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CHARACTERIZATION OF SUBSTITUTION INVARIANT WORDS CODING EXCHANGE OF THREE INTERVALS

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Abstract

We study infinite words coding an orbit under an exchange of three intervals which have full complexity $\mathcal{C}(n) = 2n+1$ for all $n \in \mathbb{N}$ (non-degenerate 3iet words). In terms of parameters of the interval exchange and the starting point of the orbit we characterize those 3iet words which are invariant under a primitive substitution. Thus, we generalize the result recently obtained for Sturmian words.

1. Introduction

We study invariance under substitution of infinite words coding exchange of three intervals with permutation $\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 1$, denoted by (3,2,1). These words, which are here called 3iet words, are one of the possible generalizations of Sturmian words to a three-letter alphabet. Our main result provides necessary and sufficient conditions on the parameters of a 3iet word to be invariant under substitution.

A Sturmian word $(u_n)_{n\in\mathbb{N}}$ over the alphabet $\{0,1\}$ is defined as

$$u_n = \lfloor (n+1)\alpha + x_0 \rfloor - \lfloor n\alpha + x_0 \rfloor \qquad \text{for all } n \in \mathbb{N},$$

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or

$$u_n = \lceil (n+1)\alpha + x_0 \rceil - \lceil n\alpha + x_0 \rceil \qquad \text{for all } n \in \mathbb{N},$$

where $\alpha \in (0,1)$ is an irrational number called the slope, and $x_0 \in [0,1)$ is called the intercept.

There are many various equivalent definitions of Sturmian words, among others also as an infinite word coding an exchange of 2 intervals of length α and $1-\alpha$. A direct generalization of this definition are infinite words coding exchange of k intervals, as introduced by Stepin and Katok [12].

Definition 1.1. *Let $\alpha_1, \ldots, \alpha_k$ be positive real numbers and let π be a permutation over the set $\{1, 2, \ldots, k\}$. Denote $I = I_1 \cup I_2 \cup \cdots \cup I_k$, where $I_j := \big[\sum_{i<j} \alpha_i, \sum_{i\leq j} \alpha_i\big)$. Put $t_j := \sum_{\pi(i)<\pi(j)} \alpha_i - \sum_{i<j} \alpha_i$. The mapping $T : I \mapsto I$ given by the prescription*

$$T(x) = x + t_j \quad \textit{for } x \in I_j$$

will be called k-interval exchange transformation (k-iet) with permutation π and parameters $\alpha_1, \ldots, \alpha_k$.

Note that usually one defines a k-iet in a less general way, where $I = [0,1)$, since scaling of the interval I does not influence properties of the corresponding transformation. On the other hand, one can give a more general definition: Having any affine transformation of the interval I, say $A(x) := \mu x + \nu$, consider the transformation ATA^{-1} instead of T. This is a modification which will be useful in our paper. It is convenient for us to study the orbit of 0 in a general interval I, instead of the orbit of a general point x_0 in $[0,1)$. This, in consequence, will allow us to express our main result in a nice way.

Keane [13] has studied under which assumptions a k-iet satisfies the so-called minimality condition, i.e., when the orbit $\{T^n(x_0) \mid n \in \mathbb{Z}\}$ of every point $x_0 \in I$ is dense in I. It is easy to see that the minimality condition can be satisfied only if the permutation π is irreducible, i.e., $\pi\{1,2,\ldots,j\} \neq \{1,2,\ldots,j\} \quad \text{for all } j < k$.

Keane has also derived a sufficient condition for minimality: Denote β_j the left boundary point of the interval I_j, i.e., $\beta_j = \sum_{i<j} \alpha_i$. If the orbits of points $\beta_1, \ldots, \beta_k$ under the transformation T are infinite and disjoint, then T satisfies the minimality property. In the literature, this sufficient condition is known under the notation i.d.o.c. However, in general, i.d.o.c. is not a necessary condition for the minimality property.

To the orbit of every point $x_0 \in I$, one can naturally associate an infinite word $u = (u_n)_{n\in\mathbb{Z}}$ in a k-letter alphabet $\mathcal{A} = \{1, 2, \ldots, k\}$. For $n \in \mathbb{Z}$ put

$$u_n = i \quad \text{if } \; T^n(x_0) \in I_i\,.$$

Infinite words coding k-iet with i.d.o.c. are called here *non-degenerate k-iet words.* Non-degenerate k-iet words are studied in [10]. The authors give a combinatorial characterization of the language of infinite words which correspond to a k-iet with the permutation

$$\pi(1) = k,\ \pi(2) = k-1,\ \ldots\ ,\ \pi(k) = 1 \tag{1}$$

or to permutations in some sense equivalent with it.

For $k = 2$, the only irreducible permutation is of the form (1). The minimality property for parameters α_1, α_2 means that they are linearly independent over $\mathbb{Q}$. Infinite words coding 2iet with the minimality property are precisely the Sturmian words.

In this paper we concentrate on infinite words coding exchange of 3 intervals under the permutation given in (1). The transformation which we study is thus given by a triple of positive parameters $\alpha_1, \alpha_2, \alpha_3$ and the prescription

$$\widetilde{T}(x) := \begin{cases} x + \alpha_2 + \alpha_3 & \text{for } \ x \in [0, \alpha_1)\,, \\ x - \alpha_1 + \alpha_3 & \text{for } \ x \in [\alpha_1, \alpha_1 + \alpha_2)\,, \\ x - \alpha_1 - \alpha_2 & \text{for } \ x \in [\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3)\,. \end{cases} \tag{2}$$

For such a transformation, the minimality property is equivalent to the following condition (as proved in [3]): numbers $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ are linearly independent over $\mathbb{Q}$.[2] It is known [1, 11] that infinite words coding (2) are non-degenerate if and only if (2) satisfies the minimality property and

$$\alpha_1 + \alpha_2 + \alpha_3 \notin (\alpha_1 + \alpha_2)\mathbb{Z} + (\alpha_2 + \alpha_3)\mathbb{Z}\,. \tag{3}$$

The central problem of this paper is the substitution invariance of given infinite words. For Sturmian words this question was extensively studied; Chapter 2. of [14] gives references to authors who gave some contributions to its solution. The complete answer to this question was first provided by Yasutomi [17] for one-directional Sturmian words, other proof of the same result is given in [6]. In [5] we have provided yet another proof valid for bidirectional Sturmian words. Crucial for stating this result is the notion of a Sturm number. The original definition of a Sturm number used continued fractions. In 1998, Allauzen [2] has provided a simple characterization of Sturm numbers: *A quadratic irrational number α with conjugate α' is called a Sturm number if $\alpha \in (0, 1)$ and $\alpha' \notin (0, 1)$.*

Theorem 1.2 ([5]). *Let α be an irrational number, $\alpha \in (0, 1)$, $x_0 \in [0, 1)$. The bidirectional Sturmian word with slope α and intercept x_0 is invariant under a primitive[3] substitution if and only if the following three conditions are satisfied:*

(i) *α is a Sturm number,*

(ii) *$x_0 \in \mathbb{Q}(\alpha)$,*

(iii) *$\min(\alpha', 1 - \alpha') \le x_0' \le \max(\alpha', 1 - \alpha')$, where x_0' denotes the image of x_0 under the Galois automorphism of the quadratic field $\mathbb{Q}(\alpha)$.*

[2]Let us mention that the question of expressing the minimality property in terms of parameters $\alpha_1, \dots, \alpha_k$ has not been solved for general k.

[3]Note that the only non-primitive substitution under which a Sturmian word can be invariant, is the identity.

Let us mention that one can also study a weaker property than substitution invariance; namely, substitutivity. For an infinite word u coding an exchange of k intervals, Boshernitzan and Carroll [8] have shown that the belonging of lengths of all intervals $I_1, \dots, I_k$ to the same quadratic field is a sufficient condition for substitutivity of u. For $k = 2$ in [7], and for $k = 3$ in [1], it is shown that such condition is also necessary.

However, quadraticity of parameters is not sufficient for the property of substitution invariance. Already in [4] it is shown that substitution invariance of 3iet words implies that a certain parameter of the 3iet, namely

$$\varepsilon = \frac{\alpha_1 + \alpha_2}{\alpha_1 + 2\alpha_2 + \alpha_3},$$

is a Sturm number. The main result of this paper is given as Theorem 6.3, where a necessary and sufficient condition for substitution invariance is expressed using simple inequalities for other parameters of the 3iet word.

