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ON TOTIENT ABUNDANT NUMBERS

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Abstract

In this note, we find an asymptotic formula for the counting function of the set of totient abundant numbers.

1. Introduction

Let $\phi(n)$ be the Euler function of the positive integer n. Put

$$k(n) = \min\{k \ge 1 : \phi^{(k)}(n) = 1\},$$

where $f^{(k)}$ denotes the kth fold iteration of the function f. Put

$$F(n) = \sum_{k=1}^{k(n)} \phi^{(k)}(n).$$

If $\sigma(n)$ is the sum of divisors function, numbers n for which $\sigma(n) = 2n$ are called *perfect*. Results on perfect numbers are well documented; as of 2007, there are only 44 known perfect numbers. By analogy, numbers n for which $F(n) = n$ are called *perfect totients*. Their distribution was studied in [4], [6], [9], [10] and [11]. Although there are infinitely many perfect totients (for instance, 3^k is a perfect totient for any k), it was shown in [11] that the set of perfect totients has asymptotic density zero.

Abundant numbers are those for which $\sigma(n) > 2n$. Analogously, let us call a number n to be *totient abundant* if $F(n) > n$ and let us put $\mathcal{A}$ for the set of all totient abundant numbers. It is known that the abundant numbers have a positive density whose value is in the interval $[.2474, .2480]$ (see [1]). It follows from Theorem 2 in [6], that $\mathcal{A}$ is of asymptotic density zero. The following table shows the frequency of the totient abundant numbers in various intervals.

Interval	Frequency	Interval	Frequency
$[1, 10^3]$	383	$[10^9, 10^9 + 10^6]$	330491
$[1, 10^4]$	3708	$[10^{12}, 10^{12} + 10^6]$	323685
$[1, 10^5]$	35731	$[10^{15}, 10^{15} + 10^6]$	319049
$[1, 10^6]$	347505	$[10^{18}, 10^{18} + 10^6]$	315789
$[1, 10^7]$	3407290	$[10^{21}, 10^{21} + 10^6]$	313195
$[1, 10^8]$	33579303	$[10^{24}, 10^{24} + 10^6]$	310836

As the proportion of totient abundant numbers stays above 0.3 for quite large values of n, it would seem interesting to find an asymptotic formula for $\#\mathcal{A}(x)$ as $x \to \infty$, where $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$, unraveling the slow convergence towards zero of this proportion. Our result is the following (here, γ is the Euler constant):

Theorem 1. *The estimate*

$$\#\mathcal{A}(x) = (e^{-\gamma} + o(1)) \frac{x}{\log \log \log \log x} \tag{1}$$

holds as $x \to \infty$.

2. The Proof

Throughout this proof, we write $c_1, c_2, \ldots$ for computable positive constants. We also write $\log_k x$ for the function defined recursively by the formula $\log_k x = \max\{1, \log(\log_{k-1} x)\}$, where log is the natural logarithm. Note that $\log_k x$ coincides with the kth fold iterate of the natural logarithm function when x is large. When $k = 1$ we omit the subscript (but still assume that all logarithms that will appear are ≥ 1).

We start by eliminating a few subsets of positive integers $n \leq x$ whose counting functions are much smaller than what is shown in the right hand side of estimate (1). On the set of remaining $n \leq x$, we then show that $F(n) > n$ holds for a set of numbers $n \leq x$ of cardinality as predicted by (1).

Lemma 2 in [7], with its proof, shows that all $n \leq x$ have the property that $p \mid \phi(n)$ for all primes $p < c_1 \log_2 x / \log_3 x$ holds with $O(x/(\log_3 x)^2)$ exceptions in n. Let $\mathcal{A}_1(x)$ be the set of these exceptional $n \leq x$. From now on, we work with $n \leq x$ not in $\mathcal{A}_1(x)$.

For a positive integer m and a positive real number z we put

$$\omega_z(m) = \sum_{\substack{p \leq z \\ p \mid m}} 1$$

for the number of distinct prime factors p of m not exceeding z. When we omit the subscript we mean the total number of distinct prime factors of m.

Put $y = \log_2 x$. Let $\mathcal{A}_2(x)$ be the set of $n \le x$ such that $\omega(\phi(n)) > y^2$. It follows from the results from [2] that $\#\mathcal{A}_2(x) \ll x/y$. It also follows from the results from [2] (see page 349 in [2], for example) that if we put $\mathcal{A}_3(x)$ for the set of n such that $\omega_{y^3}(\phi(n)) > 2\log_2 x \log_2 y$, then $\#\mathcal{A}_3(x) \ll x/y$. From now on, we work with numbers $n \le x$ not in $\mathcal{A}_1(x) \cup \mathcal{A}_2(x) \cup \mathcal{A}_3(x)$.

