# SOLITAIRE CLOBBER PLAYED ON HAMMING GRAPHS

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## Abstract

The one-player game *Solitaire Clobber* was introduced by Demaine *et al.* Beaudou *et al.* considered a variation called *SC2*. Black and white stones are located on the vertices of a given graph. A move consists in picking a stone to replace an adjacent stone of the opposite color. The objective is to minimize the number of remaining stones. The game is interesting if there is at least one stone of each color. In this paper, we investigate the case of Hamming graphs. We prove that game configurations on such graphs can always be reduced to a single stone, except for hypercubes. Nevertheless, hypercubes can be reduced to two stones.

## 1. Introduction and Definitions

We consider the one-player game *SC2* that was introduced in [3]. This game is a variation of the game *Solitaire Clobber* defined by Demaine et al. in [2]. Note that both solitaire games come from the two-player game *Clobber*, that was created and studied in [1]. One can have a look to [4] for more information about *Clobber*.

The game *SC2* is a solitaire game whose rules are described in the following. Initially, black and white stones are placed on the vertices of a given graph $G$ (one per vertex), forming what we call a *game configuration*. A move consists in picking a stone and "clobbering" (i.e.

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removing) another one of the opposite color located on an adjacent vertex. The clobbered stone is removed from the graph and is replaced by the picked one. The goal is to find a succession of moves that minimizes the number of remaining stones. A game configuration of *SC2* is said to be *k-reducible* if there exists a succession of moves that leaves at most $k$ stones on the board. The *reducibility value* of a game configuration $C$ is the smallest integer $k$ for which $C$ is k-reducible.

In [3], the game was investigated on cycles and trees. It is proved that in these cases, the reducibility value can be computed in quadratic/cubic time. In this paper, we play *SC2* on Hamming graphs.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *cartesian product* $G_1 \Box G_2$ is the graph $G = (V, E)$ where $V = V_1 \times V_2$ and $(u_1u_2, v_1v_2) \in E$ if and only if $u_1 = v_1$ and $(u_2, v_2) \in E_2$, or $u_2 = v_2$ and $(u_1, v_1) \in E_1$. One generally depicts such a graph with $|V_2|$ vertical copies of $G_1$, and $|V_1|$ horizontal copies of $G_2$, as shown on Fig. 1.

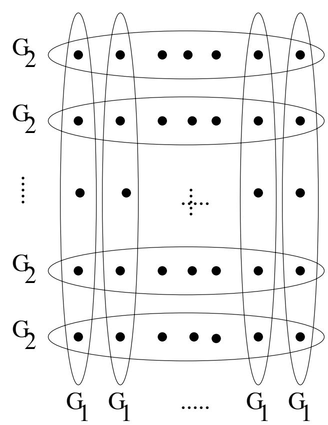


Figure 1: The cartesian product of two graphs $G_1$ and $G_2$

A *Hamming graph* is a multiple cartesian product of cliques. $K_2 \Box K_3$ and $K_4 \Box K_5 \Box K_2$ are examples of Hamming graphs. Hypercubes, defined by $\Box^n K_2$, constitute a well-known class of Hamming graphs.

For the convenience of the reader, we may often mix up a vertex and the stone that it supports. The label/color of a vertex will thus define the color of the stone on it. We may also say that "a vertex clobbers another one", instead of talking of the corresponding stones.

Given a game configuration $C$ on a graph $G$, we say that a label/color $c$ is *rare* on a subgraph $S$ of $G$ if there exists a unique vertex $v \in S$ such that $v$ is labeled $c$. On the contrary, $c$ is said to be *common* if there exist at least two vertices of this color in $S$. A configuration is said to be *monochromatic* if all the vertices have the same color. A monochromatic game configuration does not allow any move, so we now assume that a game configuration is never monochromatic.

Given $v$ a vertex of $G$, the color of the stone on $v$ will be denoted by $c(v)$. For a color $c$ (black or white), we denote by $\overline{c}$ the other color.

In this paper, we prove that we can reduce any game configuration (non monochromatic) on a Hamming graph to one or two stones. Moreover, we assert that we can choose the color and the location of the remaining stones. To facilitate the proofs, we make the following three definitions.

**Definition** We say that a graph $G$ is *strongly 1-reducible* if: for any vertex $v$, for any arrangement of the stones on $G$ (provided $G \setminus v$ is not monochromatic), for any color $c$ (black or white), there exists a way to play that yields a single stone of color $c$ on $v$.

A joker move consists of changing the color of any stone at any time during the game. It can be used only once.

Therefore, a graph $G$ is *strongly 1-reducible joker* if: for any vertex $v$, for any color $c$, for any arrangement of the stones on $G$ (provided $c(v)$ is not rare or $c(v) = c$), there exists a way to play that yields a single stone of color $c$ on $v$, with the possible use of a joker move.

**Definition** A graph $G$ is said to be *strongly 2-reducible* if: for any vertex $v$, for any arrangement of the stones on $G$ (provided $G \setminus v$ is not monochromatic), for any two colors $c$ and $c'$ (provided there exist two different vertices $u$ and $u'$ such that $c(u) = c$ and $c(u') = c'$), there exists a way to play that yields a stone of color $c$ on $v$, and (possibly) a second stone of color $c'$ somewhere else.

**Definition** Let $G$ be a graph, $v_i$ and $v_j$ two vertices of $G$, $c$ and $c'$ two colors belonging to $\{0, 1\}$. A game configuration $C$ on $G$ is said to be *1-reducible on $v_i$ with* $c$ or $(1, v_i, c)$-*reducible* if there exists a way to play that yields only one stone of color $c$ on $G$, located on $v_i$. A configuration $C$ is said to be *2-reducible on $v_i$ with $c$ and $c'$* or $(2, v_i, c, c')$-*reducible* if there exists a way to play that yields a stone of color $c$ on $v_i$, and (possibly) a second stone of color $c'$ on some other vertex. $C$ is said to be $(2, v_i, c, v_j, c')$-*reducible* if there exists a way to play that yields a stone of color $c$ on $v_i$ and a second stone of color $c'$ on $v_j$.

In the next section, we solve the case of *SC2* played on cliques. We prove in Proposition 1 that any clique of size at least 3 is strongly 1-reducible.

In Section 3, we play the game on hypercubes. We prove in Theorem 5 that hypercubes are both strongly 1-reducible joker and strongly 2-reducible, the proofs are intertwined. We also prove in Proposition 6 that any hypercube has a non-monochromatic configuration for which it is not 1-reducible. This somehow stresses the relevance ot Theorem 5.

Finally, in Section 4, we prove in Theorem 12 that all the Hamming graphs except hypercubes and $K_2 \square K_3$ are strongly 1-reducible. To prove this, we use a slightly stronger result in Theorem 8; we prove that if $G$ is a strongly 1-reducible graph containing at least 4 vertices, then the Cartesian product of $G$ with any clique is strongly 1-reducible.

## 2. *SC2* Played on Cliques

It is not very surprising that every game configuration on a clique is 1-reducible. Furthermore, we also prove that we can choose the color and the location of the single remaining stone.

**Proposition 1.** *Cliques of size $n \geq 3$ are strongly 1-reducible.*

When $n < 3$, note that cliques are 1-reducible, but we can't decide where and with which color we finish.

*Proof.* Let $C$ be a game configuration on $K_n$ ($n \geq 3$). Let $v$ be a vertex of $K_n$ such that $K_n \backslash v$ is not monochromatic. Let $c$ be any color in $\{0, 1\}$. We prove that $C$ is $(1, v, c)$-reducible:

First assume that $C$ contains no rare color. We consider two cases:

- ∗ if $c = c(v)$. By hypothesis, there exists a vertex $w$ labeled $\overline{c(v)}$. Since $c(v)$ and $c(w)$ are not rare, there exist two vertices $v'$ and $w'$ such that $c(v') = c(v)$ and $c(w') = c(w)$. The succession of moves leading to a single remaining stone is the following: $w$ clobbers $v$, $w'$ clobbers all the vertices with the label $c(v)$ except $v'$, and finally, $v'$ clobbers all the vertices labeled $\overline{c(v)}$, and ends on $v$.
- ∗ if $c = \overline{c(v)}$. As previously, there exist $w$ labeled $\overline{c(v)}$ and $v'$ labeled $c(v)$. $v'$ clobbers all the vertices labeled $\overline{c(v)}$ except $w$. Then $w$ clobbers all the vertices labeled $c(v)$ and ends on $v$.

Now assume that $C$ has a rare color located on a vertex $v_r \neq v$. If $c = c(v_r)$, then it is enough to have $v_r$ clobber all the vertices and finish on $v$. If $c = \overline{c(v_r)}$, have $v_r$ clobber all the vertices except one (call it $v' \neq v$) and finish on $v$. Then have $v'$ clobber $v$ and this concludes the proof. □

## 3. *SC2* Played on Hypercubes

In this section, we study *SC2* on hypercubes. We prove that these graphs are strongly 2-reducible.

