{\frac{2j-1}{2}}}+\frac{1}{16}\sum_{j=1}^{\infty}\frac{(-1)^j\,(2j-1)q^{\frac{j-2}{2}}}{1-(-q^{\frac{1}{2}})^{2j-1}}\\
&\quad-\frac{1}{16}\sum_{j=1}^{\infty}\frac{(-1)^{\frac{j}{2}}\,(2j-1)q^{\frac{j-2}{2}}}{1-(-q)^{\frac{2j-1}{2}}}\\
&\quad-\frac{1}{16}\sum_{j=1}^{\infty}\frac{(-1)^{\frac{3j}{2}}\,(2j-1)q^{\frac{j-2}{2}}}{1+(-q)^{\frac{2j-1}{2}}}, &(9)
\end{aligned}$$

$$\varphi(q)\varphi(q^4)\psi^2(q^4) = \frac{1}{2}\sum_{j=1}^{\infty}\frac{jq^{j-1}}{1+(-q)^j} - \frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j jq^{2j-1}}{1+q^{2j}} - \frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j(2j-1)q^{2j-2}}{1+q^{4j-2}}, \tag{10}$$

$$\varphi(q)\varphi^2(q^2)\psi(q^8) = \sum_{j=1}^{\infty}\frac{jq^{j-1}}{1+(-q)^j} + \sum_{j=1}^{\infty}\frac{(-1)^j jq^{2j-1}}{1+q^{2j}} - 2\sum_{j=1}^{\infty}\frac{(-1)^j jq^{4j-1}}{1+q^{4j}}, \tag{11}$$

$$\varphi(q)\psi^2(q^4)\psi(q^8) = \frac{1}{4}\sum_{j=1}^{\infty}\frac{jq^{j-2}}{1+(-q)^j} - \frac{1}{4}\sum_{j=1}^{\infty}\frac{(-1)^j jq^{2j-2}}{1+q^{2j}} + \frac{1}{4}\sum_{j=1}^{\infty}\frac{(-1)^j(2j-1)q^{2j-3}}{1+q^{4j-2}}. \tag{12}$$

Theorem 3

$$\varphi^4(q)\varphi^2(q^2) = 1 + 2\sum_{j=1}^{\infty}\frac{(-1)^j(2j-1)^2q^{2j-1}}{1-q^{2j-1}} + 8\sum_{j=1}^{\infty}\frac{j^2q^j}{1+q^{2j}} - 2\sum_{j=1}^{\infty}\frac{(-1)^j(2j-1)^2q^{2j-1}}{1+q^{2j-1}}, \tag{13}$$

$$\varphi^2(q)\varphi^4(q^2) = 1 + 2\sum_{j=1}^{\infty}\frac{(-1)^j(2j-1)^2q^{2j-1}}{1-q^{2j-1}} + 4\sum_{j=1}^{\infty}\frac{j^2q^j}{1+q^{2j}} - 2\sum_{j=1}^{\infty}\frac{(-1)^j(2j-1)^2q^{2j-1}}{1+q^{2j-1}}, \tag{14}$$

$$\psi^4(q)\psi^2(q^2) = \sum_{j=1}^{\infty}\frac{j^2q^{j-1}}{1+q^{2j}}, \tag{15}$$

$$\varphi^2(q)\psi^4(q) = \sum_{j=1}^{\infty}\frac{(2j-1)^2q^{j-1}}{1+q^{2j-1}}, \tag{16}$$

$$\varphi^2(q)\psi^4(q^4) = \frac{1}{4}\sum_{j=1}^{\infty}\frac{(-1)^j(2j-1)^2q^{2j-3}}{1-q^{4j-2}} + \frac{1}{4}\sum_{j=1}^{\infty}\frac{j^2q^{j-2}}{1+q^{2j}}, \tag{17}$$

$$\varphi^4(q)\psi^2(q^4) = \sum_{j=1}^{\infty}\frac{(-1)^j(2j-1)^2q^{2j-2}}{1-q^{4j-2}} + 2\sum_{j=1}^{\infty}\frac{j^2q^{j-1}}{1+q^{2j}}. \tag{18}$$

Formula (16) was given by S. Ramanujan [8, Chapter 17, Entry 17] [1, p. 139]. Proofs of (5), (6), and (16) were given by S. H. Chan [2]. In the author's thesis [7], a total 51 identities are given; only identities (1)–(18) are stated here because either they appear to be new or involve both sums of squares and triangular numbers.

