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ON THE PERIODICITY OF GENUS SEQUENCES OF QUATERNARY GAMES

M. R. Allen
Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada
ameghan@mathstat.dal.ca

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Abstract

The periodicity of the genus sequences of the heaps of finite quaternary games are examined. While the truncated genus sequence of the heaps of finite quaternary games becomes periodic, this is not true for the genus sequence in general. This contrasts with the known result that the genus sequence of the heaps of all finite subtraction games, a subset of finite quaternary games, becomes periodic.

1. Introduction

This paper assumes that the reader is familiar with the basics of combinatorial games, as presented in [5] and [7]; in particular, **impartial games**, **outcome classes**, and **disjunctive sums**.

How does a player win a combinatorial game? A player wins when she has played a winning move. In combinatorial games, it is often assumed that the winning move is to leave the other player with no moves available. However, this need not be the case. Combinatorial games can be played under two disjoint conventions, normal and misère, which differ in the choice of winning move.

Definition 1 *A game is played under the* ***normal play convention*** *if the last player to move wins. A game is played under the* ***misère play convention*** *if the last player to move loses.*

Almost all combinatorial game research has been in games played under the normal play convention due to an important result which is lacking for misère play: the Sprague-Grundy Theory for impartial normal play games ([9], [12]), which says that every impartial game played under the normal play convention is equivalent to a Nim heap. Unfortunately, there are misère games which do not behave like misère Nim, so no comparable theorem is possible.

Notation 1 *We will denote a Nim heap with n tokens by* n. *We let $\oplus$ denote the binary operation of Nim sum.*

For those unfamiliar with misère play, it may seem that a simple reversal of outcome classes is enough to form a complete theory of impartial misère games, however this is not true. The reader can check that the game $2 + 2$ is a previous player win regardless of play convention.

Every impartial misère game has a sequence of numbers associated to it, called the **Genus**. Genus is the tool traditionally used for impartial misère play analysis (for example, see [2], [3], [6], or [8]). Recently, a newer method has been developed by Plambeck and Siegel to analyse impartial misère games: the misère quotient ([10], [11]). While the misère quotient is an exciting new development in the theory, there are still questions regarding impartial misère games relevant to genus.

1.1 Quaternary Games

Much of the work done on impartial misère games has concerned itself with games in which players remove tokens from heaps based on certain rules ([2], [3], [10]). We continue with this tradition by investigating quaternary games:

Definition 2 *An **quaternary game** is an octal game $0.d_1d_2\cdots$, where $d_i \in \{0, 1, 2, 3\}$ for each $i \in$ N.*

We are often only concerned with quaternary games in which there is a limit to the number of tokens we can remove from a heap. This corresponds to a quaternary game $0.d_1d_2d_3\cdots$ such that there exists a smallest $N \in$ N such that for all $n > N$, $d_n = 0$. Quaternary games with this property are called **finite with length** N.

Definition 3 *A **subtraction game** is a quaternary game such that for all d_i, $d_i = 0$ or 3.*

Under the misère play convention, there are major differences between non-subtraction quaternary games and subtraction games. Every subtraction game behaves like misère Nim ([6], p.442), but not every quaternary game does (see [4], Appendix A). Moreover, we will show that finite subtraction games played under the misère play convention exhibit a periodicity result which finite quaternary games do not exhibit in general. The periodicity result becomes evident with the use of Genus. We quickly reproduce here the definitions and basic facts regarding genus. Full proofs of all results can be found in [4].

1.2 Genus

- The **genus** of an impartial game G, denoted by $\Gamma(G)$, is a sequence of numbers written as $g^{g_0g_1g_2g_3\cdots}$ where

$$g = \mathcal{G}^+(G),\ g_0 = \mathcal{G}^-(G),\ \text{and for } n \in \mathrm{N},\ g_n = \mathcal{G}^-\left(G + \sum_{i=1}^{n} 2\right)$$

where

$$\mathcal{G}^+(G) = \begin{cases} 0 & \textit{if } G \textit{ has no options} \\ \text{mex}\{\mathcal{G}^+(G') \mid G' \textit{ is an option of } G\} & \textit{else}, \end{cases}$$

and

$$\mathcal{G}^-(G) = \begin{cases} 1 & \textit{if } G \textit{ has no options} \\ \text{mex}\{\mathcal{G}^-(G') \mid G' \textit{ is an option of } G\} & \textit{else.} \end{cases}$$

Those familiar with impartial games will notice that $\mathcal{G}^+(G)$ is the Nim heap to which G is equivalent.

