HEREDITARY TILING SETS OF THE INTEGERS

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Abstract

A set of integers A is said to tile $\mathbb{Z}$ if there exists another set of integers B such that for every integer n there exist unique integers $a \in A$ and $b \in B$ with $n = a + b$. In this paper, we show the existence of an infinite tiling set, $A \subset \mathbb{Z}$, with the hereditary property that every infinite subset $A' \subset A$ also tiles $\mathbb{Z}$. In particular, we show that every sequence which increases "fast enough" is a tiling set.

1. Introduction

A set of integers A *tiles the integers* $\mathbb{Z}$ if there is a *complementing set* $B \subset \mathbb{Z}$ such that every integer $n \in \mathbb{Z}$ can be uniquely written as a sum $n = a + b$ with $a \in A$ and $b \in B$. We are interested in subsets A' of a tiling set A and whether or not these subsets also tile the integers. For example, consider the finite set $A = \{0, 1, 2\}$. This tiles the integers, and every (non-empty) subset of it also tiles the integers. On the other hand, consider the finite set $A = \{0, 1, 2, 3\}$. This again tiles the integers, but the subset $A' = \{0, 1, 3\}$ does not. It is easy to generalize this and show that any set of four integers that tiles has a subset of three integers that does not tile. Hence, we are only concerned with infinite sets A and infinite subsets A' of A. We will prove the following result.

Theorem 1 *There exists a subset $A \subset \mathbb{Z}$ with the hereditary property that every infinite subset $A' \subset A$ again tiles $\mathbb{Z}$.*

Theorem 1 follows immediately from the following theorem.

Theorem 2 *If the increasing sequence of integers $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ satisfies $a_i > 2a_{i-1} + 2i$ for all $i \geq 1$ then A tiles the integers.*

Before proceeding, we mention that the result does not hold for $\mathbb{N} = \{0, 1, 2, \cdots\}$. That is, if the infinite set A tiles $\mathbb{N}$ then it has subsets that do not tile $\mathbb{N}$ (and it also has subsets that do); see Theorems 7 and 8. In particular, removing any non-zero integer from A results in a subset A' which does not tile $\mathbb{N}$. We will prove this in an appendix to the paper. The methods used are entirely different and do not depend on the rest of the paper.

We note that the proof of Theorem 1 does not give any information on the complements of A or A'.

2. Hereditary Property

In this section we present Property (H) and Property (HS) which are crucial to proving the result.

Definition 1 *We say the set of integers $A = \{0 = a_0, a_1, a_2, \cdots\}$ satisfies Property (H) if the following is satisfied:*

$$(H) \quad \begin{cases} \text{for } i, j, k, l = 0, 1, 2, \cdots\ ,\ i \neq j, \quad |k'| \leq k, \quad |l'| \leq l\,, \quad \text{we have} \\ \qquad a_i - a_k + k' \neq a_j - a_l + l' \\ \ \text{if one of the following holds :} \\ \bullet \text{ any one of the indices } \{i, j, k, l\} \text{ is greater than all the others;} \\ \bullet\ i = l \text{ and } i > \{j, k\} \\ \bullet\ j = k \text{ and } j > \{i, l\}. \end{cases}$$

Remark 1 *The conditions $0 \leq |k'| \leq k$ and $0 \leq |l'| \leq l$ make Property (H) hereditary. That is, Property (H) is inherited by infinite subsequences $A' \subset A$ with the caveat that the subsequence is re-indexed $A' = \{a'_0, a'_1, a'_2, \cdots\}$ and shifted if necessary to begin at $0 = a'_0$. Neither shifting nor re-indexing has any impact on tiling.*

Notation. $A \oplus B = \{a + b : a \in A, b \in B\}$ indicates that the sum is **direct**. That is, for $a, a' \in A$ and $b, b' \in B$, $a + b = a' + b' \Rightarrow a = a'$ and $b = b'$. It is clear that shifting does not affect directness, *i.e.* $A \oplus B$ if and only if $A \oplus (B + k)$, and we will assume our sequences contain 0 as a matter of convenience.

Theorem 3 *If A satisfies Property (H), then there exists a sequence W satisfying $A \oplus W = \mathbb{Z}$. In addition, for every infinite subset $A' \subset A$ there exists a W' such that $A' \oplus W' = \mathbb{Z}$.*

Remark 2 *One should not expect too much relation between the sequences W and W'. Examine $F = \{0, 1, 2\}$. This has the complement $W = 3 \cdot \mathbb{Z}$. The subset $F' = \{0, 1\}$ has the complement $W' = 2 \cdot \mathbb{Z}$. Even though the subset F' was obtained by removing a single integer, the complement of F' is not obtained in such an easy manner from the complement of F.*

Property (H) can be strengthened in order to obtain some information on the complements.