Important tool for the proof of the theorem is the geometrical representation of an infinite word u coding an orbit of a 3iet $T : I \mapsto I$ with permutation $(3, 2, 1)$ by a cut-and-project sequence. This allows us to show that the first return map to any subinterval of I is again an exchange of intervals, namely a 3iet with permutation $(3, 2, 1)$ or a 2iet with permutation $(2, 1)$, (see Theorem 4.1). Then we use the result of [4] which states that substitution invariance of u forces T to be homothetic with the first return map of T to the interval λI, for a quadratic unit[4] λ in $\mathbb{Q}(\varepsilon)$. Fact that ε is a Sturm number is crucial in order that the orbit $T^n(0)$ be, under the Galois automorphism $x \mapsto x'$ in $\mathbb{Q}(\varepsilon)$, mapped to a strictly increasing sequence $\big(T^n(0)\big)'$. This is used to decide for which parameters of the 3iet there exists the above mentioned unit λ for which the 3iet T on I and its first return map on λI are homothetic.

2. Basic Notions of Combinatorics on Words

We will deal with infinite words over a finite alphabet, say $\mathcal{A} = \{1, 2, \dots, k\}$. We consider either right-sided infinite words

$$u = (u_n)_{n\in\mathbb{N}} = u_0u_1u_2u_3\cdots, \quad u_i \in \mathcal{A},$$

or *pointed bidirectional infinite words*,

$$u = (u_n)_{n\in\mathbb{Z}} = \cdots u_{-2}u_{-1}|u_0u_1u_2u_3\cdots, \quad u_i \in \mathcal{A},$$

A finite word $w = w_0w_1\cdots w_{n-1}$ of length $|w| = n$ is a *factor* of an infinite word $u = (u_n)$ if $w = u_iu_{i+1}\cdots u_{i+n-1}$ for some i.

[4] An algebraic number λ is a unit if both λ and λ^{-1} are algebraic integers.

The (factor) complexity of a one-sided infinite word $u = (u_n)_{n\in\mathbb{N}}$ is the function $\mathcal{C} : \mathbb{N} \mapsto \mathbb{N}$,

$$\mathcal{C}(n) := \#\{u_i \cdots u_{i+n-1} \mid i \in \mathbb{N}\}\,;$$

analogously we define it for a bidirectional infinite word $u = (u_n)_{n\in\mathbb{Z}}$. Obviously, every infinite word satisfies $1 \le \mathcal{C}(n) \le k^n$ for all $n \in \mathbb{N}$. It is not difficult to show [15] that an infinite word $u = (u_n)_{n\in\mathbb{N}}$ is eventually periodic if and only if there exists n_0 such that $\mathcal{C}(n_0) \le n_0$. Obviously, the aperiodic words of minimal complexity satisfy $\mathcal{C}(n) = n + 1$ for all $n \in \mathbb{N}$. Such infinite words are called Sturmian words. The definition of Sturmian words is extended to bidirectional infinite words $(u_n)_{n\in\mathbb{Z}}$, requiring except of $\mathcal{C}(n) = n + 1$ for all $n \in \mathbb{N}$ also the irrationality of the densities of letters.

In our paper we study invariance of infinite words under substitution. A substitution is a mapping $\varphi : \mathcal{A}^* \mapsto \mathcal{A}^*$, where $\mathcal{A}^*$ is the monoid of all finite words including the empty word, satisfying $\varphi(vw) = \varphi(v)\varphi(w)$ for all $v, w \in \mathcal{A}^*$. In fact, a substitution is a special case of a morphism $\mathcal{A}^* \mapsto \mathcal{B}^*$, where $\mathcal{A} = \mathcal{B}$. Obviously, φ is uniquely determined, if defined on all the letters of the alphabet. A substitution φ is called primitive, if there exists $n \in \mathbb{N}$ such that $\varphi^n(a)$ contains b for all letters $a, b \in \mathcal{A}$.

The action of φ can be naturally extended to infinite words. For a pointed bidirectional infinite word $u = (u_n)_{n\in\mathbb{Z}}$ we in particular have

$$\varphi(\cdots u_{-2}u_{-1}|u_0u_1u_2\cdots) \;=\; \cdots\varphi(u_{-2})\varphi(u_{-1})|\varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots$$

An infinite word u is said to be a fixed point of φ (or invariant under φ), if $\varphi(u) = u$.

3. Exchange of Three Intervals and Cut-and-project Sets

Our aim is to study substitution invariance of words coding an exchange of three intervals (2). The main tool is the fact that the orbit of an arbitrary point under this transformation can be geometrically represented by a so-called cut-and-project sequence.

Definition 3.1. *Let $\varepsilon, \eta \in \mathbb{R}$, $\varepsilon \ne -\eta$, ε, η irrational, and let $\Omega = [c, c + l)$, $c \in \mathbb{R}$, $l > 0$. The set*

$$\Sigma_{\varepsilon,\eta}(\Omega) := \{a + b\eta \mid a, b \in \mathbb{Z},\ a - b\varepsilon \in \Omega\} \tag{4}$$

is called a cut-and-project set with parameters ε, η and acceptance window Ω.

The above definition is a very special case of a general cut-and-project set, introduced in [16]. Here, points of the cut-and-project set are obtained by projection of chosen points of the square lattice $\mathbb{Z}^2$ onto the straight line $y = \varepsilon x$, along the line $y = -\eta x$. For projection we chose those points of $\mathbb{Z}^2$ which belong to a bounded strip parallel with the line $y = \varepsilon x$. The projection of such points onto the line $y = -\eta x$ along $y = \varepsilon x$ belongs to a bounded interval, determining the acceptance window. This construction is illustrated in Figure 1.

In the definition we have used an interval Ω, closed from the left and open from the right. One can also consider an interval $\hat{\Omega} = (\hat{c}, \hat{c} + \hat{l}]$. However, by doing this, we do not obtain anything new, since $\Sigma_{\varepsilon,\eta}(\Omega) = -\Sigma_{\varepsilon,\eta}(-\hat{\Omega})$.

For simplicity of notation, we denote the additive group

$$\{a + b\varepsilon \mid a, b \in \mathbb{Z}\} = \mathbb{Z} + \varepsilon\mathbb{Z} =: \mathbb{Z}[\varepsilon]$$

and analogously for $\mathbb{Z}[\eta]$. The morphism of these groups

$$x = a + b\eta \;\mapsto\; x^* = a - b\varepsilon \tag{5}$$

will be called the star map. In this formalism, the cut-and-project set $\Sigma_{\varepsilon,\eta}(\Omega)$ can be rewritten as

$$\Sigma_{\varepsilon,\eta}(\Omega) = \{x \in \mathbb{Z}[\eta] \mid x^* \in \Omega\}\,.$$

The relation between the set $\Sigma_{\varepsilon,\eta}(\Omega)$ and the exchange of 3 intervals is explained by the following theorem proved in [11].

Theorem 3.2 ([11]). *Let $\Sigma_{\varepsilon,\eta}(\Omega)$ be defined by (4). Then there exist positive numbers $\Delta_1, \Delta_2 \in \mathbb{Z}[\eta] := \mathbb{Z} + \eta\mathbb{Z}$ and a strictly increasing sequence $(s_n)_{n\in\mathbb{Z}}$ such that*

1. $\Sigma_{\varepsilon,\eta}(\Omega) = \{s_n \mid n \in \mathbb{Z}\} \subset \mathbb{Z}[\eta]$.
2. $\Delta_1^* > 0$, $\Delta_2^* < 0$, $\Delta_1^* - \Delta_2^* \geq l > \max(\Delta_1^*, -\Delta_2^*)$.
3. $s_{n+1} - s_n \in \{\Delta_1, \Delta_2, \Delta_1 + \Delta_2\}$, *for all* $n \in \mathbb{Z}$, *and, moreover,*

$$s_{n+1} = \begin{cases} s_n + \Delta_1 & \text{if} \quad s_n^* \in \; \Omega_1 := [c, c + l - \Delta_1^*)\,, \\ s_n + \Delta_1 + \Delta_2 & \text{if} \quad s_n^* \in \; \Omega_2 := [c + l - \Delta_1^*, c - \Delta_2^*)\,, \\ s_n + \Delta_2 & \text{if} \quad s_n^* \in \; \Omega_3 := [c - \Delta_2^*, c + l)\,. \end{cases}$$

4. *Numbers Δ_1 and Δ_2 depend only on parameters ε, η and the length l of the interval Ω. In particular, they do not depend on the position c of Ω on the real line.*

We see that the set $\{s_n^* \mid n \in \mathbb{Z}\}$ is an orbit under the 3iet with permutation $\pi = (3, 2, 1)$ and parameters $l - \Delta_1^*$, $\Delta_1^* - \Delta_2^* - l$ and $l + \Delta_2^*$ (if $l < \Delta_1^* - \Delta_2^*$), or it is an orbit under the 2iet with permutation $\pi = (2, 1)$ and parameters $l - \Delta_1^*$ and $l + \Delta_2^*$ (if $l = \Delta_1^* - \Delta_2^*$). Thus every cut-and-project sequence can be viewed as a geometric representation of an orbit of a point under exchange of two or three intervals. The construction of sequences $(s_n)_{n\in\mathbb{Z}}$, $(s_n^*)_{n\in\mathbb{Z}}$ and the role of numbers Δ_1, Δ_2 in the interval exchange is illustrated in Figure 1.