Let $n > 2$. Note that $Q_n$ is defined recursively as the product $K_2 \square Q_{n-1}$, $Q_0$ being a single vertex. This means that $Q_n$ is made of two copies $Q_n^l$ and $Q_n^r$ of $Q_{n-1}$, where each vertex of $Q_n^l$ is adjacent to its copy in $Q_n^r$. Let $N = 2^{n-1}$. For each $i > 1$, it is well known that $Q_i$ admits a Hamiltonian cycle. Denote by $v_1, \ldots, v_N$ the vertices of $Q_n^l$, ordered such that $(v_1, \ldots, v_N)$ form a Hamiltonian cycle. Denote by $v'_1, \ldots, v'_N$ the vertices of $Q_n^r$, such that $v_i$ is adjacent to $v'_i$ for all $i$. Note that $(v'_1, \ldots, v'_N)$ forms a Hamiltonian cycle of $Q^r_n$. Here is the diagram of the hypercube $Q_n$ that will be used in the rest of the paper:

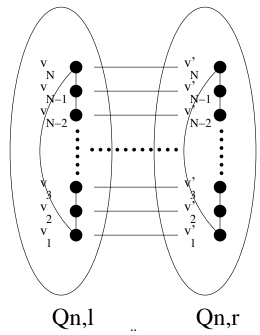


Figure 2: The hypercube $Q_n$

Let $v_i$ be a vertex of $Q_n$. Note that when referring to $v_{i+j}$, where $(i + j)$ is not in $[1, N]$, then use the appropriate subscript $i + j \pm N$ instead.

The following lemmas describe the successions of moves used to reduce a game configuration to a certain form:

**Lemma 2.** *Let $C$ be a game configuration on a Hamiltonian graph $G$ with $n$ vertices ($n > 2$). Let $(v_1, \ldots, v_n)$ be the list of the vertices ordered according to a Hamiltonian cycle of $G$. If there exists a vertex $v_i$ such that $c(v_i)$ is rare on $G$, then $C$ is both $(1, v_{i\pm1}, c(v_i))$-reducible and $(1, v_{i\pm2}, \overline{c(v_i)})$-reducible.*

*Proof.* The first reduction is obtained when $v_i$ clobbers all the stones along the Hamiltonian cycle $(v_1, \ldots, v_N)$. According to the direction in which we move around the cycle, we end either on $v_{i+1}$ or on $v_{i-1}$.

To get the second reduction, $v_i$ clobbers all the stones along the Hamiltonian cycle, except the last one. This means that $v_i$ finishes on $v_{i+2}$ or $v_{i-2}$, and is then clobbered by $v_{i+1}$ or $v_{i-1}$ respectively. □

**Lemma 3.** *Let $C$ be a game configuration on $Q_n$, with $n > 3$. If there exists a rare color on $Q^r_n$, and if $Q^l_n$ is not monochromatic, then there exists a way to play that yields no stones on $Q^r_n$ and $N$ stones on $Q^l_n$, both colors being common on $Q^l_n$. If $n = 3$, there may be a rare color on $Q^l_n$, but we can choose its location on two distinct vertices.*

*Proof.* Let $c$ be the rare color on $Q^r_n$ and denote by $v'_i$ the vertex such that $c(v'_i) = c$. We consider three cases for the stones on $Q^l_n$:

- $\overline{c}$ is rare on $Q^l_n$. Thanks to its Hamiltonian cycle and by Lemma 2, we know that $Q^r_n$ is $(1, v'_{i\pm2}, \overline{c})$-reducible. If $n > 3$, $v'_{i+2}$ and $v'_{i-2}$ are distinct vertices. Also since $\bar{c}$ is rare on $Q_n^l$, this means that either $v_{i+2}$ or $v_{i-2}$ is labeled with the color $c$. Without loss of generality, suppose that $v_{i+2}$ is labeled $c$; hence we apply a $(1, v'_{i+2}, \bar{c})$-reduction of $Q_n^r$. Then $v'_{i+2}$ clobbers $v_{i+2}$, so that $Q_n^l$ contains at least two stones of each color afterwards.

If $n = 3$ and $c(v_{i+2}) = \bar{c}$, this proof is no longer valid. In that case, there are two ways to play, each of them leaving the rare color $\bar{c}$ either on $v_{i+1}$ (diagram 1) or on $v_{i-1}$ (diagram 2).

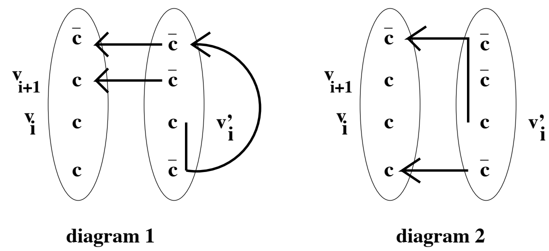


Figure 3: Lemma 3: special instance of the case $n = 3$

- $c$ is rare on $Q_n^l$. By Lemma 2, $Q_n^r$ is $(1, v'_{i\pm 1}, c)$-reducible. We know that at least one of both vertices $v_{i+1}$ and $v_{i-1}$ has the common label $\bar{c}$. Without loss of generality, assume $v_{i+1}$ does. Last, we apply a $(1, v'_{i+1}, c)$-reduction of $Q_n^r$, and then we play from $v'_{i+1}$ to $v_{i+1}$.
- Both colors are common on $Q_n^l$. We consider the four cases for the labels of $v_{i+1}$ and $v_{i+2}$:

  - $c(v_{i+1}) = c$ and $c(v_{i+2}) = \bar{c}$. Use a Hamiltonian cycle of $Q_n^r$ to have $v'_i$ clobber all the vertices except $v'_{i+1}$. This operation yields two stones on $Q_n^r$: $v'_{i+1}$ labeled $\bar{c}$, and $v'_{i+2}$ labeled $c$. Play now from $v'_{i+1}$ to $v_{i+1}$ and from $v'_{i+2}$ to $v_{i+2}$.
  - $c(v_{i+1}) = \bar{c}$ and $c(v_{i+2}) = c$. If $n > 3$, $c$ or $c'$ appears more than twice in $Q_n^l$. If it is the case of $\bar{c}$, then apply a $(1, v'_{i+1}, c)$-reduction of $Q_n^r$, and play from $v'_{i+1}$ to $v_{i+1}$. If $c$ appears more than twice in $Q_n^l$, then apply a $(1, v'_{i+2}, \bar{c})$-reduction of $Q_n^r$, and play from $v'_{i+2}$ to $v_{i+2}$. If $n = 3$, there are two possible arrangements of the stones on $Q_n^l$. In both cases, there exists a way to play that yields a rare color on $Q_n^l$, with two possible locations:
  - $c(v_{i+1}) = c$ and $c(v_{i+2}) = c$. If $c$ appears more than twice in $Q_n^l$, then apply a $(1, v'_{i+2}, \bar{c})$-reduction of $Q_n^r$, and play from $v'_{i+2}$ to $v_{i+2}$. Then play from $v'_{i+2}$ to $v_{i+2}$. Otherwise, and if $n > 3$, this means that the color $\bar{c}$ appears more than twice, in particular on $v_{i-1}$. Then apply a $(1, v'_{i+1}, c)$-reduction of $Q_n^r$, and play from $v'_{i-1}$ to $v_{i-1}$. If $n = 3$, this implies $c(v_i) = c(v_{i-1}) = \bar{c}$. It then suffices to invert the order of the vertices ($v_{i+1}$ becomes $v_{i-1}$...) to reduce to the previous case.
  - $c(v_{i+1}) = \bar{c}$ and $c(v_{i+2}) = \bar{c}$. This case is similar to the previous one. □

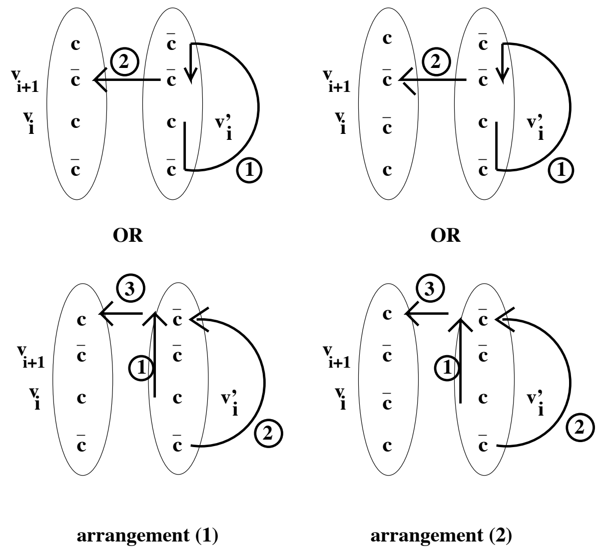


Figure 4: Lemma 3: special instances of the case $n = 3$ (2)