- ([6], p.430) Suppose G is an impartial game with options G_a, G_b, G_c,$\cdots$ such that

$$\Gamma(G_a) = a^{a_0a_1a_2a_3\cdots}, \Gamma(G_b) = b^{b_0b_1b_2b_3\cdots}, \Gamma(G_c) = c^{c_0c_1c_2c_3\cdots}, \cdots$$

Then $\Gamma(G) = g^{g_0g_1g_2g_3\cdots}$ is calculated as follows:

$$\begin{aligned} g &= \text{mex}\{a, b, c, \cdots\}, \\ g_0 &= \text{mex}\{a_0, b_0, c_0, \cdots\}, \text{ and} \\ g_n &= \text{mex}\{g_{n-1}, g_{n-1} \oplus 1, a_n, b_n, c_n, \cdots\} \text{ for } n \in \mathrm{N}. \end{aligned}$$

- The M**-truncated genus** of an impartial game G, denoted by $\Gamma_M(G)$, is the numbers in the genus up to and including g_M.

- For a game G, we say that the genus of G, $g^{g_0g_1g_2g_3\cdots}$, **stabilises** if there exists an $N \in \mathrm{Z}^{\geq 0}$ such that for all $n \geq N$,

$$g_{n+1} = g_n \oplus 2.$$

- ([6], p.422) The genus of G always stabilises and we write $\Gamma(G) = g^{g_0g_1g_2g_3\cdots}$ as $g^{g_0g_1\cdots g_N(g_N\oplus 2)}$ where N is the smallest non-negative integer such that for all $u \geq N$, $g_{u+1} = g_u \oplus 2$.

- ([6], p. 422) Given a Nim heap m,

$$\Gamma(\mathrm{m}) = \begin{cases} 0^{120} & \text{if } m = 0 \\ 1^{031} & \text{if } m = 1 \\ m^{m(m\oplus 2)} & \text{else.} \end{cases}$$

- ([7], p. 137) Suppose G is a disjunctive sum of Nim heaps. Then $\Gamma(G) = 0^{120}, 1^{031}$ or $n^{n(n\oplus 2)}$ for $n \in \mathrm{Z}^{\geq 0}$.

- Given an impartial misère game G, G is **tame** if $\Gamma(G) = 0^{120}, 1^{031}$ or $n^{n(n\oplus 2)}$ for $n \in \mathbb{Z}^{\geq 0}$ and every option of G is also tame. An impartial misère game is **wild** if it is not tame.

  Unfortunately, the definition of tame is far from standardised; [6], [7], and [11] all use varying definitions. The definition of tame used in this paper corresponds to that appearing in [7].

  If a game is tame, we say that under the misère play convention, this game behaves like misère Nim. Otherwise, if it is wild, the game does not behave like misère Nim.

For heap based games, such as quaternary games, it is often beneficial to think of the genera of the heaps as entries in a table, which we call the Γ **table**.

$$\begin{array}{lclllllll} \Gamma(h_0) & = & e & e_0 & e_1 & e_2 & e_3 & e_4 & \cdots \\ \Gamma(h_1) & = & a & a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ \Gamma(h_2) & = & b & b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\ & \vdots & & & & & & & \end{array}.$$

Restricting ourselves to the first $M+2$ columns gives us Γ_M. This is called the Γ_M **table**.

$$\begin{array}{lcllllll} \Gamma_M(h_0) & = & e & e_0 & e_1 & e_2 & \cdots & e_{M-1} & e_M \\ \Gamma_M(h_1) & = & a & a_0 & a_1 & a_2 & \cdots & a_{M-1} & a_M \\ \Gamma_M(h_2) & = & b & b_0 & b_1 & b_2 & \cdots & b_{M-1} & b_M \\ & \vdots & & & & & & & \end{array}.$$

Definition 4 *For a heap based game, the* ***genus sequence of the heaps*** *is the sequence of genus values $\Gamma(h_0)$, $\Gamma(h_1)$, $\Gamma(h_2)$, $\cdots$. Similarly, the* ***M-truncated genus sequence of the heaps*** *is the sequence of M-truncated genus values $\Gamma_M(h_0)$, $\Gamma_M(h_1)$, $\Gamma_M(h_2)$, $\cdots$.*

There are two possibilities for a periodicity results - along the rows and along the columns. Since the genus always stabilises, every row eventually becomes periodic.