Definition 2 *We say the set of integers $A = \{0 = a_0, a_1, \dots\}$ satisfies Property (HS) with the sequence B, if $0 \in B$ and the following is satisfied.*

$$(HS) \qquad \begin{cases} \text{for } i, j, k, l = 0, 1, 2, \dots\ ,\ i \neq j, \quad |k'| \le k, \quad |l'| \le l\,, \quad \text{we have} \\ \qquad\qquad (B + a_i - a_k + k') \cap (B + a_j - a_l + l') = \emptyset \\ \text{if one of the following holds :} \\ \bullet \text{ any one of the indices } \{i, j, k, l\} \text{ is greater than all the others;} \\ \bullet\ i = l \text{ and } i > \{j, k\} \\ \bullet\ j = k \text{ and } j > \{i, l\} \end{cases}$$

Remark 3 *Property (H) is a special case of Property (HS) and any set A satisfying Property (HS) also satisfies Property (H). Simply set $B = \{0\}$ in the expression*

$$(B + a_i - a_k + k') \cap (B + a_j - a_l + l') \tag{1}$$

Remark 4 *Looking ahead, when analyzing which of the $\{i, j, k, l\}$ are maximum, the cases not covered by Property (HS) are $i = k > \{j, l\}$, $j = l > \{i, k\}$, $k = l > \{i, j\}$, $i = k = l > j$ and $j = l = k > i$. (We will always have $i \neq j$, so there are no other cases.)*

Property (HS) extends Theorem 3.

Theorem 4 *If A satisfies Property (HS) with set B, then B can be extended to $W \supset B$ such that $W \oplus A = Z$. In addition, for every infinite subset $A' \subset A$, B can be extended to a complement $W' \supset B$ with $W' \oplus A' = Z$.*

Remark 5 *Property (HS) already makes the sum of A and B* direct, *$A \oplus B$. For $i \neq j$, setting $k = l = 0$ gives $B + a_i \cap B + a_j = \emptyset$. This means that $b + a_i = b' + a_j$ requires $i = j$ and $b = b'$. Similarly, Property (HS) gives $2A \oplus B$. Setting $i = l$, $j = k$, and $k' = l' = 0$ in (1) gives $(B + a_i - a_j) \cap (B + a_j - a_i)$. Either $i = l > \{j, k\}$ or $j = k > \{i, l\}$. In either case, Property (HS) makes the intersection empty. Shifting by $a_i + a_j$ gives $(B + 2a_i) \cap (B + 2a_j) = \emptyset$ and so all sums, $2a + b$, are unique.*

3. Proof of Theorems

We will prove Theorem 4 in three steps: Construct the set W, show the sum is direct, $A \oplus W$, and show the sum is complete, $A + W = \mathbb{Z}$. Theorem 3 will follow from Theorem 4. Theorem 1 will follow from Theorem 3 and Theorem 5, which is given in the next section.

3.1 Construction of the Set W

We begin the proof with the construction of the set W in Theorem 4. Put

$$W = \bigcup_{i=0}^{\infty} (C_i - a_i) \tag{2}$$

where the C_i are chosen inductively as follows:

$$C_0 = B \; ; \; C_1 = (B+1) \setminus \big(C_0 \cup (C_0 + a_1)\big) \; ; \; C_2 = (B-1) \setminus \textstyle\bigcup_{r=0}^{2} \bigcup_{s=0}^{1} (C_s - a_s + a_r).$$

Having chosen the sets $C_0, \cdots C_{p-1}$, define C_p as follows. If $p = 2q$, let

$$C_p = C_{2q} = (B - q) \setminus \bigcup_{r=0}^{p} \bigcup_{s=0}^{p-1} (C_s - a_s + a_r);$$

if $p = 2q - 1$, let

$$C_p = C_{2q-1} = (B + q) \setminus \bigcup_{r=0}^{p} \bigcup_{s=0}^{p-1} (C_s - a_s + a_r)$$

Notation. For convenience, when p is even, put $\nu(p) = -\frac{p}{2}$ and when p is odd, put $\nu(p) = \frac{p+1}{2}$. Set $B_p = C_p - \nu(p)$ which is a subset of B.