**Lemma 4.** *Let $C$ be a game configuration on $Q_n$, with $n > 2$. If there exists a rare color on $Q_n^r$, and if $Q_n^l$ is monochromatic, then there exists a way to play that yields no stones on $Q_n^r$ and $N$ stones on $Q_n^l$, which is not monochromatic. Also, if this operation yields a rare label on $Q_n^l$, we can choose its location on two distinct vertices.*

*Proof.* Let $c$ be the rare color on $Q_n^r$ and denote by $v_i'$ the vertex such that $c(v_i') = c$. We consider two cases about $Q_n^l$:

- All the vertices of $Q_n^l$ have the color $c$. Use a Hamiltonian cycle of $Q_n^r$ to have $v_i'$ clobber all the vertices except $v_{i+1}'$ and $v_{i+2}'$. It ends on $v_{i+3}'$. Then $v_{i+2}'$ clobbers $v_{i+3}'$. This operation yields two stones labeled $\overline{c}$ on $v_{i+1}'$ and $v_{i+3}'$. Then play from $v_{i+1}'$ to $v_{i+1}$ and from $v_{i+3}'$ to $v_{i+3}$. Both colors now appear at least twice on $Q_n^l$.
- All the vertices of $Q_n^l$ have the color $\overline{c}$. By Lemma 2, we can apply a $(1, v_{i\pm1}', c)$-reduction of $Q_n^r$. Then play from $v_{i+1}'$ or $v_{i-1}'$ to the corresponding vertex in $Q_n^l$. In that case, the color $c$ is rare on $Q_n^l$, but it can be located either on $v_{i+1}$ or on $v_{i-1}$.

□

We now give the main result of this section about the "strong reducibility" of the hypercube.

**Theorem 5.** *Hypercubes are strongly 1-reducible joker and strongly 2-reducible.*

Of course, the most interesting property concerns the 2-reducibility of the hypercube. However, this result is tightly linked to the strong 1-reducibilty joker. One can notice that the conditions defining the strong 2-reduction and the strong 1-reduction joker are a bit different. Indeed, the "vertex" condition of strong 2-reducibility (i.e. $G \setminus v$ must not be monochromatic) is contained in the condition of strong 1-reducibility joker. But monochromatic hypercubes and hypercubes with a rare color on $v_r$ such that $c = c(v_r)$ are also strongly 1-reducible joker, although they are not strongly 2-reducible. This explains why the conditions of strong 1-reducibility joker are "larger".

*Proof.* We proceed via induction on the dimension of the hypercube. The reader can verify that these results are true on the hypercube $Q_2$ (the square). Note that only four arrangements of the stones must be considered:

Assume that the theorem is true for the hypercube $Q_{n-1}$ and consider the hypercube $Q_n$.

$Q_n$ `is strongly 1-reducible joker.`

Without loss of generality, assume that the vertex that will support the last stone is $v_1$. Let $c$ be any color in $\{0, 1\}$. We consider any arrangement of the stones on $Q_n$ such that $c(v_1)$ is not rare or $c(v_1) = c$. Our objective consists in finding a way to yield a single stone of color $c$ on $v_1$. We are allowed to use a joker. Five cases are considered:

1. Suppose $Q_n^l$ is $(1, v_1, c)$-reducible joker, and the joker is used to change the color of some vertex $v_j$ from the color $d \in \{0, 1\}$ to $\overline{d}$. Also, we suppose that $Q_n^r$ is $(1, v_j', \overline{d})$-reducible joker.

   We first apply the $(1, v_j', \overline{d})$-reduction joker on $Q_n^r$, which yields a stone of color $\overline{d}$ on $v_j'$. We may have used a joker to do this. Then we apply a $(1, v_1, c)$-reduction joker on $Q_n^l$ with a small modification: instead of using the joker on $v_j$, we play from $v_j'$ to $v_j$. This move is indeed equivalent to the use of the joker, since $v_j'$ has the color $\overline{d}$ at this moment. At the end of the play, the joker has been used at most once.

2. $Q_n^l$ is $(1, v_1, c)$-reducible joker, and the joker is used to change the color of some vertex $v_j$ from the color $d \in \{0, 1\}$ to $\overline{d}$. Moreover, $Q_n^r$ is not $(1, v_j', \overline{d})$-reducible joker. From the conditions of the strong 1-reduction joker, this means that $c(v_j') = d$, and $c(v_i') = \overline{d}$ for all $i \neq j$.

   Since $d$ is rare on $Q_n^r$, we can apply both Lemma 3 and 4. If this yields a rare color on $Q_n^l$, we choose a location different from $v_1$ for it. Hence $c(v_1)$ is never rare and we can apply a $(1, v_1, c)$-reduction joker on $Q_n^l$.

3. $Q_n^l$ is $(1, v_1, c)$-reducible joker, but the joker is not used. We consider any arrangement of the stones on $Q_n^r$.

   We consider a succession of moves resulting from a $(1, v_1, c)$-reduction of $Q_n^l$. In this sequence, there exists a vertex $v_i$ that clobbers at least two other vertices before being (or not) clobbered. Indeed, if each vertex clobbers at most once, then $Q_n^l$ would be a star, which is not the case. Denote by $v_j$ and $v_k$ the first two vertices clobbered by $v_i$. When the moves from $v_i$ to $v_j$ and then to $v_k$ are made, let $y$ be the color of $v_i$, and $\overline{y}$ the color of $v_j$ and $v_k$. We consider four cases about the colors of $v_i'$ and $v_j'$:

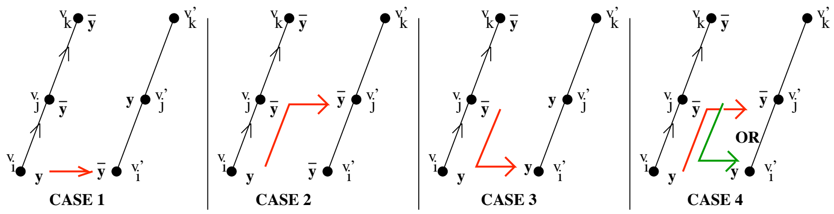


Figure 5: $Q_n^l$ is 1-reducible on $v_1$ with $c$

- CASE 1: $c(v_i') = \overline{y}$ and $c(v_j') = y$. Apply a $(1, v_1, c)$-reduction of $Q_n^l$, and when the time comes to play from $v_i$ to $v_j$, play to $v_i'$ instead. At this moment, $y$ is not rare on $Q_n^r$, so we can apply a $(1, v_j', y)$-reduction joker on $Q_n^r$. Play then from $v_j'$ to $v_j$ and continue the $(1, v_1, c)$-reduction of $Q_n^l$.
- CASE 2: $c(v_i') = c(v_j') = \overline{y}$. Begin a $(1, v_1, c)$-reduction of $Q_n^l$ up to the move from $v_j$ to $v_k$ (not included). Play to $v_j'$ instead. Since $c(v_k')$ is not rare, apply a $(1, v_k', y)$-reduction joker on $Q_n^r$. Then play from $v_k'$ to $v_k$ and continue the $(1, v_1, c)$-reduction of $Q_n^l$.
- CASE 3: $c(v_i') = c(v_j') = y$. Apply a $(1, v_1, c)$-reduction of $Q_n^l$ up to the move from $v_i$ to $v_j$ (not included). Instead of it, have $v_j$ clobber $v_i$ and then $v_i'$. The rest of the play is identical to the previous case.
- CASE 4: $c(v_i') = y$ and $c(v_j') = \overline{y}$. If $c(v_k') = y$, then play as in the second case. Otherwise, play as in the third case.

4. $Q_n^l$ is not $(1, v_1, c)$-reducible joker, and $Q_n^r$ is $(2, v_1', c, \overline{c})$-reducible.

   This implies that $c(v_1) = \overline{c}$ and $c(v_i) = c$ for all $i > 1$. If $Q_n^r$ is $(1, v_1', c)$-reducible, we apply this reduction and then play from $v_1'$ to $v_1$. $Q_n^l$ becomes monochromatic and the $(1, v_1, c)$-reduction joker can now be applied on it. If $Q_n^r$ is $(2, v_1', c, \overline{c})$-reducible, then choose the second remaining stone of color $\overline{c}$. Let $v_j'$ be the vertex on which this stone is left. Play now from $v_1'$ to $v_1$, and from $v_j'$ to $v_j$. $Q_n^l$ now satisfies the right conditions to apply a $(1, v_1, c)$-reduction joker.