Definition 5 *For a heap based game, we say that the* ***genus sequence of the heaps is periodic*** *if there exist $N, p \in \mathbb{N}$ such that for all $n \geq N$, $\Gamma(h_n) = \Gamma(h_{n+p})$. Similarly, we say the that M**-truncated genus sequence of the heaps is periodic** if there exist $T, u \in \mathbb{N}$ such that for all $t \geq T$, $\Gamma_M(h_t) = \Gamma_M(h_{t+u})$.*

This paper is concerned with the behaviour of the columns of the Γ table as well as the non periodicity/periodicity of the genus sequence of the heaps of arbitrary quaternary games.

2. The Genera of Quaternary Games

We now have all the tools necessary to begin our examination of quaternary games. We start by examining subtraction games.

Theorem 6 *For any finite subtraction game, the genus sequence of the heaps is periodic.*

Proof. Let S be a subtraction game and h_n a heap of size n. Then h_n is tame and $\Gamma(h_n) = 0^{120}, 1^{031}$, or $n^{n(n\oplus 2)}$ for $n \in \mathbb{Z}^{\geq 2}$ ([6], p. 442). Thus $\Gamma(h_n)$ depends only on $\mathcal{G}^+(h_n)$. we know that the $\mathcal{G}^+$ sequence of a finite subtraction game becomes periodic ([1], p.148). Thus the genus sequence of the heaps of a finite subtraction game is periodic. □

Even though all subtraction games are tame and have periodic genus sequence, neither of these results is true for general quaternary games. Wild quaternary games are fairly common. For example, while there are no wild quaternary games of length two or less, there is one wild quaternary game of length three, twenty-one wild quaternary games of length four, 154 wild quaternary games of length five, and 739 wild quaternary games of length six. Appendix A of [4] lists all wild finite quaternary games of length six or less.

2.1 Periodicity

We begin with the following periodicity result.

Proposition 7 *Given a finite quaternary game, there exists N, $p \in \mathbb{N}$ such that for all $n \geq N$, $\mathcal{G}^+(h_n) = \mathcal{G}^+(h_{n+p})$. Similarly, for each $v \in \mathbb{Z}^{\geq 0}$, there exists N_v, $p_v \in \mathbb{N}$ such that for all $m \geq N_v$, $\mathcal{G}^-\left(h_m + \sum_{i=1}^{v} 2\right) = \mathcal{G}^-\left(h_{m+p_v} + \sum_{i=1}^{v} 2\right)$. In other words, the values in each column in the Γ table of a finite quaternary game becomes periodic.*

Proof. Consider the finite subtraction octal game $0.d_1 d_2 \cdots d_k$ and $n \geq k+1$. From h_n, there are at most k legal moves.

We begin by examining the first column in the Γ table. That is, the $\mathcal{G}^+(h_n)$ values. Suppose $n \geq k+1$. Then $\mathcal{G}^+(h_n) = \text{mex}\{\mathcal{G}^+(h_{n-i}) \mid d_i = 2 \text{ or } 3\}$. Thus, $\mathcal{G}^+(h_n) \leq k$, since $|\{\mathcal{G}^+(h_{n-i}) \mid d_i = 2 \text{ or } 3\}| \leq k+1$, as there are at most k legal moves from any given heap, That is, $\mathcal{G}^+(h_n) = u$ for $u \in \{0, 1, \cdots, k\}$.

Let $m = \max\{i \mid d_i = 2 \text{ or } 3\}$. That is, m is the largest number of tokens which can be taken from a heap of size n. The sequence of $\mathcal{G}^+$ values from $\mathcal{G}^+(h_n)$ onwards depends only on the previous m values, $\mathcal{G}^+(h_{n-m}), \mathcal{G}^+(h_{n-m+1}), \ldots, \mathcal{G}^+(h_{n-1})$. Not all of these values will be in the mex set which determines $\mathcal{G}^+(h_n)$; the number m is an overestimation assuming that for all $i \leq m$, $d_i = 3$.