Let $m = \phi(n)$. We find upper and lower bounds for $\phi(m)/m$. On the one hand, since $n \notin \mathcal{A}_1(x)$, we have

$$\begin{aligned} \frac{\phi(m)}{m} &\le \prod_{p \le c_1 \log_2 x/\log_3 x} \left(1 - \frac{1}{p}\right) \\ &= e^{-\gamma} \frac{1}{\log(c_1 \log_2 x/\log_3 x)} \left(1 + O\left(\frac{1}{\log_3 x}\right)\right) \\ &= \frac{e^{-\gamma}}{\log_3 x}\left(1 + O\left(\frac{\log_4 x}{\log_3 x}\right)\right), \end{aligned} \tag{2}$$

where we used Mertens's estimate

$$\prod_{p \le t}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log t}\left(1 + O\left(\frac{1}{\log t}\right)\right)$$

valid for all $t \ge 2$. On the other hand,

$$\frac{\phi(m)}{m} = \prod_{\substack{p\,|\,m \\ p \le y^3}}\left(1 - \frac{1}{p}\right)\prod_{\substack{p\,|\,m \\ p > y^3}}\left(1 - \frac{1}{p}\right). \tag{3}$$

The first product above contains at most $\ell = \lfloor 2\log_2 x \log_2 y \rfloor$ primes since $n \notin \mathcal{A}_3(x)$. Letting $p_1 < p_2 < \cdots < p_k < \cdots$ be the sequence of all the prime numbers, we get that

$$\begin{aligned} \prod_{\substack{p\,|\,m \\ p \le y^3}}\left(1 - \frac{1}{p}\right) &\ge \prod_{i=1}^{\ell}\left(1 - \frac{1}{p_i}\right) > \prod_{p \le \log_2 x (\log_3 x)^2}\left(1 - \frac{1}{p}\right) \\ &= \frac{e^{-\gamma}}{\log_3 x}\left(1 + O\left(\frac{\log_4 x}{\log_3 x}\right)\right) \end{aligned}$$

for large x, where in the above inequalities we used the Prime Number Theorem to conclude that the inequality $p_\ell < \log_2 x(\log_3 x)^2$ holds when x is large, as well as Mertens's estimate. As for the second product in (3), since $n \notin \mathcal{A}_2(x)$, we have that this product contains at most y^2 primes all exceeding y^3 so

$$\prod_{\substack{p\,|\,m \\ p > y^3}}\left(1 - \frac{1}{p}\right) > \left(1 - \frac{1}{y^3}\right)^{y^2} = \exp(O(1/y)) = 1 + O\left(\frac{1}{y}\right).$$

Thus,

$$\frac{\phi(m)}{m} \geq \frac{e^{-\gamma}}{\log_3 x}\left(1+O\left(\frac{\log_4 x}{\log_3 x}\right)\right), \tag{4}$$

which together with (2) shows that

$$\frac{\phi(m)}{m} = \frac{e^{-\gamma}}{\log_3 x}\left(1+O\left(\frac{\log_4 x}{\log_3 x}\right)\right).$$

Recall that a famous theorem of Linnik asserts that there exists a positive constant L such that whenever a and $b>1$ are coprime integers, the least prime number p in the arithmetic progression $a \pmod b$ satisfies the inequality $p \ll b^L$. The best known L appears in Theorem 6 in [5] and its value is 5.5. In particular, since our m is divisible by all primes $p \leq c_1 \log_2 x/\log_3 x$, it follows that for large x, $\phi(m)$ is divisible by all primes $\leq (\log_2 x)^{1/6}$. Hence, by the Mertens's formula once again,

$$\frac{\phi(\phi(m))}{\phi(m)} \leq \prod_{p\leq(\log_2 x)^{1/6}}\left(1-\frac{1}{p}\right) \ll \frac{1}{\log_3 x}.$$

Since $\phi^{(k)}(m)$ is even for all $k < k(n)$, it follows that $\phi^{(k+1)}(m)/\phi^{(k)}(m) \leq 1/2$ for all $k < k(n)$. Hence,

$$\sum_{k=2}^{k(n)} \phi^{(k)}(m) \leq \phi(\phi(m))\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right) \ll \phi(\phi(m)),$$

therefore

$$\sum_{k=1}^{k(n)} \phi^{(k)}(m) = \phi(m) + O(\phi(\phi(m))) = \phi(m)\left(1+O\left(\frac{1}{\log_3 x}\right)\right),$$

so

$$\begin{aligned} F(n) &= m+\phi(m)+\phi(\phi(m))+\ldots = m+\phi(m)\left(1+O\left(\frac{1}{\log_3 x}\right)\right) \\ &= m\left(1+\frac{e^{-\gamma}}{\log_3 x}\left(1+O\left(\frac{\log_4 x}{\log_3 x}\right)\right)\right). \end{aligned} \tag{5}$$

Hence, for $n \leq x$ not in $\mathcal{A}_1(x)\cup\mathcal{A}_2(x)\cup\mathcal{A}_3(x)$ we have that

$$F(n) = \phi(n)\left(1+\frac{e^{-\gamma}}{\log_3 x}\left(1+O\left(\frac{\log_4 x}{\log_3 x}\right)\right)\right).$$