5. $Q_n^l$ is not $(1, v_1, c)$-reducible joker, and $Q_n^r$ is not $(2, v_1', c, \overline{c})$-reducible.

   There are four possible arrangements of the stones on $Q_n$ corresponding to these conditions:

   - The arrangement $(A)$ does not have to be considered. Indeed, this arrangement is not allowed by the conditions of the 1-reduction joker, since $c(v_1)$ is rare on $Q_n$ and $c(v_1) \neq c$.

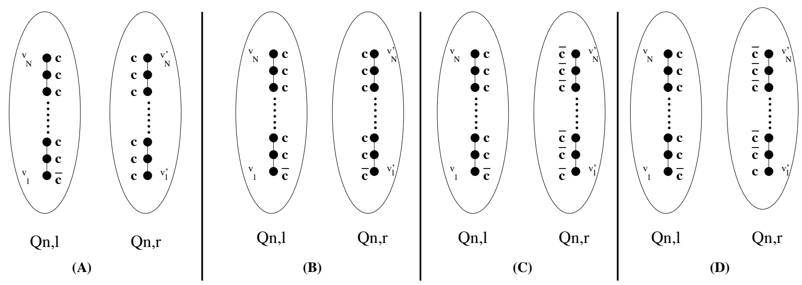


Figure 6: Strong 1-reducibility joker: case 5

- If the arrangement of the stones is $(B)$, have $v'_1$ clobber all the vertices of $Q^r_n$ and end on $v'_N$. Then $v'_N$ clobbers $v_N$, and the conditions of a $(1, v_1, c)$-reduction joker are fulfilled on $Q^l_n$.
- If the arrangement of the stones is $(C)$, have $v_i$ clobber $v'_i$ for all $2 < i < N$. Apply now a $(1, v'_1, \overline{c})$-reduction joker of $Q^r_n$. Finally, $v_1$ is clobbered by $v_2$, $v'_1$ and $v_N$ in this order.
- If the stones are placed as in $(D)$, use Lemma 2 to apply a $(1, v'_{N-1}, \overline{c})$-reduction of $Q^r_n$. Then $v'_{N-1}$ clobbers $v_{N-1}$, and we can apply a $(1, v_1, c)$-reduction joker of $Q^l_n$.

$Q_n$ `is strongly 2-reducible.`

Without loss of generality, assume that the vertex that will support the last stone is $v_1$. We consider any arrangement of the stones on $Q_n$ such that $Q_n \setminus v_1$ is not monochromatic. Let $c$ and $c'$ be any two colors in $\{0, 1\}$ such that there are two distinct vertices of $Q_n$ labeled with these values. Our objective consists in finding a way to leave a stone of color $c$ on $v_1$, and possibly another one of color $c'$ somewhere else. We consider eleven cases, starting with those where $Q^r_n$ is monochromatic (cases 1 to 5):

1. $Q^r_n$ is monochromatic of color $y \in \{0, 1\}$, and $Q^l_n$ is $(1, v_1, c)$-reducible. Consider a succession of moves resulting from a $(1, v_1, c)$-reduction of $Q^l_n$. First suppose that there exists a move from a stone of color $\overline{y}$ on some vertex $v_i$ clobbering a stone of color $y$ on the vertex $v_j$. Replace this move by having $v_i$ clobber $v'_i$. There exists an Hamiltonian cycle of $Q^r_n$ where $v'_i$ and $v'_j$ are consecutive. Have $v'_i$ clobber all the stones of $Q^r_n$ and end on $v'_j$ with the color $\overline{y}$. Finally $v'_j$ clobbers $v_j$, and we can continue the $(1, v_1, c)$-reduction of $Q^l_n$.

   Suppose now that there exist no moves clobbering a vertex labeled $y$ when applying a $(1, v_1, c)$-reduction of $Q^l_n$. Necessarily this means that $c = y$. Also, this implies that all the vertices of $Q^l_n$ are labeled $\overline{y}$, except one, namely $v_i$. The $(1, v_1, c)$-reduction of $Q^l_n$ thus consists in having $v_i$ clobber all the vertices of $Q^l_n$ and end on $v_1$. Without loss of generality, suppose that $v_2$ is the penultimate vertex which is clobbered when applying the $(1, v_1, c)$-reduction of $Q_n^l$. The following diagram shows how to apply the $(1, v_1, c)$-reduction of $Q_n$:

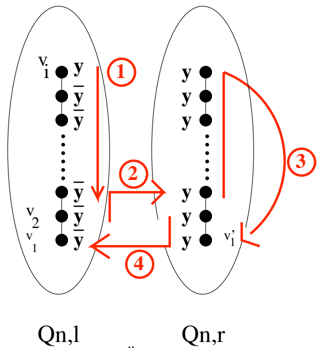


Figure 7: Strong 2-reducibility: specific instance of case 1

2. $Q_n^r$ is monochromatic of color $y \in \{0, 1\}$, and $Q_n^l$ is $(2, v_1, c, \overline{y})$-reducible.

   If $Q_n^l$ is $(1, v_1, c)$-reducible, then we are in case 1. Suppose then that the reduction yields two stones, the second one being located on some vertex $v_i$. In that case, apply a $(2, v_1, c, v_i, \overline{y})$-reduction of $Q_n^l$ and play from $v_i$ to $v_i'$. Then use Lemma 2 to yield a stone of color $c'$ either on $v_{i+1}'$ (if $c' = \overline{y}$) or on $v_{i+2}'$ (if $c' = y$).

   In cases 3, 4 and 5, we suppose that $Q_n^l$ is not $(2, v_1, c, \overline{y})$-reducible. If $Q_n^l$ is not $(2, v_1, c, \overline{y})$-reducible, then either $Q \setminus v_1$ is monochromatic, or $c = \overline{y}$ and $\overline{y}$ is rare in $Q_n^l$. But from our initial assumption that $Q_n \setminus v_1$ is not monochromatic, we know that there is at least one stone colored in $\overline{y}$ in $Q \setminus v_1$. So either $Q \setminus v_1$ is monochromatic of color $\overline{y}$ (see cases 4 and 5), or $\overline{y}$ is rare in $Q_n^l$ and $c(v_1) \neq \overline{y}$ (see case 3).

3. $Q_n^r$ is monochromatic of color $y \in \{0, 1\}$, and $\overline{y}$ is rare on $Q_n^l$ with $c(v_1) \neq \overline{y}$. If $Q_n^l$ is not $(2, v_1, c, \overline{y})$-reducible, then $c = \overline{y}$ and $c' = y$ (by our initial assumption that there are two distinct vertices of color $c$ and $c'$ respectively in $Q_n$). Let $v_i$ be the vertex of $Q_n^l$ such that $c(v_i) = \overline{y}$. See Fig.8 for the diagram of such a configuration.

   Since $c = \overline{y}$ and $c' = y$, $Q_n^l$ is $(2, v_1, c, c')$-reducible. Consider the first move of this 2-reduction: it is a move from $v_i$ to some $v_j$ since $c(v_i)$ is rare. Instead of playing it, play from $v_i$ to $v_i'$, and then have $v_i'$ clobber all the stones of $Q_n^r$ and end on $v_j'$. Then play from $v_j'$ to $v_j$ and continue the $(2, v_1, c, c')$-reduction of $Q_n^l$ to conclude this part of the proof.

4. $Q_n^r$ is monochromatic of color $y \in \{0, 1\}$ and $c(v_1) = y$ is rare on $Q_n^l$ (see Fig. 9).

   We first consider the case $c = y$. For all $2 \leq i \leq N$, play from $v_i$ to $v_i'$. Then use an Hamiltonian cycle of $Q_n^r$ to yield the second stone of the right color $c'$ (on $v_N$ or $v_{N-1}$ according to $c'$) after having clobbered all the other vertices of $Q_n^r$.

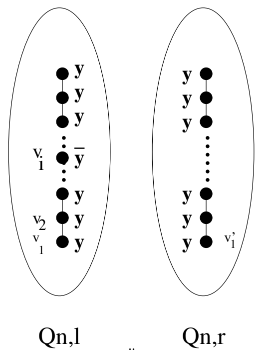


Figure 8: Strong 2-reducibility: case 3

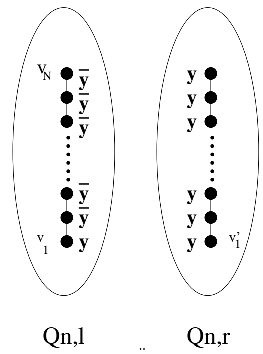


Figure 9: Strong 2-reducibility: case 4

If $c = \overline{y}$, then first $v_N$ clobbers $v_1$. Then $v_i$ clobbers $v'_i$ for all $3 \leq i \leq N-1$. We apply a $(2, v'_1, y, c')$-reduction of $Q^r_n$. The last two moves are $v'_1$ to $v_1$, and $v_2$ to $v_1$.