Consider the subsequences of length m of the $\mathcal{G}^+$ values. Eventually there will be a subsequence which repeats itself since there are only a finite number of permutations of length m with $k+1$ elements. That is, there exists $p, l \in \mathbb{Z}^{\geq 0}$ such that $\mathcal{G}^+(h_n) = \mathcal{G}^+(h_{n+p})$ for all n such that $l \leq n \leq l+m$.

We claim that $\mathcal{G}^+(h_{n+p}) = \mathcal{G}^+(h_n)$ for all $n \geq l$. To prove this, we proceed by induction on n. We have the base case from the preceding paragraph. Fix $t \in \mathbb{Z}^{\geq 0}$ and suppose that for all $u < t$, $\mathcal{G}^+(h_{(n+u)+p}) = \mathcal{G}^+(h_{n+u})$.

Consider

$$\begin{aligned}\mathcal{G}^+(h_{n+t+p}) &= \operatorname{mex}\{\mathcal{G}^+(h_{(n+t+p)-i}) \mid d_i = 2 \text{ or } 3\} \\ &= \operatorname{mex}\{\mathcal{G}^+(h_{(n+t-i)+p}) \mid d_i = 2 \text{ or } 3\} \\ &= \operatorname{mex}\{\mathcal{G}^+(h_{n+t-i}) \mid d_i = 2 \text{ or } 3\} \text{ by inductive assumption} \\ &= \operatorname{mex}\{\mathcal{G}^+(h_{(n+t)-i}) \mid d_i = 2 \text{ or } 3\} \\ &= \mathcal{G}^+(h_{n+t}).\end{aligned}$$

which completes the induction. Therefore the first column of the Γ table becomes periodic.

The argument used to show that the $\mathcal{G}^-(h_n)$ and $\mathcal{G}^-(h_n + \sum_{i=1}^{v} 2)$ sequences become periodic is virtually identical to that of $\mathcal{G}^+(h_n)$. □

Examining truncated genera, we obtain the following periodicity result.

Theorem 8 *Let G be a finite quaternary game with heaps denoted by h_n. Then the M-truncated genus sequence of the heaps is periodic.*

Proof. Consider the Γ_M table:

$$\begin{array}{lcllllllll} \Gamma_M(h_0) &=& 0 & 1 & 2 & 0 & \cdots & 2 & 0 & \\ \Gamma_M(h_1) &=& a & a_0 & a_1 & a_2 & \cdots & a_{M-1} & a_M & \\ \Gamma_M(h_2) &=& b & b_0 & b_1 & b_2 & \cdots & b_{M-1} & b_M & . \\ & \vdots & & & & & & & & \end{array}$$

By Proposition 7, each column becomes periodic. Let μ_i denote the pre-period length of column i. Let ρ_i denote the period length of column i. Then, for all $n > \max_{i=1}^{M}\{\mu_i\}$, $\Gamma_M(h_n) = \Gamma_M(h_{n+p})$ for $p = \prod_{i=0}^{M} \rho_i$. □

Corollary 9 *If there exists $M \in \mathrm{N}$ such that for all h_n, $m \geq M$,*

$$\mathcal{G}^-\left(h_n + \sum_{i=1}^{m} 2\right) = \mathcal{G}^-\left(h_n + \sum_{i=1}^{m+2} 2\right),$$

then the genus sequence of the heaps is periodic.

Proof. The given requirement means that, thinking of the genera of the heaps as columns of a table, the M^{th} column equals the $(M+2n)^{th}$ column, and the $(M+1)^{th}$ column equals the $(M+2n+1)^{th}$ column, for $n \in \mathrm{N}$. That is, each $\Gamma(h_n)$ has stabilised by the $(M+1)^{th}$ column, so $\Gamma_{M+1}(h_n)$ completely encodes all the information given in $\Gamma(h_n)$, and so for $N, p \in \mathrm{N}$ such that $\Gamma_{M+1}(h_n) = \Gamma_{M+1}(h_{n+p})$ for all $n \geq N$, we also obtain $\Gamma(h_n) = \Gamma(h_{n+p})$ for $n \geq N$. □

Once we no longer truncate the genera of the heaps, we are no longer guaranteed periodicity, as will be shown with 0.122 213.