The determination of Δ_1, Δ_2 is in general laborious; the values Δ_1, Δ_2 depend on the continued fraction expansions of parameters ε or η, according to the length l of the acceptance window $\Omega = [c, c + l)$.

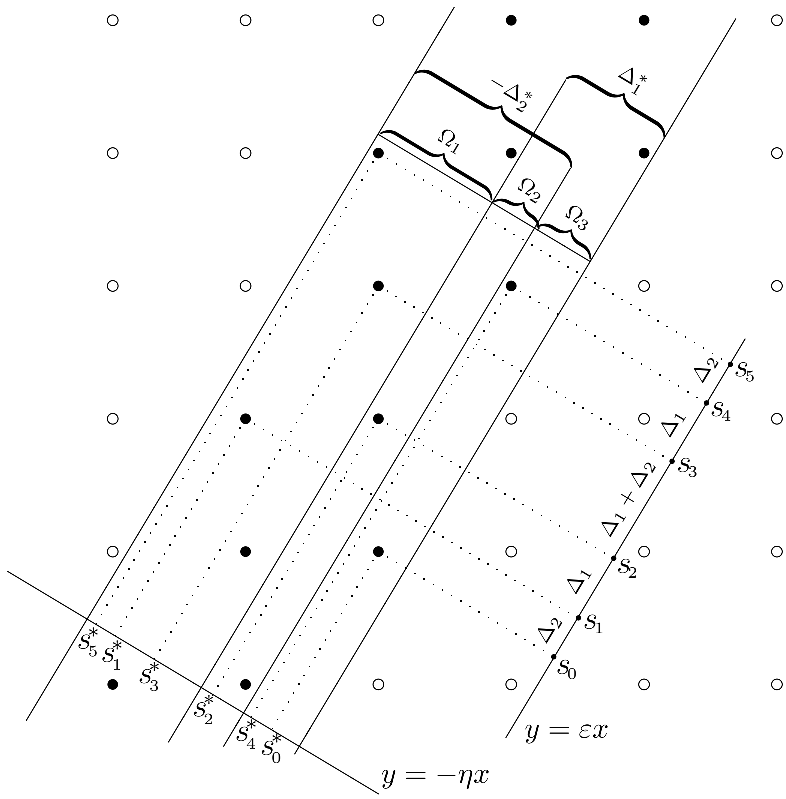

Figure 1: Construction of a cut-and-project sequence $(s_n)_{n\in\mathbb{Z}}$ and the corresponding interval exchange, see Theorem 3.2. For projection onto the line $y=\varepsilon x$ we use points of the lattice $\mathbb{Z}^2$ belonging to a bounded strip; they are marked by bullets. The strip is divided into three disjoint substrips: the presence of a lattice point in a substrip determines the distance of its projection s_n to the neighbour s_{n+1}. The projection of the entire strip onto the line $y=-\eta x$ determines the acceptance window Ω; the substrips correspond to subintervals Ω_i.

In case that

$$\varepsilon\in(0,1),\quad \eta>0\quad\text{and}\quad 1\ge l>\max(1-\varepsilon,\varepsilon)\,,\tag{6}$$

one has

$$\Delta_1=1+\eta\quad\text{and}\quad\Delta_2=\eta\,,\tag{7}$$

i.e., the corresponding triple of shifts in the prescription of the exchange of intervals is $\Delta_1^*=1-\varepsilon$, $\Delta_1^*+\Delta_2^*=1-2\varepsilon$, $\Delta_2^*=-\varepsilon$. In fact, without loss of generality, we can limit our consideration to cut-and-project sequences with parameters satisfying (6), since in [11] it is shown that every cut-and-project sequence is equal to $\mu\Sigma_{\varepsilon,\eta}(\Omega)$, where ε, η and length l of the interval Ω satisfy (6), and $\mu\in\mathbb{R}$. By that, we have shown how to interpret a cut-and-project set as an orbit under an exchange of 3 (or 2) intervals with the permutation (3,2,1) (or (2,1)).

On the other hand, let us show that every exchange of three intervals with permutation

(3,2,1) can be represented geometrically using a cut-and-project scheme. First realize that studying the orbit of a point $x_0 \in I$ under the 3iet $\widetilde{T}$ of (2), we can, without loss of generality, substitute $\widetilde{T}$ by the transformation $T(x) = \frac{1}{\mu}\widetilde{T}\big(\mu(x-c)\big) + c$ for arbitrary $\mu, c \in \mathbb{R}$, $\mu \neq 0$, and instead of the orbit of x_0 under $\widetilde{T}$ consider the orbit of the point $y_0 = c + \frac{x_0}{\mu}$ under the transformation T. In particular, putting $\mu = \alpha_1 + 2\alpha_2 + \alpha_3$ and $c = -x_0\mu^{-1}$, we have the orbit of $y_0 = 0$ under the mapping $T : [c, c+l) \mapsto [c, c+l)$

$$T(x) = \begin{cases} x + 1 - \varepsilon & \text{for } \ x \in I_1 := [c, c+l-1+\varepsilon)\,, \\ x + 1 - 2\varepsilon & \text{for } \ x \in I_2 := [c+l-1+\varepsilon, c+\varepsilon)\,, \\ x - \varepsilon & \text{for } \ x \in I_3 := [c+\varepsilon, c+l)\,, \end{cases} \tag{8}$$

where we have denoted by ε and l the new parameters

$$\varepsilon := \frac{\alpha_1 + \alpha_2}{\alpha_1 + 2\alpha_2 + \alpha_3} \quad \text{and} \quad l := \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha_1 + 2\alpha_2 + \alpha_3}\,. \tag{9}$$

Let us mention that under such parameters, the minimality property of the transformation T in (8) is equivalent to the requirement ε to be irrational.

For the above defined values of ε, l, c and arbitrary irrational $\eta > 0$ put $\Omega = [c, c+l)$ and consider the cut-and-project set $\Sigma_{\varepsilon,\eta}(\Omega)$. Since $0 \in \Omega$, we have also $0 \in \Sigma_{\varepsilon,\eta}(\Omega)$. The strictly increasing sequence $(s_n)_{n\in\mathbb{Z}}$ from Theorem 3.2 can be indexed in such a way that $s_0 = 0$. Since our parameters ε, l, η satisfy (6) (and $l < 1$), the right neighbor s_{n+1} of the point s_n is given by the position of s_n^* in the interval $[c, c+l)$, namely by the transformation $T(x)$. In particular, we have $s_{n+1}^* = T(s_n^*)$. Therefore the set

$$\{s_n^* \mid n \in \mathbb{Z}\} = \big(\Sigma_{\varepsilon,\eta}[c, c+l)\big)^* = \mathbb{Z}[\varepsilon] \cap \Omega$$

is the orbit of the point 0 under the transformation T.

Note that we have decided to consider instead of an orbit of an arbitrary point under a 3iet $\widetilde{T}$ with the domain being an interval starting at 0, the orbit of 0 under the 3iet T given by (8), with parameters ε, l, c satisfying

$$\varepsilon \in (0,1), \quad 1 > l > \max(1-\varepsilon, \varepsilon), \quad 0 \in [c, c+l)\,. \tag{10}$$

Let us summarize the advantages of this notation in the following proposition.

Proposition 3.3. *Let T be a 3iet given by* (8) *with parameters satisfying* (10).

- *For the orbit of an arbitrary point $z_0 \in [c, c+l)$ under T, one can write*

$$\{T^n(z_0) \mid n \in \mathbb{Z}\} = z_0 + \big(\mathbb{Z}[\varepsilon] \cap [c - z_0, c + l - z_0)\big) = (z_0 + \mathbb{Z}[\varepsilon]) \cap [c, c+l)\,. \tag{11}$$

  In particular, $\{T^n(0) \mid n \in \mathbb{Z}\} = \mathbb{Z}[\varepsilon] \cap [c, c+l)$.