5. $Q^r_n$ is monochromatic of color $y \in \{0, 1\}$ and $Q^l_n$ is monochromatic of color $\overline{y}$.

   We first consider the case when $c = y$. Play from $v_N$ to $v'_N$ and from $v'_{N-1}$ to $v_{N-1}$. Then use a Hamiltonian cycle of $Q^r_n$ to clobber all its vertices and yield a stone of color $c'$ on $Q^r_n$. Finally, have $v_{N-1}$ clobber all the stones of $Q^l_n$ and end on $v_1$.

   If $c = \overline{y}$, play from $v'_1$ to $v_1$, and then from $v_2$ to $v_1$. Have $v_i$ clobber $v'_i$ for all $2 < i \leq N$. Use a Hamiltonian cycle to reduce $Q^r_n$ to a single stone of color $c'$.

   In the next cases, we suppose that $Q^r_n$ is not monochromatic.

6. $Q^l_n$ is $(1, v_1, c)$-reducible, and $Q^r_n$ has a rare color.

   Apply a $(1, v_1, c)$-reduction of $Q^l_n$ and use a Hamiltonian cycle to reduce $Q^r_n$ to a single stone of color $c'$ on $v'_{i+1}$ or $v'_{i+2}$.

7. $Q^l_n$ is $(1, v_1, c)$-reducible and both colors are common on $Q^r_n$.

   We consider a sequence of moves resulting from a $(1, v_1, c)$-reduction of $Q^l_n$. In this sequence, there exists a vertex $v_i$ that clobbers at least two other vertices before being (or not) clobbered. Denote by $v_j$ and $v_k$ the first two vertices clobbered by $v_i$. When considering the moves from $v_i$ to $v_j$ and then to $v_k$, let $y$ be the color of $v_i$, and $\overline{y}$ the color of $v_j$ and $v_k$. We consider four cases according to the colors of $v'_i$ and $v'_j$:

   - CASE 1: $c(v'_i) = \overline{y}$ and $c(v'_j) = y$. Apply a $(1, v_1, c)$-reduction of $Q^l_n$ until the move from $v_i$ to $v_j$ (not included). Play now from $v_i$ to $v'_i$, and from $v_j$ to $v'_j$ instead. After this operation, both colors are still common on $Q^r_n$, so that we can apply a $(2, v'_k, y, c')$-reduction. Then play from $v'_k$ to $v_k$, and continue the $(1, v_1, c)$-reduction of $Q^l_n$.
   - CASE 2: $c(v'_i) = c(v'_j) = \overline{y}$. Apply a $(1, v_1, c)$-reduction of $Q^l_n$, and when the time comes to play from $v_j$ to $v_k$, play to $v'_j$ instead. Since $y$ is not rare on

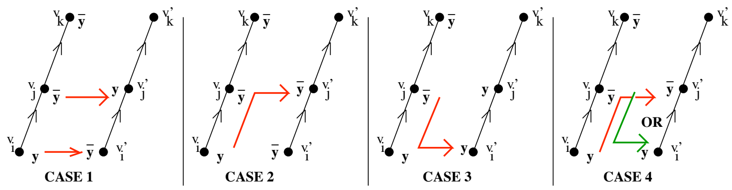


Figure 10: Strong 2-reducibility: case 7

$Q_n^r$ after this operation, apply a $(2, v'_k, y, c')$-reduction of $Q_n^r$. After this, play from $v'_k$ to $v_k$ and continue the $(1, v_1, c)$-reduction of $Q_n^l$.

- CASE 3: $c(v'_i) = c(v'_j) = y$. Apply a $(1, v_1, c)$-reduction of $Q_n^l$ until the move from $v_i$ to $v_j$ (not included). Instead of it, have $v_j$ clobber $v_i$ and then $v'_i$. If $y$ is not rare on $Q_n^r$ after this operation, then apply a $(2, v'_k, y, c')$-reduction of $Q_n^r$. If $y$ is rare on $Q_n^r$, then use a Hamiltonian path of $Q_n^r$ starting on $v'_j$ and ending on $v'_k$ to yield a stone of color $y$ on $v'_k$.

  After this, play from $v'_k$ to $v_k$ and continue the $(1, v_1, c)$-reduction of $Q_n^l$.
- CASE 4: $c(v'_i) = y$ and $c(v'_j) = \overline{y}$. If the color $\overline{y}$ appears more than twice in $Q_n^r$, or if $c(v'_k) = y$, then play as in the second case. Otherwise, this means that $c(v'_j) = c(v'_k) = \overline{y}$ and the other vertices of $Q_n^r$ have the color $y$. Play thus as in the third case.

In the next two cases, we suppose that $c(v_1)$ is not rare on $Q_n^l$ (which may be monochromatic). Hence $Q_n^l$ is $(1, v_1, c)$-reducible joker. If this reduction does not use the joker, then refer to case 6 or 7. Otherwise, assume that the joker is used to change the color of some vertex $v_j$ from $d$ to $\overline{d}$.

8. If $Q_n^r$ is $(2, v'_j, \overline{d}, c')$-reducible, we first apply a $(2, v'_j, \overline{d}, c')$-reduction of $Q_n^r$. We then apply a $(1, v_1, c)$-reduction joker of $Q_n^l$, and when the time comes to use the joker, we play from $v'_j$ to $v_j$ instead.

9. Suppose that $Q_n^r$ is not $(2, v'_j, \overline{d}, c')$-reducible. By our earlier assumption, $Q_n^r$ is not monochromatic, so this can occur in only three kinds of arrangements of the stones on $Q_n^r$, all with a rare color. The case when $Q_n^l$ is monochromatic is studied in case 10, we assume in this section that $Q_n^l$ is not monochromatic.

   - $c(v'_j) \neq \overline{d}$, $\overline{d}$ is rare on $Q_n^r$ and $c' = \overline{d}$. If $n > 3$, then use Lemma 3 to empty $Q_n^r$ and yield $N$ stones on $Q_n^l$ where both colors are common. Then we can apply a $(2, v_1, c, c')$-reduction of $Q_n^l$.

     If $n = 3$, the lemma can not be used. We thus have to consider all the configurations on $Q_3$ satisfying these conditions. Figure 11 details these five configurations (the final colors $c$ and $c'$ are detailed under each diagram):
   - $c(v'_j) = \overline{d}$, and $\overline{d}$ is rare on $Q_n^r$. If $n > 3$, we play as in the previous case. When $n = 3$, here are the configurations that must be considered:

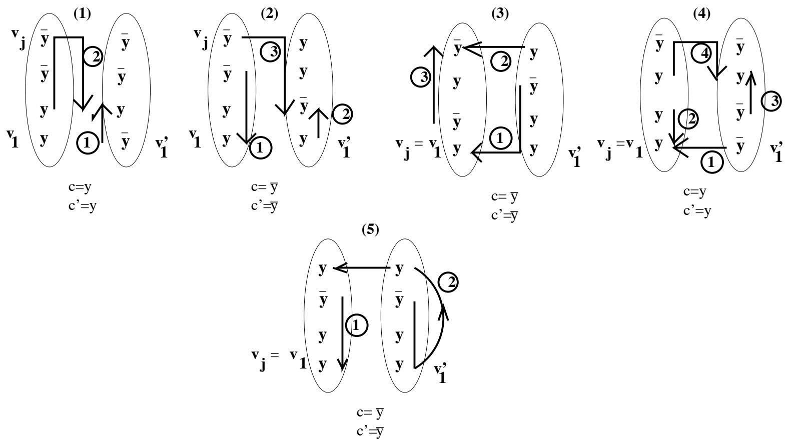


Figure 11: Case 9: arrangements on $Q_3$ (1)

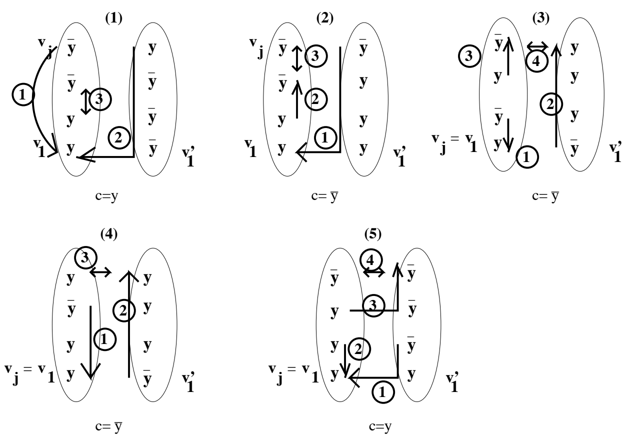


Figure 12: Case 9: arrangements on $Q_3$ (2)