Remark 6 *The set W defined above is a disjoint union. To see this, observe that in the construction of the set C_p, setting $r = p$ shows that we are removing the sets $\cup_{s=0}^{p-1} (C_s - a_s + a_p)$. Hence $C_p \cap (C_s - a_s + a_p) = \emptyset$, and shifting by $-a_p$ gives $(C_p - a_p) \cap (C_s - a_s) = \emptyset$ for all $s < p$. Similarly, the sets C_p are pairwise disjoint since the sets C_s, $s < p$ were removed in the construction of C_p.*

3.2 Directness $W \oplus A$

Lemma 1 *$W \oplus A$, i.e., the sum is direct.*

Proof. We need to show that $(W + a_i) \cap (W + a_j) = \emptyset$ for $i \neq j$. For this, it is enough to show that $(C_k - a_k + a_i) \cap (C_l - a_l + a_j) = \emptyset$ for all k and l. We analyze this by considering which of the $\{i, j, k, l\}$ are maximum. First we analyze the cases not covered by Property (HS) (see Remark 4).

Suppose $i = k > \{j, l\}$. Then $(C_k - a_k + a_i) \cap (C_l - a_l + a_j)$ becomes $C_i \cap (C_l - a_l + a_j)$. This is empty by the definition of C_i - that is, $C_l - a_l + a_j$ was removed in the construction of C_i. The case $j = l > \{i, k\}$ is symmetric and also has an empty intersection.

Suppose $k = l > \{i, j\}$. Then $(C_k - a_k + a_i) \cap (C_l - a_l + a_j)$ becomes $(C_k - a_k + a_i) \cap (C_k - a_k + a_j)$. By shifting out the common $-a_k$ term we have $(C_k + a_i) \cap (C_k + a_j)$. Using $C_k = B_k + \nu(k)$ this becomes $(B_k + \nu(k) + a_i) \cap (B_k + \nu(k) + a_j)$. Shifting out the common $\nu(k)$ term leaves $(B_k + a_i) \cap (B_k + a_j) \subset (B + a_i) \cap (B + a_j)$ which is empty (Remark 5).

Finally, suppose $i = k = l > j$ and again using $C_i = B_i + \nu(i)$ we have

$$(C_k - a_k + a_i) \cap (C_l - a_l + a_j) = C_i \cap (C_i - a_i + a_j) = (B_i + \nu(i)) \cap (B_i + \nu(i) - a_i + a_j).$$

Shifting by $\nu(i) - a_i$ gives $(B_i + a_i) \cap (B_i + a_j)$. Since $B_i \subset B$ this intersection is empty, (Remark 5). The symmetric case $j = l = k > i$ also has an empty intersection.

All other cases will follow from Property (HS). Specifically, start with $(C_k - a_k + a_i) \cap (C_l - a_l + a_j)$ and substitute as before $C_p = B_p + \nu(p)$ to get

$$(B_k + \nu(k) - a_k + a_i) \cap (B_l + \nu(l) - a_l + a_j)$$

The sets B_k, B_l are subsets of B. From the definition of $\nu(p)$ it follows $0 \le |\nu(p)| \le p$. Hence, Property (HS) applies and the lemma is proved. $\square$

3.3 Completeness, $W + A = \mathbb{Z}$

Lemma 2 $W + A = \mathbb{Z}$

Proof. We have

$$\begin{aligned} W + A \supset \bigcup_{r=0}^{2q} (W + a_r) &\supset \bigcup_{r=0}^{2q} \left(\bigcup_{i=0}^{2q} (C_i - a_i) + a_r \right) \\ &\supset \bigcup_{i=0}^{2q} C_i \cup \bigcup_{i \neq r}^{p-1} \left(C_i - a_i + a_r \right) \\ &\supset \bigcup_{j=-q}^{q} (B + j). \end{aligned}$$

Hence, $W + A \supset \bigcup_{j=-\infty}^{\infty} (B + j) = \mathbb{Z}$. $\square$

4. Sequences Increasing "Fast Enough"

In this section we show there are many sequences satisfying Property (H).

Theorem 5 *Let* $A = \{0 = a_0 < a_1 < a_2 \cdots\}$. *Suppose* $a_i > 2 \cdot a_{i-1} + 2 \cdot i$ for all $i \geq 1$. *Then A satisfies Property (H).*

We want to show the following inequality holds under the conditions stipulated by Property (H):

$$a_i - a_k + k' \neq a_j - a_l + l', \quad \text{for } i \neq j \tag{3}$$

In particular, we are interested in the cases (and their symmetric variations)

- $i > \{j, k, l\}$ (symmetric: $j > \{i, k, l\}$)
- $k > \{i, j, l\}$ (symmetric: $l > \{i, j, k\}$
- $i = l > \{j, k\}$ (symmetric: $j = k > \{i, l\}$.