- *For arbitrary irrational $\eta > 0$, denote the mapping $-* : \mathbb{Z}[\varepsilon] \mapsto \mathbb{Z}[\eta]$ given by*

$$x = a + b\varepsilon \ \mapsto \ x^{-*} = a - b\eta\,. \tag{12}$$

  Then the sequence $(s_n)_{n\in\mathbb{Z}}$ defined by $s_n = \big(T^n(0)\big)^{-*}$ is strictly increasing.

Further advantages of the presented point of view on 3iets by cut-and-project sequences will be clear from the following section.

Remark 3.4. *To conclude the section, let us stress that for the 3iet T the parameter η was chosen arbitrarily, except the requirement of irrationality and positiveness. Then adjacency of points x, y, $x < y$, in the set $\Sigma_{\varepsilon,\eta}(\Omega)$ indicates that their star map images x^*, y^* are consecutive iterations of T, i.e., $T(x^*) = y^*$. Choosing the parameter $\eta < 0$, we obtain again a cut-and-project set $\Sigma_{\varepsilon,\eta}(\Omega)$ but with different Δ_1, Δ_2. Therefore the corresponding 3iet is different from T. From the definition of a cut-and-project set, it can be easily shown that*

$$\Sigma_{\varepsilon,\eta}(\Omega) = \Sigma_{1-\varepsilon,1-\eta}(\Omega).$$

Therefore in case that $\eta < -1$, the corresponding cut-and-project set represents a 3iet, in which we interchange the lengths of the first and last intervals, i.e., the mapping T^{-1}. In fact, the 'dangerous' choice for the irrational parameter η is $\eta \in (-1, 0)$.

4. First Return Map

Let $T : I \mapsto I$ be a k-interval exchange transformation with minimality property and let J be an interval $J \subset I$, J closed from the left and open from the right, say $[\hat{c}, \hat{c} + \hat{l})$.

The minimality property of T ensures that for every $z \in J$ there exists a positive integer $i \in \mathbb{N}$ such that $T^i(z) \in J$. The minimal such i is called the return time of z and denoted by $r(z)$.

To every $z \in J$ we associate a 'return name', i.e., a finite word $w = v_0 v_1 \cdots v_{r(z)-1}$ in the alphabet $\{1, \ldots, k\}$, whose length is equal to the return time of z and for all i, $0 \le i < r(z)$ we have

$$v_i = X \quad \text{if} \quad T^i(z) \in I_X \, .$$

To the given subinterval J of I, we define the map $T_J : J \mapsto J$ by the prescription

$$T_J(z) = T^{r(z)}(z) \, ,$$

which is called the first return map.

Since for a fixed interval J the return time $r(z)$ is bounded, there exist only finitely many return names. It is obvious, that points $z \in J$ with the same return name form an interval, and J is thus a finite disjoint union of such subintervals, say $J_1, \ldots, J_p$. The boundary points of these intervals can be easily described by the notion of ancestor in J.

The minimality property of T ensures that for every $y \in I$ there exists $z \in J$ such that $y \in \{z, T(z), \ldots, T^{r(z)-1}(z)\}$. Such z is uniquely determined and we call it the ancestor of y in the interval J. We denote $z = \text{anc}_J(y)$.

The boundary points of the intervals $J_1, \ldots, J_p$ are then exactly the following points:

$$\begin{aligned} &\bullet\ \hat{c},\ \hat{c}+\hat{l} \text{ (i.e., the boundary points of } J \text{ itself)};\\ &\bullet\ \mathrm{anc}_J(\hat{c}+\hat{l});\\ &\bullet\ \mathrm{anc}_J(\alpha_1+\alpha_2+\cdots+\alpha_i) \ \text{ for } i=1,2,\ldots,k-1;\\ &\bullet\ \text{and the point } z \in J \text{ such that } T^{r(z)}(z)=\hat{c}. \end{aligned} \tag{13}$$

This implies that for a k-iet the number of different return names is at most $k+2$. It is obvious, that the first return map T_J is again a m-iet for some $m \le k+2$. In fact, it is known that $m \le k+1$ (see [9], Chap. 5). For a 3iet which we study in this paper, we can say even more. The following theorem is a consequence of Theorem 3.2 and Proposition 3.3.

Theorem 4.1. *Let $T : I \mapsto I$ be a 3iet with permutation (3,2,1) and satisfying minimality property, and let $J \subset I$ be an interval. Then the first return map T_J is either a 3iet with permutation (3,2,1) or a 2iet with permutation (2,1).*

Proof. Without loss of generality, consider a 3iet T given by (8), i.e. $I = [c, c+l)$, and let $\eta > 0$ be an arbitrary irrational. Then $\Sigma_{\varepsilon,\eta}[c, c+l)$ is a geometric representation of the orbit of 0 under T, and using Theorem 3.2 there exists a strictly increasing sequence $(s_n)_{n\in\mathbb{Z}}$ such that

$$\Sigma_{\varepsilon,\eta}[c, c+l) = \{s_n \mid n \in \mathbb{Z}\}.$$

Moreover, by Proposition 3.3, we have

$$\{T^n(0) \mid n \in \mathbb{Z}\} = \mathbb{Z}[\varepsilon] \cap [c, c+l) = \{s_n^* \mid n \in \mathbb{Z}\}\,.$$

Consider a subinterval $J := [\tilde{c}, \tilde{c}+\tilde{l}) \subset I$, and a cut-and-project set with parameters ε, η and acceptance window J, as in Definition 3.1. According to Theorem 3.2, there exists a strictly increasing sequence $(r_n)_{n\in\mathbb{Z}}$ such that

$$\Sigma_{\varepsilon,\eta}(J) = \{r_n \mid n \in \mathbb{Z}\}\,,$$

and r_{n+1}^* is the iteration of the point r_n^* under a transformation $\tilde{T} : J \mapsto J$. This transformation $\tilde{T}$ is either a 3iet with permutation $(3, 2, 1)$ or a 2iet with permutation $(2, 1)$. Formally, we have

$$r_{n+1}^* = \tilde{T}(r_n^*)\,.$$

Since $J \subset I$ implies $\{r_n \mid n \in \mathbb{Z}\} = \Sigma_{\varepsilon,\eta}(J) \ \subset\ \Sigma_{\varepsilon,\eta}(I) = \{s_n \mid n \in \mathbb{Z}\}$, there exists a strictly increasing sequence of indices $(k_n)_{n\in\mathbb{Z}}$ such that $r_n = s_{k_n}$, for all $n \in \mathbb{Z}$.

As $\{r_n^* \mid n \in \mathbb{Z}\} = \mathbb{Z}[\varepsilon] \cap J$, we must have $T^i(0) \in J$ if and only if $i = k_n$ for some $n \in \mathbb{Z}$. We see that the image of x under the first return map $T_J(x)$ coincides with $\tilde{T}(x)$ for arbitrary $x \in \mathbb{Z}[\varepsilon] \cap J$. Since $\mathbb{Z}[\varepsilon] \cap J$ is dense in J and a first return map to any interval is an interval exchange, we must have $T_J(x) = \tilde{T}(x)$ for all $x \in J$. $\square$

5. First Return Map and Substitution Invariance

Let us now see how the notions of first return map, return time and return name are related to substitution invariance of words coding 3iet. We will focus on non-degenerate 3iet words. Let us mention that non-degeneracy in terms of parameters ε, l of (9) means that $l \notin \mathbb{Z}[\varepsilon]$, cf. (3).

Consider a 3iet $T : [c, c+l) \mapsto [c, c+l)$ of (8) with parameters (10) and an interval $J \subset [c, c+l)$ such that $0 \in J$. Let $w_1, \ldots, w_p$ be all possible return names of points $z \in J$. Then the infinite word $u = (u_n)_{n\in\mathbb{Z}}$ coding 0 under the transformation T can be written as a concatenation

$$u = \cdots w_{j_{-2}} w_{j_{-1}} | w_{j_0} w_{j_1} w_{j_2} \cdots , \qquad \text{with} \;\; j_i \in \{1, \ldots, p\} . \tag{14}$$

The starting positions of the blocks w_{j_m} correspond to positions n in the infinite word u if and only if $T^n(0) \in J$.

Suppose we have an interval $J \subset I$, $0 \in J$ such that the first return map T_J satisfies

P1. T_J is homothetic with T, i.e.,

$$T_J(x) = \nu T(\tfrac{x}{\nu}) , \quad \text{for} \;\; x \in J \text{ and some } \;\; \nu \in (0, 1) ,$$

which means that T_J is an exchange of intervals $J_1 = \nu I_1$, $J_2 = \nu I_2$, and $J_3 = \nu I_3$;

P2. the set of return names defined by J has three elements.