- $d$ is rare on $Q_n^r$ and $c(v_j') = d$. If $n > 3$, we play as in the previous case. If $n = 3$, here are the configurations that must be considered:

10. Assume that $c(v_1) = \overline{y}$ is rare on $Q_n^l$ or that $Q_n^l$ is monochromatic, and that $Q_n^r$ has a rare label. This induces four possible cases:

    - CASE 1: We suppose that $c(v_1) = \overline{y}$ is rare on $Q_n^l$ and $Q_n^r$. Let $v_i'$ be the vertex such that $c(v_i') = \overline{y}$. Either $v_{i+1}'$ or $v_{i-1}'$ (or both) is different from $v_1'$. Without loss of generality, assume $v_{i+1}'$ is. Apply a $(1, v_{i+1}', \overline{y})$-reduction of $Q_n^r$ in the way of Lemma 2. Then play from $v_{i+1}'$ to $v_{i+1}$. Both colors are now common on $Q_n^l$, which becomes $(2, v_1, c, c')$-reducible.
    - CASE 2: $c(v_1) = \overline{y}$ is rare on $Q_n^l$ and $y$ is rare on some vertex $v_i'$ of $Q_n^r$. By Lemma 2, apply a $(1, v_{i\pm2}', \overline{y})$-reduction of $Q_n^r$ (choose to finish on a vertex different from $v_1'$). Play then as in the previous case. This operation is not possible if $n = 3$ and when the arrangement of the stones is the following:

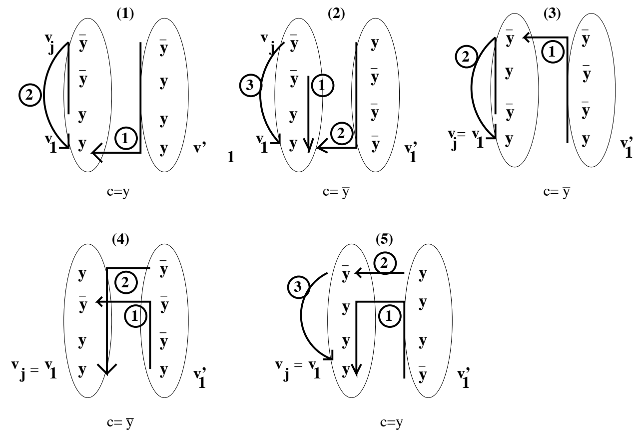

Figure 13: Case 9: arrangements on $Q_3$ (3)

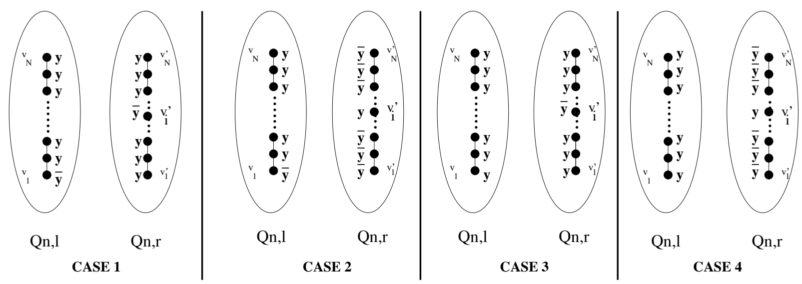

Figure 14: Possible arrangements in case 10

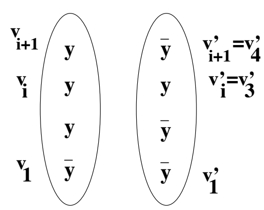

Figure 15: Special instance of the case 10.2

In that case, if $(c, c') \neq (y, y)$, then consider the following succession of moves: $v'_{i+1}$ to $v_{i+1}$, $v'_i$ to $v'_2$, $v'_1$ to $v'_2$, $v'_2$ to $v_2$. Use then a Hamiltonian cycle of $Q_n^l$ to conclude. If $(c, c') = (y, y)$, then play like this: Use a Hamiltonian cycle of $Q_n^r$ to apply a $(1, v'_1, \overline{y})$-reduction. Then move from $v_2$ to $v_1$, from $v'_1$ to $v_1$, and from $v_N$ to $v_1$.

- CASE 3: $Q_n^l$ is monochromatic of color $y$ and $\overline{y}$ is rare on some $v'_i$ of $Q_n^r$. This case is identical to the first case (note that $c = c' = \overline{y}$ is not allowed since $\overline{y}$ is rare on $Q_n$).
- CASE 4: $Q_n^l$ is monochromatic of color $y$ and $y$ is rare on some $v'_i$. Have $v'_i$ clobber all the vertices of $Q_n^r$ except $v'_{i+1}$ and $v'_{i+2}$,and end on $v'_{i+3}$. Then play from $v'_{i+2}$ to $v'_{i+3}$, from $v'_{i+3}$ to $v_{i+3}$, and from $v'_{i+1}$ to $v_{i+1}$. All the stones

of $Q_n^r$ have been removed and both colors are now common on $Q_n^l$. Apply now a $(2, v_1, c, c')$-reduction of $Q_n^l$.

11. Assume that $c(v_1) = \overline{y}$ is rare on $Q_n^l$ and that both colors are common on $Q_n^r$.

    If $Q_n^r$ is $(1, v'_{N-1}, \overline{y})$-reducible, then apply this reduction and move from $v'_{N-1}$ to $v_{N-1}$. Both colors are now common on $Q_n^l$, and we can conclude to the right result.

    Otherwise, $Q_n^r$ is 2-reducible on $v'_{N-1}$ with $\overline{y}$, and $\overline{y}$ on some other vertex called $v'_i$. Apply this reduction. If $v'_i \neq v'_1$, move from $v'_{N-1}$ to $v_{N-1}$, and from $v'_i$ to $v_i$. If $n > 3$, then both colors are common on $Q_n^l$, and we can conclude the proof. If $n = 3$, then $y$ is rare on $Q_n^l$, and located either on $v_2$, or on $v_N$. Clobbering along the Hamiltonian cycle of $Q_n^l$ permits a 2-reduction.

    If $v'_i = v'_1$, we distinguish two cases. If $c = y$, then play from $v_2$ to $v_1$, $v'_1$ to $v_1$ and $v_N$ to $v_1$. Then have $v'_{N-1}$ clobber $v_{N-1}$ and follow a Hamiltonian cycle of $Q_n^l$ to leave the last stone of color $c'$. If $c = \overline{y}$, then play from $v_N$ to $v_1$, and from $v'_1$ to $v_1$. Have $v'_{N-1}$ clobber $v_{N-1}$ and use a Hamiltonian cycle of $Q_n^l$ to leave the last stone of color $c'$. □

This theorem ensures that hypercubes are 2-reducible. Besides, as next proposition shows, non 1-reducible configurations exist. We use to prove it the invariant $\delta$ given by Demaine et al. in [2], defined below.

**Proposition 6.** *For each integer $n$, there exists a non-monochromatic configuration on $Q_n$ which is not 1-reducible.*

*Proof.* We prove this result thanks to the invariant defined by Demaine et al. in [2]. On a bipartite graph $G$, vertices of both partitions are respectively labeled '0' and '1'. Now consider a game configuration $C$ of Solitaire Clobber on $G$, with stones labeled '0' and '1'. A stone is said to be "clashing" if its label differs from the label of the vertex it occupies. Denote by $\delta(C)$ the following quantity:

$$\delta(C) = \text{number of stones plus number of clashing stones.}$$

In their paper, Demaine et al. proved that $\delta(C) \pmod 3$ never changes during the game.

Let $n > 1$ and consider $Q_n = Q_{n-1} \square K_2$. As previously, denote by $Q_n^l$ and $Q_n^r$ both copies of $Q_{n-1}$. Hypercubes are bipartite graphs. Choose a bipartition of $Q_n$ such that half the vertices of $Q_n^l$ are labeled '0', and the other ones are labeled '1'. Ditto for $Q_n^r$. Now choose an arrangement of the stones on $Q_n$ such that all the stones labeled '0' belong to $Q_n^l$, and all the stones labeled '1' belong to $Q_n^r$. In that case, we have

$$\delta(C) = 2^n + 2^{n-1} = 3 \cdot 2^{n-1}$$

Hence $\delta(C)$ $(mod\ 3) = 0$. Since a single stone configuration never satisfies $\delta(C) \pmod 3 = 0$ (see [2]), this concludes the proof. □

Proposition 6 shows that our result is sharp. Nevertheless, it is still an open problem to determine if a given configuration in a hypercube satisfying $\delta = 1$ is 1-reducible.