Proof of Theorem 5. The first two cases are when there is one unique maximum among the i, j, k, l. By symmetry, there are only two cases.

Case $i > \{j, k, l\}$. We have

$$\begin{aligned} a_i &> 2 \cdot a_{i-1} + 2 \cdot i \\ &= a_{i-1} + a_{i-1} + i + i \\ &\geq a_j + a_k + i + i \quad \text{by } i > \{j, k\},\ a_{i-1} \geq \{a_j, a_k\} \\ &\geq a_j + a_k - a_l + i + i \quad \text{by } a_l \geq 0 \\ &> a_j + a_k - a_l + l' + i \quad \text{by } l' \leq l < i \\ &> a_j + a_k - a_l + l' - k' \quad \text{by } |k'| \leq k < i. \end{aligned}$$

Rearranging, we get $a_i - a_k + k' > a_j - a_l + l'$, so they are not equal.

Case $k > \{j, i, l\}$ is similar. We have

$$\begin{aligned} a_k &> 2 \cdot a_{k-1} + 2 \cdot k \\ &= a_{k-1} + a_{k-1} + k + k \\ &\geq a_i + a_l + k + k \quad \text{by } k > \{i, l\} \\ &\geq a_i + a_l - a_j + k + k \quad \text{by } a_j \geq 0 \\ &\geq a_i + a_l - a_j + k' + k \quad \text{by } k' \leq k \\ &> a_i + a_l - a_j + k' - l' \quad \text{by } |l'| \leq l < k, \end{aligned}$$

which by rearranging gives $a_j - a_l + l' > a_i - a_k + k'$.

Case $i = l > \{j, k\}$. Obviously $i = l \neq 0$ so $2a_i > a_i$. Hence we start with

$$\begin{aligned} 2a_i &> a_i \\ a_i + a_i &> 2 \cdot a_{i-1} + 2 \cdot i \\ a_i + a_l &> a_{i-1} + a_{i-1} + i + l \\ &\geq a_j + a_k + i + l \\ &\geq a_j + a_k + i + l' \\ &> a_j + a_k - k' + l'. \end{aligned}$$

Rearrangment yields $a_i - a_k + k' > a_j - a_l + l'$, so they are not equal. □

5. Appendix

In this appendix, we prove a couple of results mentioned in the introduction for which hereditary results do not apply.

5.1 Finite A

As mentioned in the beginning, the set $A = \{0, 1, 2, 3\}$ tiles the integers $\mathbb{Z}$ but the subset $A' = \{0, 1, 3\}$ does not. This can be generalized.

Theorem 6 *Any set of four integers, $A = \{0 = a_0, a_1, a_2, a_3\}$ has a subset of three integers which does not tile $\mathbb{Z}$.*

The proof uses Theorem 1 from Newman [3]. We don't require the full theorem just the following.

Proposition 1 (Newman) *Let $A = \{a, b, c\}$. Then A tiles the integers if and only if there is an integer $e \geq 0$ such that 3^e divides $a - b$, $a - c$ and $b - c$, but 3^{e+1} does not divide any of them.*

If the set A were normalized so that $A = \{0, a, b\}$, $gcd(a, b) = 1$, then the proposition says $A \equiv \{0, 1, 2\}$ *mod* 3 in some order.

Proof of Theorem 6. The proof is straightforward. If $g = gcd(a_1, a_2, a_3)$, then $A = \{0, a_1, a_2, a_3\}$ tiles the integers if and only if $\{0, \frac{a_1}{g}, \frac{a_2}{g}, \frac{a_3}{g}\}$ tiles the integers. Hence we can assume that $g = 1$.

Examine $\{0, a_1, a_2, a_3\}$ modulo 3. These cannot all be the same since the $gcd = 1$, nor can the four of them all be different. Therefore, we can choose a subset $\{a_i, a_j, a_k\}$ so that modulo 3, two of them are the same, $a_j \equiv a_k \; mod\, 3$, and the third is different $a_i \not\equiv a_j \; mod\, 3$. By the proposition, this subset does not tile the integers. □

5.2 $\mathbb{N}$ Has No Hereditary Result

We now show that there is no hereditary result similar to Theorem 1 for $\mathbb{N} = \{0, 1, 2, \cdots\}$. Specifically, we show the following result.