Then the sequence of indices $(j_m)_{m\in\mathbb{Z}}$ defining the ordering of finite words w_1, w_2, w_3 in the concatenation (14) equals to the infinite word u. In particular, it means that u is invariant under the substitution

$$\begin{array}{lcl} 1 & \mapsto & \varphi(1) = w_1, \\ 2 & \mapsto & \varphi(2) = w_2, \\ 3 & \mapsto & \varphi(3) = w_3. \end{array}$$

The following example shows that a 3iet T with the domain I and a subinterval $J \subset I$ with properties P1. and P2. exist.

Example 5.1. *Consider $\varepsilon = \frac{1}{2}(\sqrt{5} - 1)$ and $l = \frac{1}{2}(1 + \varepsilon)$, and $c = -\varepsilon$. The transformation $T : I \mapsto I$, where $I = \big[-\varepsilon, \frac{1}{2}(1 - \varepsilon)\big)$, is thus the exchange of intervals*

$$I_1 = \big[-\varepsilon, -\tfrac{1}{2}(1 - \varepsilon)\big) , \quad I_2 = \big[-\tfrac{1}{2}(1 - \varepsilon), 0\big) , \quad \textit{and} \quad I_3 = \big[0, \tfrac{1}{2}(1 - \varepsilon)\big) .$$

Choosing the subinterval $J = \varepsilon^6 I$, we can easily verify that T_J is homothetic with T and the set of return names has three elements, namely

$$\begin{array}{lll} \textit{for } z \in J_1 = \varepsilon^6 I_1 & \textit{the return name is} & w_1 = 213121311131213121 ; \\ \textit{for } z \in J_2 = \varepsilon^6 I_2 & \textit{the return name is} & w_2 = 21312131212131213111312131 21 ; \\ \textit{for } z \in J_3 = \varepsilon^6 I_3 & \textit{the return name is} & w_3 = 31131213121 . \end{array}$$

Therefore the infinite word u coding the orbit of 0 under the transformation T is invariant under the substitution $\varphi(i) \mapsto w_i$, $i = 1, 2, 3$.

We stand therefore in front of the following questions: How to decide, for which 3iets a subinterval $J \subset I$ with properties P1. and P2. exists? What can be said in case that such J does not exist?

In case that $u = (u_n)_{n\in\mathbb{Z}}$ is a non-degenerate 3iet word coding the orbit of 0 under the transformation T defined by (8), the second question is solved by the paper [4], as follows.

The existence of a substitution φ over the alphabet $\{1, 2, 3\}$, under which the word u is invariant, means that u can be written as a concatenation of blocks $\varphi(1)$, $\varphi(2)$, $\varphi(3)$, i.e.,

$$u \quad = \quad \cdots u_{-2}u_{-1}|u_0u_1u_2\cdots \quad = \quad \cdots\varphi(u_{-2})\varphi(u_{-1}) \mid \varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots . \tag{15}$$

In [4] one considers a non-degenerate 3iet word u invariant under a primitive substitution φ and studies for $i = 1, 2, 3$ the set $E_{\varphi(i)}$ of points $T^n(0)$ such that the block $\varphi(i)$ starts at position n in the concatenation (15). Formally,

$$E_{\varphi(i)} = \{T^n(0) \mid \exists m \in \mathbb{Z},\ u_m = i \text{ and } \varphi(u_m)\varphi(u_{m+1})\varphi(u_{m+2})\cdots = u_nu_{n+1}u_{n+2}\cdots\}\,.$$

As a result, several properties of a matrix of substitution φ are described. Recall that for a substitution φ over the alphabet $\mathcal{A} = \{1, 2, \ldots, k\}$ one defines the substitution matrix M_φ by

$$(M_\varphi)_{ij} = \text{ number of letters } i \text{ in the word } \varphi(j)\,, \quad 1 \le i, j \le k\,.$$

Such matrix has obviously non-negative integer entries and if the substitution φ is primitive, the matrix M_φ is primitive as well, and therefore one can apply the Perron-Frobenius theorem.

We summarize several statements of [4] in the following theorem.

Theorem 5.2 ([4]). *Let $u = (u_n)_{n\in\mathbb{Z}}$ be a non-degenerate 3iet word with parameters ε, l, c satisfying* (10)*. Let φ be a primitive substitution such that $\varphi(u) = u$. Then*

(i) ε is a Sturm number, i.e., ε is a quadratic irrational in $(0, 1)$ such that its algebraic conjugate ε' satisfies $\varepsilon' \notin (0, 1)$;

(ii) the dominant eigenvalue Λ of the matrix M_φ of the substitution φ is a quadratic unit in $\mathbb{Q}(\varepsilon)$;

(iii) the column vector $(1-\varepsilon, 1-2\varepsilon, -\varepsilon)^T$ is a right eigenvector of M_φ corresponding to the eigenvalue Λ', i.e., to the algebraic conjugate of Λ;

(iv) parameters c, l belong to $\mathbb{Q}(\varepsilon)$;

(v) $E_{\varphi(i)} = \Lambda'\big(I_i \cap \mathbb{Z}[\varepsilon]\big)$ for $i = 1, 2, 3$, $\mathbb{Z}[\varepsilon] := \mathbb{Z} + \varepsilon\mathbb{Z}$.

The statement (v) in particular says that the existence of a substitution φ under which a non-degenerate 3iet word u is invariant forces existence of an interval $J \subset I$ with properties P1. and P2. We have already explained that existence of an interval J with properties P1. and P2. forces substitution invariance. We have thus the following statement.

Proposition 5.3. *Let $u = (u_n)_{n\in\mathbb{Z}}$ be a non-degenerate 3iet word with parameters ε, l, c satisfying* (10). *Then there exists a primitive substitution φ under which u is invariant, if and only if there exists an interval $J \subset I$ with properties P1. and P2.*

Let us first derive two simple observations which complement results of [4].

Lemma 5.4. *For Λ, Λ' and ε from Theorem 5.2 we have*

$$\Lambda\mathbb{Z}[\varepsilon] = \Lambda'\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon]\,.$$

Proof. Statement (iii) of Theorem 5.2 implies

$$M_\varphi \begin{pmatrix} 1-\varepsilon \\ 1-2\varepsilon \\ -\varepsilon \end{pmatrix} = \Lambda' \begin{pmatrix} 1-\varepsilon \\ 1-2\varepsilon \\ -\varepsilon \end{pmatrix}\,.$$

Since M_φ is an integer matrix, we obtain from the third row of the above equality that $\Lambda'\varepsilon \in \mathbb{Z}[\varepsilon]$. Subtracting third row from the first one we get $\Lambda' \in \mathbb{Z}[\varepsilon]$. As $\mathbb{Z}[\varepsilon]$ is closed under addition, we have $\Lambda'\mathbb{Z}[\varepsilon] \subseteq \mathbb{Z}[\varepsilon]$.

Since Λ is a quadratic integer, we have $\Lambda + \Lambda' \in \mathbb{Z}$. This implies that $\Lambda \in \mathbb{Z} - \Lambda' \subset \mathbb{Z}[\varepsilon]$, whence $\Lambda\varepsilon \in \varepsilon\mathbb{Z} - \Lambda'\varepsilon \subset \mathbb{Z}[\varepsilon]$, and thus $\Lambda\mathbb{Z}[\varepsilon] \subseteq \mathbb{Z}[\varepsilon]$.