Suppose now that $F(n) > n$. Then putting $p(n)$ for the smallest prime factor of n, we have that

$$1+\frac{e^{-\gamma}}{\log_3 x}\left(1+O\left(\frac{\log_4 x}{\log_3 x}\right)\right) > \frac{n}{\phi(n)} \geq 1+\frac{1}{p(n)-1},$$

giving $p(n) \ge c_2 \log_3 x$ for large x, where one can take c_2 to be any constant smaller than e^γ. Hence, $n \le x$ is coprime to all primes $p < c_2 \log_3 x$, and the number of such numbers is, via Eratosthenes's sieve and Mertens's formula,

$$= (1+o(1))x \prod_{p<c_2\log_3 x}\left(1-\frac{1}{p}\right) = (e^{-\gamma}+o(1))\frac{x}{\log_4 x},$$

which proves the upper bound (1) on $\mathcal{A}(x)$. Finally, for the lower bound on the set $\mathcal{A}(x)$, consider the set $\mathcal{A}_4(x)$ of $n \le x$ such that either $\omega(n) > 2y$, or $\omega_{y^2}(n) > (\log_2 y)^2$. The Túran-Kubilius inequalities (see, for example, [12]) assert that the estimate

$$\sum_{n\le x}(\omega(n)-\log_2 t)^2 = O(x\log_2 t)$$

holds uniformly in $2 \le t \le x$. Applying this with $t = x$ and $t = y^2$, we get easily that

$$\#\mathcal{A}_4(x) \ll \frac{x}{y} + \frac{x}{(\log_2 y)^3} \ll \frac{x}{(\log_4 x)^3}.$$

Put now $z = \log_3 x$. Consider numbers $n \le x$ coprime to all primes $p \le z(\log z)^{10}$ which do not belong to $\mathcal{A}_1(x) \cup \mathcal{A}_2(x) \cup \mathcal{A}_3(x) \cup \mathcal{A}_4(x)$. By the Eratosthenes's sieve and Mertens's formula, the number of such numbers n is

$$\begin{aligned} &\ge (1+o(1))x \prod_{p\le z(\log z)^{10}}\left(1-\frac{1}{p}\right) - \sum_{i=1}^{4}\#\mathcal{A}_i(x) \\ &= (e^{-\gamma}+o(1))\frac{x}{\log_4 x} + O\left(\frac{x}{(\log_4 x)^3}\right) \\ &= (e^{-\gamma}+o(1))\frac{x}{\log_4 x}, \qquad \text{as } x \to \infty. \end{aligned}$$

For such numbers,

$$\frac{n}{\phi(n)} = \prod_{\substack{p\le y^2\\ p\,|\,n}}\left(1+\frac{1}{p-1}\right)\prod_{\substack{p>y^2\\ p\,|\,n}}\left(1+\frac{1}{p-1}\right).$$

The first product contains at most $(\log_2 y)^2 < 2(\log z)^2$ primes all exceeding $z(\log z)^{10}$, therefore

$$\prod_{\substack{p\le y^2\\ p\,|\,n}}\left(1+\frac{1}{p-1}\right) < \exp\left(\frac{2(\log z)^2}{z(\log z)^{10}-1}\right) < 1+\frac{5}{z(\log z)^8}$$

for large x, where we used the fact that $1+t > e^{t/2}$ when $t \in (0,1/2)$. The second product contains at most $2y$ primes all exceeding y^2 so

$$\prod_{\substack{p>y^2\\ p\,|\,n}}\left(1+\frac{1}{p-1}\right) < \exp\left(\frac{2y}{y^2-1}\right) < 1+\frac{5}{y}.$$

Thus,

$$\frac{n}{\phi(n)} < \left(1 + \frac{5}{z(\log z)^8}\right)\left(1 + \frac{5}{y}\right) < 1 + \frac{1}{\log_3 x \log_4 x}$$

for large x, which together with estimate (5) shows that the numbers n such constructed are indeed totient abundant. This completes the proof of our theorem.

3. Comments

Let $\mathcal{F} = \{F(n) : n \in \mathbb{N}\}$ and put $\mathcal{F}(x) = \mathcal{F} \cap [1, x]$. In [11], Shparlinski observed that since the image of the map

$$\Psi : \{\phi(n) : n \in \mathbb{N}\} \longrightarrow \mathbb{N} : \qquad v \mapsto v + \phi(v) + \cdots + \phi^{k(v)}(v)$$

is the range of $\mathcal{F}$, it follows that the order of magnitude of $\#\mathcal{F}(x)$ is at most the order of magnitude of the cardinality of the set of totients not exceeding x, which is known to be

$$\frac{x}{\log x} \exp((c_3 + o(1))(\log \log \log x)^2)$$

with some positive constant c_3 (see [3] and [8]) as $x \to \infty$. Note that the above argument is not enough to decide whether the series

$$\sum_{f \in \mathcal{F}} \frac{1}{f}$$

is convergent or divergent, which is a problem we leave for the reader. It will also seem interesting to give a sharp lower bound on $\#\mathcal{F}(x)$. Pomerance, in a personal communication, notes that since $\phi(n) = F(n) - F(\phi(n))$, it follows that $\#\mathcal{F}(x) \gg x^{1/2+o(1)}$ as $x \to \infty$. It would seem interesting to improve the exponent 1/2.

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