Finally, we will demonstrate an arithmetic interpretation of Theorems 1–3 in terms of divisor sums. For example, (4) implies that the number of solutions in integers x_1, x_2, x_3, and y_1 of $x_1^2 + x_2^2 + x_3^2 + (2y_1+1)^2 = m$, is

$$k(m) \sum_{\substack{d|m \\ d \text{ odd}}} d, \tag{19}$$

where

$$k(m) = \begin{cases} 1 & : m \equiv 1 \pmod 4, \\ 6 & : m \equiv 2 \pmod 4, \\ 3 & : m \equiv 3 \pmod 4, \\ 8 & : m \equiv 4 \pmod 8, \\ 0 & : m \equiv 0 \pmod 8. \end{cases}$$

An arithmetic interpretation of identity (3) appeared in M. D. Hirschhorn [6].

2. Preliminary Results

Following [4, pp. 120–121], we define $f_1(\theta)$, $f_2(\theta)$, and $f_3(\theta)$ by

$$f_1(\theta) = f_1(\theta; q) = \frac{1}{2}\cot\frac{\theta}{2} - 2\sum_{j=1}^{\infty} \frac{q^{2j}}{1+q^{2j}} \sin j\theta, \tag{20}$$

$$f_2(\theta) = f_2(\theta; q) = \frac{1}{2}\csc\frac{\theta}{2} + 2\sum_{j=1}^{\infty} \frac{q^{2j-1}}{1-q^{2j-1}} \sin\left(j-\frac{1}{2}\right)\theta, \tag{21}$$

$$f_3(\theta) = f_3(\theta; q) = \frac{1}{2}\csc\frac{\theta}{2} - 2\sum_{j=1}^{\infty} \frac{q^{2j-1}}{1+q^{2j-1}} \sin\left(j-\frac{1}{2}\right)\theta. \tag{22}$$

It can be shown [4, p. 121] that the series in (20)–(22) converge for $-\mathrm{Im}\,(2\pi\tau) < \mathrm{Im}\,(\theta) < \mathrm{Im}\,(2\pi\tau)$. By Ramanujan's ${}_1\psi_1$ summation formula [8, Chapter 16, Entry 17], we have [4, p. 121]:

$$f_1(\theta) = \frac{1}{2i}\prod_{k=1}^{\infty} \frac{\left(1+q^{2k-2}e^{i\theta}\right)\left(1+q^{2k}e^{-i\theta}\right)\left(1-q^{2k}\right)^2}{\left(1-q^{2k-2}e^{i\theta}\right)\left(1-q^{2k}e^{-i\theta}\right)\left(1+q^{2k}\right)^2}, \tag{23}$$

$$f_2(\theta) = \frac{e^{i\theta/2}}{i}\prod_{k=1}^{\infty} \frac{\left(1-q^{2k-1}e^{i\theta}\right)\left(1-q^{2k-1}e^{-i\theta}\right)\left(1-q^{2k}\right)^2}{\left(1-q^{2k-2}e^{i\theta}\right)\left(1-q^{2k}e^{-i\theta}\right)\left(1-q^{2k-1}\right)^2}, \tag{24}$$

$$f_3(\theta) = \frac{e^{i\theta/2}}{i}\prod_{k=1}^{\infty} \frac{\left(1+q^{2k-1}e^{i\theta}\right)\left(1+q^{2k-1}e^{-i\theta}\right)\left(1-q^{2k}\right)^2}{\left(1-q^{2k-2}e^{i\theta}\right)\left(1-q^{2k}e^{-i\theta}\right)\left(1+q^{2k-1}\right)^2}. \tag{25}$$

These are valid for all values of θ except $\theta = 2m\pi + 2n\pi\tau$, where there are poles of order 1. Equations (23)–(25) provide an analytic continuation for the functions f_1, f_2, f_3. The functions f_1, f_2, f_3 are the Jacobian elliptic functions cs, ns, and ds, respectively. See [3, p. 77] for precise identification.