2.2 The Quaternary Game 0.122 213: A Counterexample

The counterexample presented for the claim that the genus sequence of the heaps is periodic for every finite quaternary game is the game 0.122 213.

We begin with some notation.

Notation 2 *Let* $\{a_1a_2\cdots a_n\}^m$ *denote the string* $a_1a_2\cdots a_n$ *repeated* m *times. For example,* $\{a_1a_2a_3\}^6 = a_1a_2a_3\ a_1a_2a_3\ a_1a_2a_3\ a_1a_2a_3\ a_1a_2a_3\ a_1a_2a_3$.

Proposition 10 *Let* $G = 0.122\ 213$*. For* $n \in \mathrm{Z}^{\geq 0}$*, let* h_n *denote a heap of size* n*. Then*

heap	genus	heap	genus
h_{24+40n}	$2^{\{\{43131\}^2\{42020\}^2\}^n 43131\,431}$	h_{44+40n}	$3^{\{\{43131\}^2\{42020\}^2\}^n\{43131\}^2 42020\,420}$
h_{25+40n}	$2^{131\{43131\{42020\}^2 43131\}^n 431}$	h_{45+40n}	$1^{13143\,131\{\{42020\}^2\{43131\}^2\}^n 42020\,420}$
h_{26+40n}	$0^{0\{43131\{42020\}^2 43131\}^n 43131\,420}$	h_{46+40n}	$1^{04313\,1\{\{42020\}^2\{43131\}^2\}^n\{42020\}^2 431}$
h_{27+40n}	$0^{3131\{\{42020\}^2\{43131\}^2\}^n 420}$	h_{47+40n}	$4^{3131\{\{42020\}^2\{43131\}^2\}^n\{42020\}^2 431}$
h_{28+40n}	$3^{31\{\{42020\}^2\{43131\}^2\}^n 420}$	h_{48+40n}	$2^{31\{\{42020\}^2\{43131\}^2\}^n\{42020\}^2 43131\,431}$
h_{29+40n}	$1^{\{\{42020\}^2\{43131\}^2\}^n 42020\,420}$	h_{49+40n}	$2^{\{\{42020\}^2\{43131\}^2\}^n\{42020\}^2 43131\,431}$
h_{30+40n}	$1^{120\{42020\{43131\}^2 42020\}^n 42020\,431}$	h_{50+40n}	$0^{12042\,020\{\{43131\}^2\{42020\}^2\}^n\{43131\}^2 420}$
h_{31+40n}	$4^{0\{42020\{43131\}^2 42020\}^n 42020\,431}$	h_{51+40n}	$0^{04202\,0\{\{43131\}^2\{42020\}^2\}^n\{43131\}^2 420}$
h_{32+40n}	$2^{2020\{\{43131\}^2\{42020\}^2\}^n 43131\,431}$	h_{52+40n}	$3^{2020\{\{43131\}^2\{42020\}^2\}^n\{43131\}^2 42020\,420}$
h_{33+40n}	$2^{20\{\{43131\}^2\{42020\}^2\}^n 43131\,431}$	h_{53+40n}	$1^{20\{\{43131\}^2\{42020\}^2\}^n\{43131\}^2 42020\,420}$
h_{34+40n}	$0^{\{\{43131\}^2\{42020\}^2\}^n\{43131\}^2 420}$	h_{54+40n}	$1^{\{\{43131\}^2\{42020\}^2\}^{n+1} 431}$
h_{35+40n}	$0^{13143\,131\{\{42020\}^2\{43131\}^2\}^n 420}$	h_{55+40n}	$4^{131\{\{43131\}^2\{42020\}^2\}^n\{42020\}^2 431}$
h_{36+40n}	$3^{04313\,1\{\{42020\}^2\{43131\}^2\}^n 42020\,420}$	h_{56+40n}	$2^{0\{43131\{42020\}^2 43131\}^{n+1} 431}$
h_{37+40n}	$1^{3131\{\{42020\}^2\{43131\}^2\}^n 42020\,420}$	h_{57+40n}	$2^{3131\{\{42020\}^2\{43131\}^2\}^n\{42020\}^2 43131\,431}$
h_{38+40n}	$1^{31\{\{42020\}^2\{43131\}^2\}^n\{42020\}^2 431}$	h_{58+40n}	$0^{31\{\{42020\}^2\{43131\}^2\}^{n+1} 420}$
h_{39+40n}	$4^{\{\{42020\}^2\{43131\}^2\}^n\{42020\}^2 431}$	h_{59+40n}	$0^{\{\{42020\}^2\{43131\}^2\}^{n+1} 420}$
h_{40+40n}	$2^{12042\,020\{\{43131\}^2\{42020\}^2\}^n 43131\,431}$	h_{60+40n}	$3^{120\{42020\{43131\}^2 42020\}^{n+1} 420}$
h_{41+40n}	$2^{04202\,0\{\{43131\}^2\{42020\}^2\}^n 43131\,431}$	h_{61+40n}	$1^{0\{42020\{43131\}^2 42020\}^{n+1} 420}$
h_{42+40n}	$0^{2020\{\{43131\}^2\{42020\}^2\}^n\{43131\}^2 420}$	h_{62+40n}	$1^{2020\{\{43131\}^2\{42020\}^2\}^{n+1} 431}$
h_{43+40n}	$0^{20\{\{43131\}\{42020\}\}^n\{43131\}^2 420}$	h_{63+40n}	$4^{20\{\{43131\}^2\{42020\}^2\}^{n+1} 431}$