Now since Λ is a unit, we have $\Lambda\Lambda' = \pm 1$, and therefore multiplying $\Lambda\mathbb{Z}[\varepsilon] \subseteq \mathbb{Z}[\varepsilon]$ by Λ' we obtain $\mathbb{Z}[\varepsilon] \subseteq \Lambda'\mathbb{Z}[\varepsilon]$. $\square$

It is obvious that in our considerations, ε must be a quadratic irrational. When putting a 3iet with such a parameter into context of cut-and-project sets, we need to specify the slope of the second projection, i.e., the parameter η. Choosing $\eta = -\varepsilon'$, where ε' is the algebraic conjugate of ε, the star map $x = a + b\eta \mapsto x^* = a - b\varepsilon$ becomes the Galois automorphism in $\mathbb{Q}(\varepsilon)$. The Galois automorphism is a mapping of order 2, therefore $*$ and $-*$ given by (5) and (12) coincide. Instead of x^* we will use the notation $x' = a + b\varepsilon'$ if $x = a + b\varepsilon$, $a, b \in \mathbb{Q}$, as usual. Recall that for $x, y \in \mathbb{Q}(\varepsilon)$ we have

$$(x+y)' = x' + y' \quad \text{and} \quad (xy)' = x'y'\,.$$

With such notation, $\Sigma_{\varepsilon,-\varepsilon'}(\Omega)$ can be rewritten in the form

$$\Sigma_{\varepsilon,-\varepsilon'}(\Omega) = \{x \in \mathbb{Z}[\varepsilon'] \mid x' \in \Omega\}\,. \tag{16}$$

Lemma 5.5. *Let ε be a quadratic irrational and let Λ be a quadratic unit in $\mathbb{Q}(\varepsilon)$ such that*

$$\Lambda\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon] := \mathbb{Z} + \varepsilon\mathbb{Z}\,. \tag{17}$$

- *Then for any acceptance window Ω we have*

$$\Lambda\Sigma_{\varepsilon,-\varepsilon'}(\Omega) = \Sigma_{\varepsilon,-\varepsilon'}(\Lambda'\Omega)\,.$$

- *If moreover $\varepsilon' < 0$, $\Lambda > 1$, $\Lambda' \in (0, 1)$ and $T : [c, c + l) \mapsto [c, c + l)$ is a 3iet with parameters satisfying* (10), *then the first return map T_J for the interval $J = \Lambda'[c, c+l)$ is a 3iet homothetic with T.*

Proof. Since $\Lambda\Lambda' = \pm 1$, multiplying of (17) by Λ' leads to $\Lambda'\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon] = \mathbb{Z}[-\varepsilon]$. By algebraic conjugation we obtain $\Lambda\mathbb{Z}[\varepsilon'] = \mathbb{Z}[\varepsilon'] = \mathbb{Z}[-\varepsilon']$. Note that in general $\mathbb{Z}[\varepsilon] \neq \mathbb{Z}[\varepsilon']$. From (16) we obtain

$$\begin{aligned}\Lambda\Sigma_{\varepsilon,-\varepsilon'}(\Omega) = \Lambda\{x \in \mathbb{Z}[\varepsilon'] \mid x' \in \Omega\} = \{\Lambda x \in \mathbb{Z}[\varepsilon'] \mid \Lambda' x' \in \Lambda'\Omega\} = \\ = \{y \in \mathbb{Z}[\varepsilon'] \mid y' \in \Lambda'\Omega\} = \Sigma_{\varepsilon,-\varepsilon'}(\Lambda'\Omega).\end{aligned}$$

This however means that the distances between adjacent elements of the cut-and-project set $\Sigma_{\varepsilon,-\varepsilon'}(\Lambda'\Omega)$ are Λ multiples of the distances between adjacent elements of the cut-and-project set $\Sigma_{\varepsilon,-\varepsilon'}(\Omega)$. Since the star map images (in our case the images under the Galois automorphism) of the distances between neighbors in a cut-and-project set correspond to translations in the corresponding 3iet (see Theorem 3.2), the factor of homothety between the two 3iets is Λ.

If the parameter $\eta = -\varepsilon' > 0$, the 3iet mappings corresponding to $\Sigma_{\varepsilon,-\varepsilon'}(\Omega)$ and $\Sigma_{\varepsilon,-\varepsilon'}(\Lambda'\Omega)$ are precisely T and T_J respectively, see Remark 3.4. ☐

Using Lemma 5.4 and statement (v) of Theorem 5.2,we obtain

$$E_{\varphi(i)} = (\Lambda' I_i) \cap \mathbb{Z}[\varepsilon] = (\Lambda' I_i) \cap \{T^n(0) \mid n \in \mathbb{Z}\}\,. \tag{18}$$

We are now in position to prove the main theorem of this section, which provides a necessary and sufficient condition for substitution invariance of a non-degenerate 3iet word.

Proposition 5.6. *Let u be a non-degenerate 3iet word with parameters ε, l, c, such that ε is a Sturm number having $\varepsilon' < 0$ and $l, c \in \mathbb{Q}(\varepsilon)$, $l \notin \mathbb{Z}[\varepsilon] := \mathbb{Z} + \varepsilon\mathbb{Z}$. Then u is invariant under a primitive substitution if and only if there exists a quadratic unit $\Lambda \in \mathbb{Q}(\varepsilon)$, $\Lambda > 1$, with conjugate $\Lambda' \in (0, 1)$, such that*

C1. $\Lambda\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon]$, *and*

C2. for the interval $J = \Lambda'[c, c+l)$*, one has*

$$\mathrm{anc}_J(c+\varepsilon), \mathrm{anc}_J(c+l-(1-\varepsilon)) \in \big\{\Lambda' c, \Lambda'(c+\varepsilon), \Lambda'(c+l-(1-\varepsilon))\big\}.$$

Proof. Let u be invariant under a primitive substitution φ. We search for Λ with properties C1. and C2. of the proposition. According to Theorem 5.2, the dominant eigenvalue of the matrix M_φ is a quadratic unit in $\mathbb{Q}(\varepsilon)$, i.e., its conjugate belongs to the interval $(-1, 1)$. If the conjugate is positive, we use for Λ the dominant eigenvalue of M_φ. Otherwise, since u is invariant also under the substitution φ^2, we take for Λ the dominant eigenvalue of the matrix $M_{\varphi^2} = M_\varphi^2$.

The validity of property C1. follows from Lemma 5.4. Equation (18) states that the interval $J = \Lambda' I$ defines only three return names and that the subintervals corresponding to these return names are $\Lambda' I_1$, $\Lambda' I_2$ and $\Lambda' I_3$. Since $I = [c, c+l)$, these are $\Lambda' I_1 = \big[\Lambda' c, \Lambda'(c+l-1+\varepsilon)\big)$, $\Lambda' I_2 = \big[\Lambda'(c+l-1+\varepsilon), \Lambda'(c+\varepsilon)\big)$, and $\Lambda' I_1 = \big[\Lambda'(c+\varepsilon), \Lambda'(c+l)\big)$. The list (13) defines the boundary points of subintervals determining the return names. Property C2. follows.

For the opposite implication, realize that by Lemma 5.5 property C1. ensures that T_J is a 3iet with subintervals $\Lambda'[c, c+l-1+\varepsilon)$ $\Lambda'[c+l-1+\varepsilon, c+\varepsilon)$, and $\Lambda'[c+\varepsilon, c+l)$. This, together with property C2., forces that points of the list (13) belong to the set $\{\Lambda' c, \Lambda'(c+\varepsilon), \Lambda'(c+l-1+\varepsilon)\}$, and thus the interval $J = \Lambda' I$ defines three return names. Hence according to Proposition 5.3, the infinite word u is invariant under a primitive substitution. □

Remark 5.7. *The proof of the above proposition directly implies that in case that* u *is invariant under a substitution* φ*, the scaling factor* Λ *from Proposition 5.6 can be taken to be the dominant eigenvalue of the substitution matrix* M_φ *or* $M_{\varphi^2} = M_\varphi^2$*.*

6. Characterization of Substitution Invariant 3iet Words

We now have to solve the question, when for a given Sturm number ε and parameters $c, l \in \mathbb{Q}(\varepsilon)$ satisfying (10) there exists Λ with properties C1. and C2. of Proposition 5.6. Finding Λ having the first of the properties is simple. For the comfort of the reader, we provide the following lemma with a short proof. More detailed demonstration can be found as Lemma 7.1 in [5].