From [4, p. 124] we have

$$f_1'(\theta) = -f_2(\theta)\, f_3(\theta), \tag{26}$$
$$f_2'(\theta) = -f_1(\theta)\, f_3(\theta), \tag{27}$$
$$f_3'(\theta) = -f_1(\theta)\, f_2(\theta). \tag{28}$$

Letting

$$z = z(q) = \prod_{k=1}^{\infty} \left(1+q^{2k-1}\right)^4 \left(1-q^{2k}\right)^2, \tag{29}$$
$$x = x(q) = 16q \prod_{k=1}^{\infty} \frac{\left(1+q^{2k}\right)^8}{\left(1+q^{2k-1}\right)^8}, \tag{30}$$
$$x' = x'(q) = \prod_{k=1}^{\infty} \frac{\left(1-q^{2k-1}\right)^8}{\left(1+q^{2k-1}\right)^8}, \tag{31}$$

we have ([4, p. 124–134]):

$$x + x' = 1, \tag{32}$$
$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2} = \sqrt{z}, \tag{33}$$
$$\psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2} = \frac{\sqrt{z}\,x^{\frac{1}{8}}}{\sqrt{2}\,q^{\frac{1}{8}}}. \tag{34}$$

Using the infinite products for f_1, f_2, f_3, and comparing with (29)–(31), we obtain [4, p. 129] the values in Table 1.

Table 1. Values of f_1, f_2, and f_3.

	π	$\pi\tau$	$\pi+\pi\tau$
$f_1(\theta)$	0	$\frac{z}{2i}$	$\frac{z\sqrt{x'}}{2i}$
$f_2(\theta)$	$\frac{z}{2}$	0	$\frac{z\sqrt{x}}{2}$
$f_3(\theta)$	$\frac{z\sqrt{x'}}{2}$	$\frac{z\sqrt{x}}{2i}$	0

We also summarize 3 transformations of the functions x', x, and z from [1, pp. 125–126] in Table 2.

Table 2. Three transformations of the functions x', x, and z.

q	x'	x	z
$q \to -q$	$\frac{1}{x'}$	$-\frac{x}{x'}$	$z\sqrt{x'}$
$q \to q^{\frac{1}{2}}$	$\frac{(1-\sqrt{x})^2}{(1+\sqrt{x})^2}$	$\frac{4\sqrt{x}}{(1+\sqrt{x})^2}$	$z\left(1+\sqrt{x}\right)$
$q \to q^2$	$\frac{4\sqrt{x'}}{\left(1+\sqrt{x'}\right)^2}$	$\frac{\left(1-\sqrt{x'}\right)^2}{\left(1+\sqrt{x'}\right)^2}$	$\frac{1}{2}z\left(1+\sqrt{x'}\right)$

We shall give some explanation for Table 2. For example, if we apply the transformation $q \to -q$ to the functions x', x, and z, then the second row of Table 2 implies that $x'(-q) = \frac{1}{x'}$, $x(-q) = -\frac{x}{x'}$, and $z(-q) = z\sqrt{x'}$. The transformations $q \to q^{\frac{1}{2}}$ and $q \to q^2$ can be read similarly.

The results in Table 3 can be easily obtained by applying the results of Table 2.

Table 3. Four transformations of the functions x', x, and z.

q	x'	x	z
$q \to -q^{\frac{1}{2}}$	$\frac{(1+\sqrt{x})^2}{(1-\sqrt{x})^2}$	$\frac{-4\sqrt{x}}{(1-\sqrt{x})^2}$	$z\left(1-\sqrt{x}\right)$
$q \to -q^2$	$\frac{\left(1+\sqrt{x'}\right)^2}{4\sqrt{x'}}$	$\frac{-\left(1-\sqrt{x'}\right)^2}{4\sqrt{x'}}$	$z\sqrt[4]{x'}$
$q \to iq^{\frac{1}{2}}$	$\frac{\left(\sqrt{x'}-i\sqrt{x}\right)^2}{\left(\sqrt{x'}+i\sqrt{x}\right)^2}$	$\frac{4i\sqrt{xx'}}{\left(\sqrt{x'}+i\sqrt{x}\right)^2}$	$z\left(\sqrt{x'}+i\sqrt{x}\right)$
$q \to -iq^{\frac{1}{2}}$	$\frac{\left(\sqrt{x'}+i\sqrt{x}\right)^2}{\left(\sqrt{x'}-i\sqrt{x}\right)^2}$	$\frac{-4i\sqrt{xx'}}{\left(\sqrt{x'}-i\sqrt{x}\right)^2}$	$z\left(\sqrt{x'}-i\sqrt{x}\right)$

3. Proofs of Theorems 1–3

The following 3 lemmas are required to prove Theorems 1–3.