Proof. We proceed by induction on n. For $n = 0$, calculations give us

heap	*genus*	*heap*	*genus*
h_{24}	$2^{43131\ 431}$	h_{44}	$3^{43131\ 43131\ 42020\ 420}$
h_{25}	$2^{13143\ 1}$	h_{45}	$1^{13143\ 13142\ 02042\ 0}$
h_{26}	$0^{04313\ 1420}$	h_{46}	$1^{04313\ 14202\ 04202\ 0431}$
h_{27}	$0^{31314\ 20}$	h_{47}	$4^{31314\ 20204\ 20204\ 31}$

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heap	*genus*	*heap*	*genus*
h_{28}	$3^{31420\ 20420}$	h_{48}	$2^{31420\ 20420\ 20431\ 31431}$
h_{29}	$1^{42020\ 420}$	h_{49}	$2^{42020\ 42020\ 43131\ 431}$
h_{30}	$1^{12042\ 02043\ 1}$	h_{50}	$0^{12042\ 02043\ 13143\ 13142\ 0}$
h_{31}	$4^{04202\ 0431}$	h_{51}	$0^{04202\ 04313\ 14313\ 1420}$
h_{32}	$2^{20204\ 31314\ 31}$	h_{52}	$3^{20204\ 31314\ 31314\ 20204\ 20}$
h_{33}	$2^{20431\ 31431}$	h_{53}	$1^{20431\ 31431\ 31420\ 20420}$
h_{34}	$0^{43131\ 43131\ 420}$	h_{54}	$1^{43131\ 43131\ 42020\ 42020\ 431}$
h_{35}	$0^{13143\ 13142\ 0}$	h_{55}	$4^{13143\ 13142\ 02042\ 02043\ 1}$
h_{36}	$3^{04313\ 14202\ 0420}$	h_{56}	$2^{04313\ 14202\ 04202\ 04313\ 1431}$
h_{37}	$1^{31314\ 20204\ 20}$	h_{57}	$2^{31314\ 20204\ 20204\ 31314\ 31}$
h_{38}	$1^{31420\ 20420\ 20431}$	h_{58}	$0^{31420\ 20420\ 20431\ 31431\ 31420}$
h_{39}	$4^{42020\ 42020\ 431}$	h_{59}	$0^{42020\ 42020\ 43131\ 43131\ 420}$
h_{40}	$2^{12042\ 02043\ 13143\ 1}$	h_{60}	$3^{12042\ 02043\ 13143\ 13142\ 02042\ 0}$
h_{41}	$2^{04202\ 04313\ 1431}$	h_{61}	$1^{04202\ 04313\ 14313\ 14202\ 0420}$
h_{42}	$0^{20204\ 31314\ 31314\ 20}$	h_{62}	$1^{20204\ 31314\ 31314\ 20204\ 20204\ 31}$
h_{43}	$0^{20431\ 31431\ 31420}$	h_{63}	$4^{20431\ 31431\ 31420\ 20420\ 20431}$

which shows the base case.