Lemma 6.1. *Let* ε *be irrational, solution of the equation* $Ax^2 + Bx + C = 0$*. Then there exists a quadratic unit* $\Lambda \in \mathbb{Q}(\varepsilon)$ *such that*

$$\Lambda > 1,\ \Lambda' \in (0,1), \quad \textit{and} \quad \Lambda\mathbb{Z}[\varepsilon] = \Lambda'\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon]\,, \tag{19}$$

where $\mathbb{Z}[\varepsilon] := \mathbb{Z} + \varepsilon\mathbb{Z}$*.*

Proof. Let the pair of integers X, Y be a non-trivial solution of the Pell equation

$$X^2 - (B^2 - 4AC)Y^2 = 1\,.$$

Put $\gamma := X + BY + 2AY\varepsilon$. Using $A\varepsilon^2 = -B\varepsilon - C$, we easily verify that $\gamma\varepsilon \in \mathbb{Z}[\varepsilon]$. Using $A(\varepsilon + \varepsilon') = -B$ and $A\varepsilon\varepsilon' = C$, we derive that $\gamma\gamma' = 1$. This implies

$$\gamma\mathbb{Z}[\varepsilon] = \gamma'\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon].$$

Finally, we put $\Lambda = \max\{|\gamma|, |\gamma'|\}$. □

In Lemma 6.1 we have found Λ with property C1. It is more difficult to decide when Λ satisfies also property C2. of Proposition 5.6. By definition of the map T, it follows that x and $T(x)$ differ by an element of $\mathbb{Z}[\varepsilon]$. Therefore for arbitrary z_0 and its ancestor $\mathrm{anc}_J(z_0)$ we have $z_0 - \mathrm{anc}_J(z_0) \in \mathbb{Z}[\varepsilon]$. It is useful to introduce an equivalence on $\mathbb{Q}(\varepsilon)$ as follows. We say that elements $x, y \in \mathbb{Q}(\varepsilon)$ are equivalent if their difference belongs to $\mathbb{Z}[\varepsilon]$. Formally,

$$x - y \in \mathbb{Z}[\varepsilon] \qquad \Longleftrightarrow \qquad x \sim y\,.$$

For the parameters $c, l \in \mathbb{Q}(\varepsilon)$, one can find $q \in \mathbb{N}$ such that $c, l \in \frac{1}{q}\mathbb{Z}[\varepsilon]$. Clearly, $\mathrm{anc}_J(c + \varepsilon)$ and $\mathrm{anc}_J(c + l - 1 + \varepsilon)$ also belong to the set $\frac{1}{q}\mathbb{Z}[\varepsilon]$. The set to which belong ancestors of $c + \varepsilon$ and $c + l - 1 + \varepsilon$ can be restricted even more. For, the equivalence $\sim$ divides the set $\frac{1}{q}\mathbb{Z}[\varepsilon]$ into q^2 classes of equivalence of the form

$$T_{ij} := \frac{i + j\varepsilon}{q} + \mathbb{Z}[\varepsilon]\,, \qquad \text{where } 0 \le i, j \le q - 1\,.$$

Relation $\Lambda'\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon]$ implies

$$z \in \mathbb{Z}[\varepsilon] \qquad \Longleftrightarrow \qquad \Lambda' z \in \mathbb{Z}[\varepsilon]\,.$$

Therefore the mapping $\psi(T_{ij}) = \Lambda' T_{ij}$ is a bijection on the set of q^2 classes of equivalence. For every bijection ψ on a finite set, there exists an iteration $s \in \mathbb{N}$, $s \ge 1$, such that $\psi^s = \mathrm{id}$. Denoting $L := \Lambda^s$, the number L has obviously similar properties as Λ, namely

a) L is a quadratic unit in $\mathbb{Q}(\varepsilon)$;

b) $L > 1$, $L' \in (0, 1)$;

c) $L\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon]$;

and moreover

d) $L'\big(\frac{i+j\varepsilon}{q} + \mathbb{Z}[\varepsilon]\big) = \frac{i+j\varepsilon}{q} + \mathbb{Z}[\varepsilon]$, for all i, j, with $0 \le i, j \le q - 1$.

Having a quadratic unit Λ with properties of the number L in items a) – d), it is less difficult to decide about validity of the condition

$$\mathrm{anc}_J(c+\varepsilon), \mathrm{anc}_J(c+l-1+\varepsilon) \ \in \ \left\{\Lambda' c, \Lambda'(c+\varepsilon), \Lambda'(c+l-1+\varepsilon)\right\}. \tag{20}$$

Non-degeneracy of the infinite word u implies that $l \notin \mathbb{Z}[\varepsilon]$, and therefore $c+\varepsilon \ \not\sim \ c+l-1+\varepsilon$. Since for every $z_0 \in \frac{1}{q}\mathbb{Z}[\varepsilon]$ we have now

$$z_0 \ \sim \ \mathrm{anc}_J(z_0) \ \sim \ \Lambda' z_0\,,$$

the condition (20) in fact means

$$\mathrm{anc}_J(c+l-1+\varepsilon) = \Lambda'(c+l-1+\varepsilon) \tag{21}$$

and

$$\mathrm{anc}_J(c+\varepsilon) \ \in \ \left\{\Lambda' c, \Lambda'(c+\varepsilon)\right\}. \tag{22}$$

Lemma 6.2. *Let ε be a Sturm number with $\varepsilon' < 0$. Let $l, c \in \frac{1}{q}\mathbb{Z}[\varepsilon]$. Let Λ satisfy properties of L in a) – d) and let $J = \Lambda'[c, c+l)$. Then for arbitrary $z_0 \in \frac{1}{q}\mathbb{Z}[\varepsilon] \cap [c, c+l)$, one has*

$$\mathrm{anc}_J(z_0) = \Lambda' z_0 \qquad \Longleftrightarrow \qquad z_0' \le 0 \le (T(z_0))'\,.$$

Proof. The transformation T preserves the classes of equivalence and thus for the orbit of a point z_0 it holds that

$$\{T^n(z_0) \mid n \in \mathbb{Z}\} \ \subset \ z_0 + \mathbb{Z}[\varepsilon]\,.$$

As $(T^{n+1}(z_0) - T^n(z_0))' \in \{1-\varepsilon', 1-2\varepsilon', -\varepsilon'\}$, the assumption $\varepsilon' < 0$ implies that the sequence $(s_n)_{n\in\mathbb{Z}}$,

$$s_n := (T^n(z_0))'$$

is strictly increasing. By (11) we have moreover

$$\{T^n(z_0) \mid n \in \mathbb{Z}\} \ = \ \{s_n' \mid n \in \mathbb{Z}\} \ = \ (z_0 + \mathbb{Z}[\varepsilon]) \cap [c, c+l)\,.$$

Since $0 \in [c, c+l)$ and $\Lambda' \in (0,1)$, we have $\Lambda'[c, c+l) \subset [c, c+l)$. This inclusion together with property d) implies

$$\{s_n' \mid n \in \mathbb{Z}\} \ \supset \ \Lambda'\Big((z_0 + \mathbb{Z}[\varepsilon]) \cap [c, c+l)\Big) \ = \ \{\Lambda' s_n' \mid n \in \mathbb{Z}\}\,.$$

The strictly increasing sequence $(\Lambda s_n)_{n\in\mathbb{Z}}$ is therefore a subsequence of the strictly increasing sequence $(s_n)_{n\in\mathbb{Z}}$. Thus there exists a unique index m such that

$$\Lambda s_m \le s_0 < s_1 \le \Lambda s_{m+1}\,. \tag{23}$$

For determination of the ancestor of the point $z_0 = s_0'$ by definition, we search for the maximal non-positive exponent $k \in \mathbb{Z}$ such that $T^k(z_0) \in \Lambda'[c, c+l)$, i.e., such that $T^k(z_0)$ is an element of the sequence $(\Lambda' s_n')_{n\in\mathbb{Z}}$. Since both $(s_n)_{n\in\mathbb{Z}}$ and $(\Lambda s_n)_{n\in\mathbb{Z}}$ are strictly increasing, we have $(T^k(z_0))' = \Lambda s_m$ and thus $\mathrm{anc}_J(s_0') = \Lambda' s_m'$. Denoting $s_m' = y_0$, equation (23) can be rewritten

$$\Lambda y_0' \leq z_0' < (T(z_0))' \leq \Lambda (T(y_0))' . \tag{24}$$

On the other hand, recall that $\{s_n' \mid n \in \mathbb{Z}\} = (z_0 + \mathbb{Z}[\varepsilon]) \cap [c, c+l)$ and the index m for which (23) holds, is determined uniquely. Therefore we can claim that $\mathrm{anc}_J(z_0) = \Lambda' y_0$ if and only if y_0 verifies inequalities (24). Thus $\mathrm{anc}_J(z_0) = \Lambda' z_0$ if and only if

$$\Lambda z_0' \leq z_0' < (T(z_0))' \leq \Lambda (T(z_0))' . \tag{25}$$

Note that strict inequality in the middle is trivial and it is satisfied by arbitrary z_0. Since $\Lambda > 1$ we have $\Lambda z_0' \leq z_0' \Leftrightarrow z_0' \leq 0$ and $(T(z_0))' \leq \Lambda(T(z_0))' \Leftrightarrow 0 \leq (T(z_0))'$, which completes the proof. $\square$

Theorem 6.3. *Let u be a non-degenerate 3iet word coding the orbit of the point x_0 under a 3iet with permutation (3,2,1) and parameters $\alpha_1, \alpha_2, \alpha_3$. Put*

$$\varepsilon := \frac{\alpha_1 + \alpha_2}{\alpha_1 + 2\alpha_2 + \alpha_3}, \quad l := \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha_1 + 2\alpha_2 + \alpha_3}, \quad \text{and} \quad c := \frac{-x_0}{\alpha_1 + 2\alpha_2 + \alpha_3} .$$

Then u is invariant under a primitive substitution if and only if

1. *ε is a Sturm number;*
2. *$c, l \in \mathbb{Q}(\varepsilon)$;*
3. $\min(\varepsilon', 1-\varepsilon') \leq -c' \leq \max(\varepsilon', 1-\varepsilon')$ *and* $\min(\varepsilon', 1-\varepsilon') \leq c' + l' \leq \max(\varepsilon', 1-\varepsilon')$.