Lemma 1

$$\varphi(q)\varphi(q^4) = f_2(\pi) + i f_1(\pi+\pi\tau;q^2), \tag{35}$$

$$\varphi(q)\psi(q^8) = \frac{1}{2q}\left[f_2(\pi\tau) - i f_1(\pi+\pi\tau;q^2)\right], \tag{36}$$

$$\varphi(q^4)\psi(q^2) = \frac{1}{4q^{\frac{1}{4}}}\left[f_2^{(0)}\left(\pi+\pi\tau;q^{\frac{1}{2}}\right) + \frac{1}{i^{\frac{1}{2}}} f_2^{(0)}\left(\pi+\pi\tau;iq^{\frac{1}{2}}\right)\right]. \tag{37}$$

Lemma 2

$$\varphi^3(q)\psi(q^8) = -\frac{1}{q}f_1'\left(\pi;iq^{\frac{1}{2}}\right) + \frac{1}{q}f_1'(\pi;q^2) + \frac{i}{q}f_3'(\pi+\pi\tau;q^2), \tag{38}$$

$$\varphi^2(q)\psi^2(q) = \frac{1}{2q^{\frac{1}{4}}}\left[f_2'\left(\pi\tau;q^{\frac{1}{2}}\right) - i f_3'\left(\pi+\pi\tau;q^{\frac{1}{2}}\right)\right], \tag{39}$$

$$\varphi^2(q)\psi^2(q^4) = \frac{-1}{2q}f_1'\left(\pi;iq^{\frac{1}{2}}\right) + \frac{1}{2q}f_1'(\pi), \tag{40}$$

$$\varphi^2(q)\psi^2(q^8) = \frac{-1}{4q^2}f_1'\left(\pi;iq^{\frac{1}{2}}\right) + \frac{1}{2q^2}f_1'(\pi;q^2) + \frac{1}{2q^2}if_3'(\pi+\pi\tau;q^2) - \frac{1}{4q^2}f_1'(\pi), \tag{41}$$

$$\varphi(q)\psi^3(q^8) = -\frac{1}{16q^3}f_1'\left(\pi;iq^{\frac{1}{2}}\right) - \frac{3}{16q^3}f_1'(\pi) + \frac{1}{4q^3}f_1'(\pi;q^2) + \frac{i}{8q^3}f_3'(\pi+\pi\tau;q^2), \tag{42}$$

$$\varphi(q^4)\psi^3(q^2) = \frac{1}{16q^{\frac{3}{4}}}f_2'\left(\pi\tau;q^{\frac{1}{2}}\right) + \frac{i}{16q^{\frac{3}{4}}}f_2'\left(\pi\tau;-q^{\frac{1}{2}}\right) - \frac{i^{\frac{1}{2}}}{16q^{\frac{3}{4}}}f_2'\left(\pi\tau;iq^{\frac{1}{2}}\right) + \frac{1}{16i^{\frac{1}{2}}q^{\frac{3}{4}}}f_3'\left(\pi+\pi\tau;iq^{\frac{1}{2}}\right), \tag{43}$$

$$\varphi(q)\psi^2(q^4)\phi(q^4) = -\frac{1}{4q}f_1'\left(\pi;iq^{\frac{1}{2}}\right) + \frac{1}{4q}f_1'(\pi) - \frac{i}{2q}f_3'(\pi+\pi\tau;q^2), \tag{44}$$

$$\varphi(q)\varphi^2(q^2)\psi(q^8) = -\frac{1}{2q}f_1'\left(\pi;iq^{\frac{1}{2}}\right) - \frac{1}{2q}f_1'(\pi) + \frac{1}{q}f_1'(\pi;q^2), \tag{45}$$

$$\varphi(q)\psi^2(q^4)\psi(q^8) = -\frac{1}{8q^2}f_1'\left(\pi;iq^{\frac{1}{2}}\right) + \frac{1}{8q^2}f_1'(\pi) + \frac{i}{4q^2}f_3'(\pi+\pi\tau;q^2). \tag{46}$$