Theorem 7 *Given two infinite sequences of integers $A \oplus B = \mathbb{N}$, let $\tilde{a} \in A$. Define $A' = A \backslash \{\tilde{a}\}$. Then A' does not tile $\mathbb{N}$; that is, there does not exist C such that $A' \oplus C = \mathbb{N}$.*

The proof does not depend on any of the previous results but does depend on the structure of the set A. N. G. De Bruijn, [1], C. T. Long [2] and others, have studied the case $A \oplus B = \mathbb{N}$ and determined that A and B have a well defined structure. To illustrate, consider the dyadic representation of $\mathbb{N}$; that is, every $n \in \mathbb{N}$ can be written as $n = \epsilon_0 + \epsilon_1 2 + \epsilon_2 2^2 + \cdots + \epsilon_k 2^k$, $\epsilon_i = 0$ or 1 for $i = 0, 1, 2, \cdots, k$. Set $O = \{n \in \mathbb{N} :$ in the dyadic representation of $n,\ \epsilon_i = 0$ if $i =$ odd$\}$, and $E = \{n \in \mathbb{N} :$ in the dyadic representation of $n,\ \epsilon_i = 0$ if $i =$ even$\}$. Then, it is clear that $O \oplus E = \mathbb{N}$.

This example extends to more general bases. Namely, for $i \geq 1$, let $m_i \geq 2$ be a sequence of integers. Define $M_0 = 1$, and $M_k = \prod_{i=1}^{k} m_i$ for $k \geq 1$. This gives an $\{m_i\}$-adic representation; every $n \in \mathbb{N}$ can be written as $n = \epsilon_0 M_0 + \epsilon_1 M_1 + \epsilon_2 M_2 + \cdots + \epsilon_k M_k$, $0 \leq \epsilon_i < m_{i+1}$ for $0 \leq i \leq k$. Again, let $A = \{n \in \mathbb{N} :$ in the $\{m_i\}$-adic representation of $n,\ \epsilon_i = 0$ if $i =$ odd$\}$, and $B = \{n \in \mathbb{N} :$ in the $\{m_i\}$-adic representation of $n,\ \epsilon_i = 0$ if $i =$ even$\}$. Once more, $A \oplus B = \mathbb{N}$. The converse of this is also true. It has been shown that when $A \oplus B = \mathbb{N}$, then in some $\{m_i\}$-adic representation, the subsets A and B are obtained as described [1, 2].

Besides the structure described above, the proof of Theorem 7 uses the difference sets $A - A = \{a - a' : a, a' \in A\}$ and similarly $B - B$. Each sum $a + b$ being unique is equivalent to the intersection of the two difference sets being the singleton $\{0\}$, *i.e.*,

$$(A - A) \cap (B - B) = \{0\}.$$

Lemma 3 *If A' and $\tilde{a}$ are as above, then $A - A = A' - A'$.*

Proof. We need only show that $\tilde{a} - a \in (A' - A)$ for any $a \in A$. Suppose $\tilde{a} = \sum_{i=0}^{s} \epsilon_i M_i$ and let $a = \sum_{i=0}^{t} \epsilon_i M_i$. By the definition of A we may assume s, t are both even and $\epsilon_s, \epsilon_t \neq 0$. Let $r = max(s, t) + 2$. Then $a' = \tilde{a} + M_r$ and $a'' = a + M_r$ are both in A'. Thus, the difference $\tilde{a} - a = a' - a'' \in (A' - A')$. □

Proof of Theorem 7. If $A' \oplus C = \mathbb{N}$, then $\tilde{a} = a + c$ for some $a \in A'$ and $c \in C$. Since $\tilde{a} \notin A'$, we must have $c \neq 0$. Thus $\tilde{a} - a = c$. But, $\tilde{a} - a = a' - a'' = c = c - 0$ this means $c \in (A' - A') \cap (C - C)$ contradicting each sum in $A' \oplus C$ being unique. □

The same proof also shows the following.

Corollary 1 *The set A' does not tile $\mathbb{Z}$.*

The proof of Theorem 7 is easily extended to show that the removal of any finite set results in a non-tile. On the other hand, the structure of the set A when $A \oplus B = \mathbb{N}$, immediately supplies one method for finding infinite subsets of A which also tile. To illustrate we return to the dyadic example at the beginning of the section. define $O_1 = \{a \in O : \epsilon_i = 0 \text{ if } i \text{ MOD } 4 = 0\}$ and $O_2 = \{a \in O : \epsilon_i = 0 \text{ if } i \text{ MOD } 4 = 2\}$. Then $O = O_1 \oplus O_2$ and $O_1 \oplus (O_2 \oplus E) = \mathbb{N}$. We summarize this in the following.

Theorem 8 *If A is an infinite set which tiles $\mathbb{N}$, then there exists proper infinite subsets A' that also tile $\mathbb{N}$.*

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