Proof. Theorem 5.2 claims that items 1. and 2. are necessary conditions for existence of a primitive substitution under which u be invariant. Therefore we shall prove the following statement:

If ε is a Sturm number and $c, l \in \mathbb{Q}(\varepsilon)$, then u is invariant under a primitive substitution if and only if condition 3. holds.

Note that the infinite word

$$\cdots u_{-3} u_{-2} u_{-1} | u_0 u_1 u_2 \cdots$$

is substitution invariant if and only if

$$\cdots u_2 u_1 u_0 | u_{-1} u_{-2} u_{-3} \cdots$$

is substitution invariant. At the same time, $\cdots u_2 u_1 u_0 | u_{-1} u_{-2} u_{-3} \cdots$ is a 3iet word coding the transformation T^{-1}, i.e. the 3iet with parameters $1-\varepsilon, l, c$. The fact that ε is a Sturm number means either $\varepsilon' < 0$ or $\varepsilon' > 1$. Instead of parameters ε, l, c we can thus have $1-\varepsilon, l, c$, and therefore limit our study (without loss of generality) to Sturm number ε satisfying $\varepsilon' < 0$. In that case, inequalities in item 3. of the theorem are of the form

$$\varepsilon' \le c' + l' \le 1 - \varepsilon' \quad \text{and} \quad \varepsilon' \le -c' \le 1 - \varepsilon' \,. \tag{26}$$

Denote $q \in \mathbb{N}$, such that $c, l \in \frac{1}{q}\mathbb{Z}[\varepsilon]$. With the use of Lemma 6.1 we find Λ with properties of L given in a) – d). Put $J = \Lambda'[c, c+l)$. According to Proposition 5.6, for such Λ, substitution invariance of the infinite word u is equivalent to validity of relations (21) and (22). We show that these relations are equivalent to inequalities (26).

The first of inequalities (26) can be rewritten as

$$c' + l' - 1 + \varepsilon' \le 0 \le c' + l' - \varepsilon' = \big(T(c + l - 1 + \varepsilon)\big)' \,.$$

which is, applying Lemma 6.2, equivalent to

$$\operatorname{anc}_J(c + l - 1 + \varepsilon) = \Lambda'(c + l - 1 + \varepsilon) \,,$$

which is (21).

The second of inequalities (26), namely $\varepsilon' \le -c' \le 1 - \varepsilon'$, can be rewritten as

$$\varepsilon' \le -c' \le 0 \qquad \text{or} \qquad 0 < -c' \le 1 - \varepsilon' \,,$$

equivalently,

$$c' + \varepsilon' \le 0 \le c' = \big(T(c + \varepsilon)\big)' \qquad \text{or} \qquad c' < 0 \le c' + 1 - \varepsilon' = \big(T(c)\big)' \,.$$

Again, by Lemma 6.2,

$$\operatorname{anc}_J(c + \varepsilon) = \Lambda'(c + \varepsilon) \qquad \text{or} \qquad c \neq 0 \ \text{ and } \ \operatorname{anc}_J(c) = \Lambda'(c) \,.$$

For $c = 0$, it is obvious that condition (22) is automatically satisfied. For $c \neq 0$, it suffices to realize that $\operatorname{anc}_J(c) = \operatorname{anc}_J(c + \varepsilon)$, since $T(c + \varepsilon) = c$. □

Example 6.4. *In Example 5.1, we have seen a non-degenerate 3iet word invariant under a primitive substitution. Its parameters are* $\varepsilon = \frac{1}{2}(\sqrt{5} - 1)$*,* $l = \frac{1}{2}(1 + \varepsilon)$*, and* $c = -\varepsilon$*. In this case* $\min(\varepsilon', 1 - \varepsilon') = \varepsilon'$*, and* $\max(\varepsilon', 1 - \varepsilon') = 1 - \varepsilon'$*. It is straightforward to verify that* $-c' = \varepsilon'$ *and* $c' + l' = \frac{1}{2}(1 - \varepsilon')$ *satisfy the inequalities of Theorem 6.3.*

It is clear that the factor Λ of homothety influences the length of words used in the substitution. Whenever an infinite word u is invariant under a substitution φ, it is invariant also under all its powers, for which the substituted words are obviously longer. It can be shown that in Example 5.1 we have given the shortest substitution possible for given parameters. The following remark may be helpful when searching for such substitutions.

Remark 6.5. *Note that in the proof of the main theorem we have applied Lemma 6.2 only to points* $z_0 = c + l - 1 + \varepsilon$ *and* $z_0 = c + \varepsilon$*. Realize that in fact, we do not need that* Λ *satisfies property d) of* L*, i.e. that all classes of the equivalence* $\sim$ *are preserved when multiplied by* Λ*. It is sufficient that two of the classes are preserved, namely*

$$\Lambda'\big(c + \mathbb{Z}[\varepsilon]\big) = c + \mathbb{Z}[\varepsilon] \quad \text{and} \quad \Lambda'\big(c + l + \mathbb{Z}[\varepsilon]\big) = c + l + \mathbb{Z}[\varepsilon] \,.$$

This can be important when we search for minimal $\Lambda > 1$ *with desired properties.*

7. Characterization of Substitution Invariant 3iet Words Using Sturmian Words

Comparing Theorems 6.3 and 1.2 we immediately see a striking narrow connection between 3iet words and Sturmian words, namely that the 3iet word $u = (u_n)_{n\in\mathbb{Z}}$ is invariant under a primitive substitution if and only if the Sturmian word with slope ε and intercept $-c$ and the Sturmian word with slope ε and intercept $l + c$ are both invariant under a primitive substitution.

In fact, as shown in [3], these two Sturmian words appear naturally as images of the given 3iet word by the following morphisms.

Let us denote by $\sigma_{01} : \{1, 2, 3\}^* \to \{0, 1\}^*$ the morphism given by

$$1 \mapsto 0\,, \qquad 2 \mapsto 01\,, \qquad 3 \mapsto 1\,, \tag{27}$$

and by $\sigma_{10} : \{1, 2, 3\}^* \to \{0, 1\}^*$ the morphism given by

$$1 \mapsto 0\,, \qquad 2 \mapsto 10\,, \qquad 3 \mapsto 1\,. \tag{28}$$

One verifies in [3] that if $u = (u_n)_{n\in\mathbb{Z}}$ is a non-degenerate 3iet word with parameters ε, l, c satisfying (10), then the infinite word

$$\sigma_{01}(u) \;=\; \cdots \sigma_{01}(u_{-2})\sigma_{01}(u_{-1}) \mid \sigma_{01}(u_0)\sigma_{01}(u_1)\sigma_{01}(u_2)\cdots$$

is the Sturmian word with slope ε and intercept $-c$ and the infinite word $\sigma_{10}(u)$ is the Sturmian word with slope $1 - \varepsilon$ and intercept $l + c$.

With the definition of morphisms σ_{01} and σ_{10}, we can give a characterization of substitution invariant non-degenerate 3iet words without use of any parameters.

Corollary 7.1. *Let $u = (u_n)_{n\in\mathbb{Z}}$ be a non-degenerate 3iet word coding an orbit under a 3iet with permutation (3,2,1). Then u is invariant under a primitive substitution if and only if both Sturmian words $\sigma_{10}(u)$ and $\sigma_{01}(u)$ are invariant under a primitive substitution.*

Let us mention that morphisms σ_{01} and σ_{10} can be used for characterizing 3iet words using Sturmian words, as proven in [4].

Theorem 7.2 ([4]). *Let u be a sequence on the alphabet $\{1, 2, 3\}$ whose letters have positive densities. The sequence u is an aperiodic 3iet word if and only if $\sigma_{01}(u)$ and $\sigma_{10}(u)$ are Sturmian words.*

Since the whole paper [4] is devoted to the study of one-directional infinite words, the above theorem applies to words $u = (u_n)_{n\in\mathbb{N}}$. However, a slight modification of the proof can be made so that the statement holds also for bidirectional words.

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