Suppose that for all $n < k$, the genus of a heap of size h_{i+40n}, for $i \in \{24, 25, \cdots, 63\}$, equals the genus given in the chart in the statement of the theorem. Call this (IH1). Consider $n = k$. We will only show the result for h_{24+40k}, as the method of proof is similar for all 39 other cases.

The moves available from h_{24+40k} are

$$\begin{aligned} h_{24+40k} &\xrightarrow{-2} h_{24+40k-2} = h_{62+40(k-1)} \\ &\xrightarrow{-3} h_{24+40k-3} = h_{61+40(k-1)} \\ &\xrightarrow{-4} h_{24+40k-4} = h_{60+40(k-1)} \\ &\xrightarrow{-6} h_{24+40k-6} = h_{58+40(k-1)}, \end{aligned}$$

where each of the options falls under the induction hypothesis (IH1). That is,

$$\begin{aligned} \Gamma\left(h_{61+40(k-1)}\right) &= 1^{2020\{\{43131\}^2\{42020\}^2\}^k431}, \\ \Gamma\left(h_{61+40(k-1)}\right) &= 1^{0\{42020\{43131\}^2 42020\}^k420}, \\ \Gamma\left(h_{60+40(k-1)}\right) &= 3^{120\{42020\{43131\}^2 42020\}^k420}, \\ \Gamma\left(h_{58+40(k-1)}\right) &= 0^{31\{\{42020\}^2\{43131\}^2\}^k420}. \end{aligned}$$

We want

$$\Gamma\left(h_{24+40k}\right) = 2^{\{\{43131\}^2\{42020\}^2\}^k 43131\ 431}. \quad (1)$$

We begin with $\mathcal{G}^+(h_{24+40k})$ and $\mathcal{G}^-(h_{24+40k})$:

$$\begin{aligned}\mathcal{G}^+(h_{24+40k}) &= \text{mex}\{1,1,3,0\} = 2,\\ \mathcal{G}^-(h_{24+40k}) &= \text{mex}\{2,0,1,3\} = 4.\end{aligned}$$

Therefore the base and the first superscript equal the desired result.

Consider $\mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{m} 2\right)$ for $m \in \mathrm{N}$. We claim that this equals the $(m+1)^{th}$ digit in the superscript of the genus on the RHS of Equation (1). We proceed by induction on m. Suppose $m = 1$. Then

$$\begin{aligned}\mathcal{G}^-(h_{24+40k}+2) &= \text{mex}\{\mathcal{G}^-(h_{24+40k}), \mathcal{G}^-(h_{24+40k}) \oplus 1, \mathcal{G}^-(h_{62+40(k-1)}+2),\\ &\quad \mathcal{G}^-(h_{61+40(k-1)}+2), \mathcal{G}^-(h_{60+40(k-1)}+2), \mathcal{G}^-(h_{58+40(k-1)}+2)\}\\ &= \text{mex}\{4, 5, \mathcal{G}^-(h_{24+40k}) \oplus 1, \mathcal{G}^-(h_{62+40(k-1)}+2),\\ &\quad \mathcal{G}^-(h_{61+40(k-1)}+2), \mathcal{G}^-(h_{60+40(k-1)}+2), \mathcal{G}^-(h_{58+40(k-1)}+2)\}\\ &= \text{mex}\{4,5,0,4,2,1\} \text{ by (IH1)}\\ &= 3.\end{aligned}$$

Now suppose the result holds for all $m < 10k+6$, i.e.,

$$\Gamma\left(h_{24+40k}\right) = 2^{\{\{43131\}^2\{42020\}^2\}^n 43131\ 4\ g_{10k+6}\ g_{10k+7}\ g_{10k+8}\cdots} \tag{2}$$

with $g_{10k+i} \in \mathrm{Z}^{\geq 0}$, $i \in \{6, 7, \cdots\}$.