Lemma 3

$$\begin{aligned}
\varphi^4(q)\,\varphi^2\left(q^2\right) &= 4f_2''(\pi) - 4if_1''(\pi\tau) + 4f_3''(\pi), && (47)\\
\varphi^2(q)\,\varphi^4\left(q^2\right) &= 4f_2''(\pi) - 2if_1''(\pi\tau) + 4f_3''(\pi), && (48)\\
\psi^4(q)\,\psi^2\left(q^2\right) &= -\frac{i}{2q}f_1''(\pi\tau), && (49)\\
\varphi^2(q)\,\psi^4(q) &= -\frac{2i}{q^{\frac{1}{2}}}f_3''(\pi\tau), && (50)\\
\varphi^2(q)\,\psi^4\left(q^4\right) &= \frac{1}{4q^2}f_2''(\pi) - \frac{i}{8q^2}f_1''(\pi\tau) - \frac{1}{4q^2}f_3''(\pi), && (51)\\
\varphi^4(q)\,\psi^2\left(q^4\right) &= \frac{1}{q}f_2''(\pi) - \frac{i}{q}f_1''(\pi\tau) - \frac{1}{q}f_3''(\pi). && (52)
\end{aligned}$$

Proof. The proofs of Lemmas 1–3 follow by using (26)–(28), (32)–(34), and Tables 1–3. We express both sides of (35)–(52) in terms of z and x. We give complete details for $\varphi\left(q^4\right)\psi^3\left(q^2\right)$ only; the other formulae can be proved in a similar way.
By employing (33), (34), and Table 2, the left hand side of (38) can be rewritten as

$$\varphi^3(q)\,\psi\left(q^8\right) = \frac{1}{4q}z^2\left(1 - x'^{\frac{1}{4}}\right). \tag{53}$$

Next by substituting the values of $\theta = \pi$ and $\pi + \pi\tau$ into (26) and (28), respectively, and then using Table 1, we find that

$$\begin{aligned}
f_1'(\pi) &= -\frac{z^2\sqrt{x'}}{4}, && (54)\\
f_3'(\pi + \pi\tau) &= -\frac{iz^2\sqrt{xx'}}{4}. && (55)
\end{aligned}$$

Now if we employ the results of Table 2, Table 3, (54), and (55) in the right hand side of (38) we get

$$\begin{aligned}
&-\frac{1}{q}f_1'\left(\pi; iq^{\frac{1}{2}}\right) + \frac{1}{q}f_1'\left(\pi; q^2\right) + \frac{i}{q}f_3'\left(\pi + \pi\tau; q^2\right)\\
&= \frac{1}{4q}z^2 - \frac{1}{8q}z^2x'^{\frac{1}{4}}\left(1 + x'^{\frac{1}{2}}\right) - \frac{1}{8q}z^2x'^{\frac{1}{4}}\left(1 - x'^{\frac{1}{2}}\right)\\
&= \frac{1}{4q}z^2\left(1 - x'^{\frac{1}{4}}\right). && (56)
\end{aligned}$$

Combining (53) and (56) we obtain (38). This completes the proof of Lemmas 1–3. □

We now prove Theorems 1–3.

Proofs of Theorems 1–3. We use the series expansions of $f_1(\theta)$, $f_2(\theta)$, and $f_3(\theta)$ in (20), (21), and (22), while the right hand sides of the results in Lemmas 1–3 can be represented explicitly as Lambert series. We give complete details for (4) only; the others can be proved in a similar way.

First, differentiating (21) and (22) with respect to θ we have

$$f_1'(\theta) = -\frac{1}{4} - \frac{1}{4}\cot^2\frac{\theta}{2} - 2\sum_{j=1}^{\infty}\frac{jq^{2j}}{1+q^{2j}}\cos j\theta, \tag{57}$$

$$f_3'(\theta) = -\frac{1}{4}\csc\frac{\theta}{2}\cot\frac{\theta}{2} - \sum_{j=1}^{\infty}\frac{(2j-1)\,q^{2j-1}}{1+q^{2j-1}}\cos\left(j-\frac{1}{2}\right)\theta. \tag{58}$$

Substituting $\theta = \pi$ into (57) we have

$$f_1'(\pi) = -\frac{1}{4} - 2\sum_{j=1}^{\infty}\frac{(-1)^j\, jq^{2j}}{1+q^{2j}}. \tag{59}$$