## 4. On the Other Hamming Graphs

Hypercubes are strongly 2-reducible. In this section, we prove that *almost* all the other Hamming graphs are strongly 1-reducible. This induction is initialized by Lemmas 10 and 11, and the property is proved to be hereditary by Theorem 8.

In the following, we prove that the cartesian product of a strongly 1-reducible graph $G$ with a clique $K_n$ is strongly 1-reducible. This product contains $n$ copies of $G$, that we denote by $G_1, \ldots, G_n$. For any vertex $v$ of $G$, we denote by $v_i$ the corresponding vertex in the copy $G_i$. Then, denote by $v_1$ any vertex of $G_1$.

**Lemma 7.** *Let $G$ be a strongly 1-reducible graph containing at least 4 vertices. $K_2 \square G$ is strongly 1-reducible.*

*Proof.* Let $G$ be a strongly 1-reducible graph with at least 4 vertices. Without loss of generality, assume that the vertex on which we will leave the last stone is $v_1$. Let $c$ be any color in $\{0, 1\}$. We consider any arrangement of the stones on $K_2 \square G$ such that $K_2 \square G \setminus v_1$ is not monochromatic. Let us prove that $K_2 \square G$ is $(1, v_1, c)$-reducible. We split the problem into three cases.

1. **$G_2$ is not monochromatic.**
   Since $G$ is of size at least 4, there exist 2 vertices of the same color in $G_1 \setminus v_1$. We denote them by $a_1$ and $b_1$. Similarly, $c(a_2)$ or $c(b_2)$ (or both) is common in $G_2$. Without loss of generality, we suppose $c(a_2)$ is. One applies a $(1, a_2, \overline{c(a_1)})$-reduction of $G_2$, and then have $a_2$ clobber $a_1$. $G_2$ is now empty. $a_1$ and $b_1$ are now of different colors on $G_1$, so we can apply a $(1, v_1, c)$-reduction of $G_1$.

2. **$G_2$ is monochromatic of color $y$ and $G_1 \setminus v_1$ is not monochromatic.**

   This means that $G_1$ is $(1, v_1, c)$-reducible. We consider two cases:

   - Suppose that when one applies a $(1, v_1, c)$-reduction of $G_1$, there exists a vertex $a_1$ colored in $\overline{y}$ clobbering another vertex $b_1$ of color $y$. We then choose to apply this reduction, and when the time comes to play from $a_1$ to $b_1$, play to $a_2$ instead. We then apply a $(1, b_2, \overline{y})$-reduction of $Q_2$. $b_2$ then clobbers $b_1$ and we can continue the $(1, v_1, c)$-reduction of $G_1$.
   - Otherwise, there is exactly one vertex $a_1$ colored in $y$ in $G_1$. Since there are at least 4 vertices in $G_1$, $a_1$ has to clobber consecutively 2 vertices during the $(1, v_1, c)$-reduction of $G_1$. Denote them by $b_1$ and $c_1$. We replace these two consecutive moves by these ones: $b_1$ clobbers $a_1$ and then $a_2$. We then apply a $(1, c_2, y)$-reduction of $G_2$. It finally suffices to play from $c_2$ to $c_1$, and continue the $(1, v_1, c)$-reduction of $G_1$.

3. **$G_2$ is monochromatic of color $y$ and $G_1 \setminus v_1$ is monochromatic.**
   Since $K_2 \square G \setminus v_1$ is not monochromatic, $G_1 \setminus v_1$ is necessarily colored $\overline{y}$. Let $a_1$ be any vertex of $G_1$ different from $v_1$. Act now as if $a_1$ was colored $y$. We can thus consider a $(1, v_1, c)$-reduction of $G_1$. The first step of such a reduction would be "$a_1$ clobbers some vertex $b_1$." We use this reduction, replacing this step by "$a_1$ (which is actually colored $\overline{y}$) clobbers $a_2$, then we do a $(1, b_2, y)$-reduction of $G_2$, followed by $b_2$ clobbers $b_1$". □

**Theorem 8.** *Let $G$ be a strongly 1-reducible graph containing at least 4 vertices. Then for any positive integer $n$, $K_n \square G$ is strongly 1-reducible.*

*Proof.* Let $G$ be a strongly 1-reducible graph with at least 4 vertices. We prove the theorem by induction on $n$. If $n = 2$, see Lemma 7. Suppose $n \geq 3$ and $K_{n-1} \square G$ is strongly 1-reducible. Without loss of generality, assume that the vertex on which we will leave the last stone is $v_1$. Let $c$ be any color in $\{0, 1\}$. We consider any arrangement of the stones on $K_2 \square G$ such that $K_2 \square G \setminus v_1$ is not monochromatic. Let us give a $(1, v_1, c)$-reduction of $K_n \square G$.

We consider 3 different cases:

1. **There exists $i \in [2 \ldots n]$ such that $G_i$ is not monochromatic.**
   Since $G$ contains at least 4 vertices, there are 2 vertices $a_i$ and $b_i$ such that $G_i \setminus \{a_i, b_i\}$ is not monochromatic. For the same reasons, in any other copy $G_j$, $c(a_j)$ or $c(b_j)$ (or both) is not rare. Without loss of generality, we can suppose that $c(a_j)$ is common on $G_j$. Start by applying a $(1, a_i, \overline{c(a_j)})$-reduction of $G_i$, and then play from $a_i$ to $a_j$. We can proceed with a $(1, v_1, c)$-reduction of the remaining non monochromatic $K_{n-1} \square G$.

2. **For all $i \in [2 \ldots n]$, $G_i$ is monochromatic of color $y$.**
   If $G_n$ is deleted from the graph, then the configuration is $(1, v_1, c)$-reductible according to the induction hypothesis. In this reduction, there exists a move from some $a_i$ to some $b_i$ of color $y$, where $1 < i < n$. When considering the graph with $G_n$, we apply the $(1, v_1, c)$-reduction as if $G_n$ was not there. And when the time comes to play from $a_i$ to $b_i$, we play to $a_n$ instead. We then do a $(1, b_n, \overline{y})$-reduction of $G_n$ and have $b_n$ clobber $b_i$. We can finally continue the execution of the $(1, v_1, c)$-reduction.

3. **For all $i \in [2 \ldots n]$, $G_i$ is monochromatic, but all the copies do not have the same color.**
   Let $y$ be the color of some vertex of $G_1 \setminus v_1$. Let $G_i$ $(i > 1)$ be a copy of color $y$ and $G_j$ $(j > 1)$ a copy of color $\overline{y}$. We start by having all the vertices of $G_j$ clobber the corresponding vertices of $G_i$. Hence there remains a $K_{n-1} \square G$ where $K_{n-1} \square G \setminus v_1$ is not monochromatic. We can apply the induction hypothesis to conclude the proof. □

With these results, we can assert that any Hamming graph containing a $K_4$ is strongly 1-reducible. What about Hamming graphs that are the product of $K_2$ and $K_3$ only?

We begin by studying configurations on $K_2 \Box K_3$. Such a graph will be considered as two adjacent copies $G_1$ and $G_2$ of $K_3$.

**Lemma 9.** *Let $G = K_3 \Box K_2$ and $i \in \{1,2\}$. For any vertex $a_i$ of $G$, for any color $c \in \{0,1\}$ and for any configuration $C$ on $G$ such that: (i) $c(a_i)$ is not rare on $G_i$ and (ii) $K_3 \Box K_2 \setminus a_i$ is not monochromatic, $C$ is $(1, a_i, c)$-reducible.*

*Proof.* For $i \in \{1,2\}$, let $v_i$, $u_i$, and $w_i$ be the vertices of each copy $G_i$. Without loss of generality, assume that we will leave the last stone on $v_1$. By $(i)$, one may assume that $v_1$ and $u_1$ have the same color $y$. Let $c \in \{0,1\}$. Our goal is now to prove that any configuration satisfying $(i)$ and $(ii)$ is $(1, v_1, c)$-reducible. We consider several cases:

- $c(w_1) = y$ and $G_2$ is not monochromatic. By Proposition 1, $G_2$ is either $(1, u_2, \overline{y})$-reducible, or $(1, w_2, \overline{y})$-reducible. Without loss of generality, suppose that $G_2$ is $(1, u_2, \overline{y})$-reducible. Apply this reduction and play from $u_2$ to $u_1$. The conditions are now fulfilled on the clique $G_1$ to apply a $(1, v_1, c)$-reduction.
- $c(w_1) = y$ and $G_2$ is monochromatic. From $(ii)$, $G_2$ is monochromatic of color $\overline{y}$. According to $c$, play as shown on diagrams $(a)$ $(c = y)$ or $(b)$ $(c = \overline{y})$ of Figure 16.
- $c(w_1) = \overline{y}$ and $G_2$ is $(1, v_2, \overline{y})$-reducible. Apply this reduction, and then play from $v_2$ to $v_1$. Now $G_1$ is $(1, v_1, c)$-reducible by Proposition 1.
- $c(w_1) = \overline{y}$ and $G_2$ is monochromatic. Play according to Figure 16. On diagrams $(c)$ and $(e)$, we have $c = y$. On diagrams $(d)$ and $(f)$, we end with the color $c = \overline{y}$.
- $c(w_1) = \overline{y}$ and $c(v_2)$ is rare on $G_2$. In both cases, we play from $v_2$ either to $u_2$ or to $w_2$, such that $c(u_2) \neq c(u_1)$ and $c(w_2) \neq c(w_1)$ after this operation. We then play from $u_2$ to $u_1$, and from $w_2$ to $w_1$. Use Proposition 1 to apply a $(1, v_1, c)$-reduction of $G_1$.