Examining g_{10k+6}, we have:

$$\begin{aligned}g_{10k+6} &= \mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+6} 2\right)\\ &= \text{mex}\left\{\mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+5} 2\right), \mathcal{G}^-\left(h_{24+40k} + \sum_{i=1}^{10k+5} 2\right) \oplus 1,\right.\\ &\quad \mathcal{G}^-\left(h_{62+40(k-1)} + \sum_{i=1}^{10k+6} 2\right), \mathcal{G}^-\left(h_{61+40(k-1)} + \sum_{i=1}^{10k+6} 2\right),\\ &\quad \left.\mathcal{G}^-\left(h_{60+40(k-1)} + \sum_{i=1}^{10k+6} 2\right), \mathcal{G}^-\left(h_{58+40(k-1)} + \sum_{i=1}^{10k+6} 2\right)\right\}\\ &= \text{mex}\left\{4, 5, \mathcal{G}^-\left(h_{62+40(k-1)} + \sum_{i=1}^{10k+6} 2\right), \mathcal{G}^-\left(h_{61+40(k-1)} + \sum_{i=1}^{10k+6} 2\right),\right.\\ &\quad \left.\mathcal{G}^-\left(h_{60+40(k-1)} + \sum_{i=1}^{10k+6} 2\right), \mathcal{G}^-\left(h_{58+40(k-1)} + \sum_{i=1}^{10k+6} 2\right)\right\}\\ &\quad \text{by Equation (2)}\end{aligned}$$

$$= \text{mex}\{4,5,1,2,2,0\} \text{ by (IH1)}$$
$$= 3,$$

as required. Similarly, $g_{10k+7} = 1$ and $g_{10k+8} = 3$.

By (IH1), the genus of each of the options has stabilised by this index and the genus of h_{24+40k} has exhibited stabilising behaviour. Thus, $h_{24+40k} = 2^{\left\{\{43131\}^2\{42020\}^2\right\}^n 43131\ 431}$. □

Theorem 11 *Let $G = 0.122\ 213$. Then the genus sequence of the heaps is not periodic.*

Proof. Consider heaps h_{24+40n} for $n \in \mathrm{Z}^{\geq 0}$. By Proposition 10,

$$\Gamma\left(h_{24+40n}\right) = 2^{\left\{\{43131\}^2\{42020\}^2\right\}^n 43131\ 431},$$

and for heaps of size twenty-four or greater, the only heaps whose genera have 2^{43} as their starting digits are heaps of the form h_{24+40k} for some $k \in \mathrm{Z}^{\geq 0}$. Thus, if the genus sequence of the heaps is periodic, there exists $N \in \mathrm{Z}^{\geq 0}$, such that for all $n, m \geq N$, $\Gamma\left(h_{24+40n}\right) = \Gamma\left(h_{24+40m}\right)$.

We claim that for $n \in \mathrm{Z}^{\geq 0}$, there does not exist $m \in \mathrm{Z}^{\geq 0}$ with $m \neq n$ such that $\Gamma\left(h_{24+40n}\right) = \Gamma\left(h_{24+40m}\right)$. Fix n, $m \in \mathrm{Z}^{\geq 0}$ with $n \neq m$, and suppose, without loss of generality, that $n < m$. By Proposition 10,

$$\mathcal{G}^-\left(h_{24+40m} + \sum_{i=1}^{10m+5} 2\right) = 4,$$

while

$$\mathcal{G}^-\left(h_{24+40n} + \sum_{i=1}^{10m+5} 2\right) = 1.$$

Therefore the genera of h_{24+40m} and h_{24+40n} cannot be the same if n does not equal m as there exists digits genera of h_{24+40m} and h_{24+40n} which are not equal. Hence, the genus sequence of the heaps is not periodic for 0.122 213. □

We see now that there can be no comparable periodicity result to Theorem 6 for quaternary games in general.

3. Conclusion

We conclude with an open question regarding the periodicity of the genus sequence of the heaps for finite quaternary games: Is there a method of classification to determine which finite quaternary games have their genus sequence of the heaps periodic versus those which do not other than through manual calculations similar to those given in the proof of Proposition 10? Perhaps an analysis of quaternary games under the misère quotient ([10], [11]) will yield the answer.

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