Substituting $\theta = \pi + \pi\tau$ into (58) and recalling that $q = e^{i\pi\tau}$, we have

$$\begin{aligned} f_3'(\pi+\pi\tau) &= \frac{e^{i(\pi+\pi\tau)/2} + e^{-i(\pi+\pi\tau)/2}}{2\left(e^{i(\pi+\pi\tau)/2} - e^{-i(\pi+\pi\tau)/2}\right)^2} \\ &\quad -\frac{1}{2}\sum_{j=1}^{\infty}\frac{(2j-1)\,q^{2j-1}}{1+q^{2j-1}}\left(e^{i(j-1/2)(\pi+\pi\tau)} - e^{-i(j-1/2)(\pi+\pi\tau)}\right) \\ &= \frac{iq^{1/2}(1-q)}{2(1+q)^2} - \frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^{j-1/2}(2j-1)\,q^{j-1/2}(1-1+q^{2j-1})}{1+q^{2j-1}} \\ &\quad +\frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^{j-1/2}(2j-1)\,q^{j-1/2}}{1+q^{2j-1}} \\ &= \frac{iq^{1/2}(1-q)}{2(1+q)^2} - \frac{1}{2}\sum_{j=1}^{\infty}(-1)^{j-1/2}(2j-1)\,q^{j-1/2} \\ &\quad +\sum_{j=1}^{\infty}\frac{(-1)^{j-1/2}(2j-1)\,q^{j-1/2}}{1+q^{2j-1}}. \end{aligned} \tag{60}$$

We observe that

$$\begin{aligned} \sum_{j=1}^{\infty}(-1)^{j-1/2}(2j-1)\,q^{j-1/2} &= \sum_{j=1}^{\infty} j(-q)^{j/2} - \sum_{j=1}^{\infty} 2j(-q)^j \\ &= \frac{(-q)^{1/2}}{\left(1-(-q)^{1/2}\right)^2} - \frac{2(-q)}{\left(1-(-q)\right)^2} \\ &= \frac{iq^{1/2}(1-q)}{(1+q)^2}. \end{aligned}$$

Substituting these into (60) we obtain

$$f_3'(\pi+\pi\tau) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1/2}(2j-1)\,q^{j-1/2}}{1+q^{2j-1}}. \tag{61}$$

Using (59) and (61) in the right hand side of (38) and simplifying the results we obtain (4). This completes the proofs of Theorems 1–3. □

Next we present an arithmetic interpretation of Theorems 1–3 in terms of divisor sums.

Let k and m be positive integers. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mu_1, \mu_2, \ldots, \mu_m$ be positive integers, where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$. The function

$$r(\lambda_1\Box + \lambda_2\Box + \cdots + \lambda_k\Box + \mu_1\triangle + \mu_2\triangle + \cdots + \mu_m\triangle)(n)$$

will denote the number of solutions in integers of

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_k x_k^2 + \mu_1 \frac{y_1(y_1+1)}{2} + \mu_2 \frac{y_2(y_2+1)}{2} + \cdots + \mu_m \frac{y_m(y_m+1)}{2} = n, \quad (62)$$

where $n = 0, 1, 2, 3, \ldots$. We also define $r(\lambda_1\Box + \cdots + \lambda_k\Box + \mu_1\triangle + \cdots + \mu_m\triangle)(0) = 1$.

Then the generating function for $r(\lambda_1\Box + \lambda_2\Box + \cdots + \lambda_k\Box + \mu_1\triangle + \mu_2\triangle + \cdots + \mu_m\triangle)(n)$ is

$$\sum_{n=0}^{\infty} r(\lambda_1\Box + \lambda_2\Box + \cdots + \lambda_k\Box + \mu_1\triangle + \mu_2\triangle + \cdots + \mu_m\triangle)(n)\, q^n$$
$$= \varphi\left(q^{\lambda_1}\right)\varphi\left(q^{\lambda_2}\right)\cdots\varphi\left(q^{\lambda_k}\right)\psi\left(q^{\mu_1}\right)\psi\left(q^{\mu_2}\right)\cdots\psi\left(q^{\mu_m}\right). \quad (63)$$

We remark that since (62) is equivalent to

$$2\lambda_1 x_1^2 + 2\lambda_2 x_2^2 + \cdots + 2\lambda_k x_k^2 + \mu_1\left(y_1 + \frac{1}{2}\right)^2 + \mu_2\left(y_2 + \frac{1}{2}\right)^2 + \cdots + \mu_m\left(y_m + \frac{1}{2}\right)^2 = 2n + \frac{m}{4}, \quad (64)$$

then geometrically, $2^m r(\lambda_1\Box + \lambda_2\Box + \cdots + \lambda_k\Box + \mu_1\triangle + \mu_2\triangle + \cdots + \mu_m\triangle)(n)$ counts the number of lattice points on the $k+m$ dimensional ellipsoid centred at $(0, 0, \ldots, 0, -\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2})$, the point whose first k coordinates are 0 and remaining m coordinates are $-\frac{1}{2}$, with radius $\sqrt{2n + \frac{m}{4}}$.

Now we give complete details for an arithmetic interpretation of (4) in terms of divisor sums.

Corollary 1 *For $n \geq 1$,*

$$r(\Box + \Box + \Box + 8\triangle)(n) = k(n) \sum_{\substack{d \mid n+1 \\ d \text{ odd}}} d, \quad (65)$$

where

$$k(n) = \begin{cases} 6 & : n \equiv 1 \pmod{4}, \\ 3 & : n \equiv 2 \pmod{4}, \\ 8 & : n \equiv 3 \pmod{8}, \\ 1 & : n \equiv 0 \pmod{4}, \\ 0 & : n \equiv 7 \pmod{8}. \end{cases}$$

Proof. First use (63) and expand the right hand side using the geometric series in (4) to get

$$\begin{aligned}
&\sum_{n=0}^{\infty} r\left(\square+\square+\square+8\triangle\right)(n)\, q^n \\
=\ & 2\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{mj}\,(j+1)q^{(j+1)(m+1)-1} \\
&+2\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+j}\,(j+1)q^{4(j+1)(m+1)-1} \\
&-\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+j}\,(2j+1)q^{(2j+1)(2m+1)-1} \\
=\ & 2\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(2j+1)q^{(2j+1)(m+1)-1}+2\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m}\,(2j+2)q^{(2j+2)(m+1)-1} \\
&-2\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{(m+1)}\,(2j+1)q^{4(2j+1)(m+1)-1} \\
&+2\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+1}\,(2j+2)q^{4(2j+2)(m+1)-1} \\
&-\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+j}\,(2j+1)q^{(2j+1)(2m+1)-1}
\end{aligned}$$

$$\begin{aligned}
=\ & \sum_{n=0}^{\infty}\Bigg[2\sum_{(2j+1)(m+1)=n+1}(2j+1)+2\sum_{(2j+2)(m+1)=n+1}(-1)^{m}\,(2j+1) \\
&-2\sum_{4(2j+1)(m+1)=n+1}(-1)^{(m+1)}\,(2j+1) \\
&+2\sum_{4(2j+2)(m+1)=n+1}(-1)^{(m+1)}\,(2j+2) \\
&-\sum_{(2j+1)(2m+1)=n+1}(-1)^{(m+j)}\,(2j+1)\Bigg]q^n \\
=\ & \sum_{n=0}^{\infty}\Bigg[2\sum_{\substack{d\mid n+1\\ d\text{ odd}}}d+4\sum_{\substack{d\mid n+1\\ n,d\text{ odd}}}d-2\sum_{\substack{d\mid n+1\\ n\equiv 3\ (\mathrm{mod}\ 4),d\text{ odd}}}(-1)^{\frac{n+1}{4d}}\,d-4\sum_{\substack{d\mid n+1\\ n\equiv 7\ (\mathrm{mod}\ 8),d\text{ odd}}}d \\
&-\sum_{\substack{d\mid n+1\\ n\text{ even},\ d\text{ odd}}}(-1)^{\frac{n}{2}}\,d\Bigg]q^n.
\end{aligned}$$

By comparing coefficients of q^n on both sides we obtain

$$\begin{aligned} r\left(\Box+\Box+\Box+8\triangle\right)(n) &= 2\sum_{\substack{d|n+1\\ d\text{ odd}}} d \,+\, 4\sum_{\substack{d|n+1\\ n,d\text{ odd}}} d \,-\, 2\sum_{\substack{d|n+1\\ n\equiv 3\,(\mathrm{mod}\ 4),d\text{ odd}}} (-1)^{\frac{n+1}{4d}}\, d \\ &\quad -4\sum_{\substack{d|n+1\\ n\equiv 7\,(\mathrm{mod}\ 8),d\text{ odd}}} d \,-\, \sum_{\substack{d|n+1\\ n\text{ even},\ d\text{ odd}}} (-1)^{\frac{n}{2}}\, d. \end{aligned}$$

If $n \equiv 1 \pmod 4$, then $n+1 = 4k+2$ and so

$$\begin{aligned} r\left(\Box+\Box+\Box+8\triangle\right)(n) &= 2\sum_{\substack{d|n+1\\ d\text{ odd}}} d + 4\sum_{\substack{d|4k+2\\ d\text{ odd}}} d \\ &= 2\sum_{\substack{d|n+1\\ d\text{ odd}}} d + 4\sum_{\substack{d|n+1\\ d\text{ odd}}} d \\ &= 6\sum_{\substack{d|n+1\\ d\text{ odd}}} d. \end{aligned} \tag{66}$$

Similarly, if $n \equiv 2 \pmod 4$, then $n+1 = 4k+3$ and so

$$\begin{aligned} r\left(\Box+\Box+\Box+8\triangle\right)(n) &= 2\sum_{\substack{d|n+1\\ d\text{ odd}}} d - \sum_{\substack{d|4k+3\\ d\text{ odd}}} (-1)^{\frac{4k+2}{2}}\, d \\ &= 2\sum_{\substack{d|n+1\\ d\text{ odd}}} d + \sum_{\substack{d|n+1\\ d\text{ odd}}} d \\ &= 3\sum_{\substack{d|n+1\\ d\text{ odd}}} d. \end{aligned} \tag{67}$$

If $n \equiv 3 \pmod 8$, then $n+1 = 8k+4$ and so

$$\begin{aligned} r\left(\Box+\Box+\Box+8\triangle\right)(n) &= 2\sum_{\substack{d|n+1\\ d\text{ odd}}} d + 4\sum_{\substack{d|8k+4\\ d\text{ odd}}} d - 2\sum_{\substack{d|8k+4\\ d\text{ odd}}} (-1)^{\frac{8k+4}{4d}}\, d \\ &= 2\sum_{\substack{d|n+1\\ d\text{ odd}}} d + 4\sum_{\substack{d|n+1\\ d\text{ odd}}} d + 2\sum_{\substack{d|n+1\\ d\text{ odd}}} d \\ &= 8\sum_{\substack{d|n+1\\ d\text{ odd}}} d. \end{aligned} \tag{68}$$

If $n \equiv 0 \pmod 4$, then $n+1 = 4k+1$ and so

$$\begin{aligned} r\left(\Box+\Box+\Box+8\triangle\right)(n) &= 2\sum_{\substack{d|n+1\\ d\text{ odd}}} d - \sum_{\substack{d|4k+1\\ d\text{ odd}}} (-1)^{\frac{4k}{2}}\, d \\ &= 2\sum_{\substack{d|n+1\\ d\text{ odd}}} d - \sum_{\substack{d|n+1\\ d\text{ odd}}} d \\ &= \sum_{\substack{d|n+1\\ d\text{ odd}}} d. \end{aligned} \tag{69}$$

If $n \equiv 7 \pmod 8$, then $n+1 = 8k+8$ and so

$$\begin{aligned} r\left(\square+\square+\square+8\triangle\right)(n) &= 2\sum_{\substack{d\mid n+1\\ d \text{ odd}}} d + 4\sum_{\substack{d\mid 8k+8\\ d \text{ odd}}} d - 2\sum_{\substack{d\mid 8k+8\\ d \text{ odd}}} (-1)^{\frac{8k+8}{4d}} d - 4\sum_{\substack{d\mid 8k+8\\ d \text{ odd}}} d \\ &= 0. \end{aligned} \tag{70}$$

Combining (66)–(70), we obtain (65). □

By (64), formula (65) is equivalent to (19).

4. Remarks

The results in this paper can also be proved using the theory of modular forms. I thank the referee for his/her permission to reproduce the following remark.

Consider identity (15). We have that $q\psi^4(q)\,\psi^2(q^2)$ is a modular form of weight 3 on $\Gamma_0(4)$. On the other hand, it is easy to check that

$$F(q) = \sum_{j=-\infty}^{\infty} \frac{j^2 q^j}{1+q^{2j}}$$

is an Eisenstein series in that same space. To see this, let χ be the non-trivial Dirichlet character mod 4, and let $\sigma(n) := \sum_{d\mid n} \chi(n/d)\, d^2$. Define

$$E(q) = \sum_{n=1}^{\infty} \sigma(n)\, q^n.$$

It is known that $E(q)$ is in the space (see [5, chapter 4] for a complete discussion). It is not hard to check that $E(q) = F(q)$ directly. The identity $q\psi^4(q)\,\psi^2(q^2) = F(q)$ then follows by checking that the first few terms agree.

An arithmetic interpretation of other identities can be found in the author's thesis [7]. We shall remark that Hirschhorn [6] also presented many others which give the number of representations of an integer n by various quadratic forms in terms of divisor sums.

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