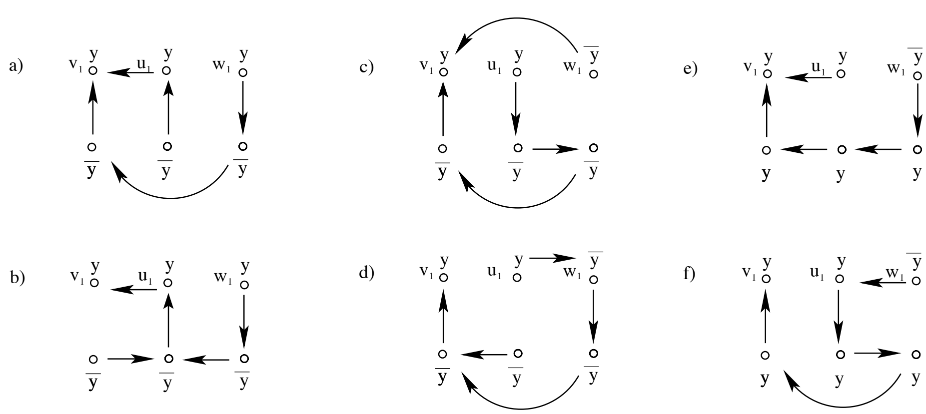


Figure 16: reduction of $K_2 \Box K_3$

□

**Lemma 10.** *$K_3 \square K_3$ is strongly 1-reducible.*

*Proof.* Let us consider the graph $K_3 \square K_3$, $v_1$ being any vertex of it. Assume that we will leave the last stone on $v_1$. Let $c \in \{0, 1\}$. We consider any arrangement of the stones such that $K_3 \square K_3 \setminus v_1$ is not monochromatic. Let us prove that this configuration is $(1, v_1, c)$-reducible.

Among the six copies of $K_3$ constituting the product $K_3 \square K_3$ (three horizontal and three vertical), one of them is not monochromatic and does not contain $v_1$: call it $G_3$. Denote by $G_1$ the parallel copy of $G_3$ containing $v_1$, and $G_2$ the last parallel copy. $G_3$ is then 1-reducible with any color on two possible vertices: $a_3$ and $b_3$. At least one of these is different from $v_3$ ($v_3$ being the copy of $v_1$ in $G_3$). Without loss of generality, assume $a_3 \neq v_3$.

If $G_1 \setminus v_1$ is not monochromatic, we apply a $(1, a_3, \overline{c(a_2)})$-reduction of $G_3$ and then play from $a_3$ to $a_2$. Otherwise, we apply a $(1, a_3, \overline{c(a_1)})$-reduction of $G_3$ and then play from $a_3$ to $a_1$. In both cases, we finally get a configuration on $K_2 \square K_3$ that we can reduce from Lemma 9. □

**Lemma 11.** *$K_3 \square K_2 \square K_2$ is strongly 1-reducible.*

*Proof.* Consider the graph $K_3 \square K_2 \square K_2$. Let $v_1$ be any vertex of it and let $c$ be any color. Assume that we will leave the last stone on $v_1$. We consider any arrangement of the stones such that $K_3 \square K_2 \square K_2 \setminus v_1$ is not monochromatic.

Let $G_1$ be the copy of $K_3$ containing $v_1$. We call $G_2$, $G_3$, and $G_4$ the other copies of $K_3$, $G_3$ being the copy containing no neighbour of $v_1$. We distinguish two cases:

- **The graph without $G_1$ is not monochromatic**
  There exists a non monochromatic copy of $K_2 \square K_3$ that does not contain $G_1$. Without loss of generality, suppose it is the one made of $G_3$ and $G_4$. We can 1-reduce it to various places.

  We first suppose that both vertices $a_1$ and $b_1$ of $G_1 \setminus v_1$ have the same color. At least one of the corresponding vertex $a_4$ and $b_4$ in $G_4$ has a common color in $G_4$. Assume it is the case of $a_4$. The conditions of Lemma 9 are fulfilled so that we are able to apply a $(1, a_4, \overline{c(a_1)})$-reduction of $G_3 \cup G_4$; then we have $a_4$ clobber $a_1$. Now, $G_1 \cup G_2 \setminus v_1$ is not monochromatic, and $c(v_1)$ is common on $G_1$. By Lemma 9, $G_1 \cup G_2$ is $(1, v_1, c)$-reducible.

  Suppose now that the vertices $a_1$ and $b_1$ of $G_1 \setminus v_1$ have different colors. At least one vertex of $a_3$ and $b_3$ has a common color in $G_3$. Assume it is $a_3$. The conditions of Lemma 9 are fulfilled to apply a $(1, a_3, \overline{c(a_2)})$-reduction of $G_3 \cup G_4$; then have $a_3$ clobber $a_2$. Now, $G_1 \cup G_2 \setminus v_1$ is not monochromatic, and $c(v_1)$ is common on $G_1$. By Lemma 9, $G_1 \cup G_2$ is $(1, v_1, c)$-reducible.

- **The graph without $G_1$ is monochromatic of color $y$**
  Then $G_1 \setminus v_1$ contains a stone of color $\overline{y}$. Denote by $z$ the initial color of $v_1$. We describe the way to play on Figure 17.

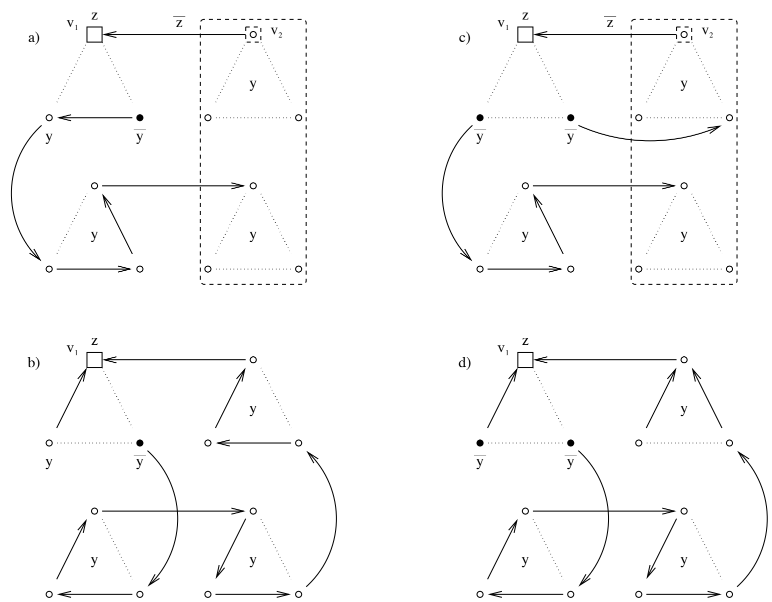

Figure 17: 1-reduction of $K_3 \square K_2 \square K_2$

In cases $(a)$ and $(c)$, we have $c = \overline{z}$. We execute the moves described by the figure, leaving $v_1$ and a copy of $K_2 \square K_3$. We can apply a $(1, v_2, \overline{z})$-reduction of this copy (from Lemma 9), and conclude by playing from $v_2$ to $v_1$. In cases $(b)$ and $(d)$, we have $c = z$. Just follow the moves on the figure as soon as they are possible. □

From all these results, we can deduce the following theorem about Hamming graphs.

**Theorem 12.** *Any Hamming graph that is neither $K_2 \square K_3$ nor a hypercube is strongly 1-reducible.*

Note that $K_2 \square K_3$ is 1-reducible for any coloration, and is also strongly 1-reducible joker.

## References


[1] Michael H. Albert, J. P. Grossman, Richard J. Nowakowski, and David Wolfe, *An introduction to Clobber*, Integers **5** (2005).

[2] Erik D. Demaine, Martin L. Demaine, and Rudolf Fleischer, *Solitaire Clobber*, Theor. Comput. Sci. **313** (2004), 325-338.

[3] L. Beaudou, E. Duchêne, L. Faria and S. Gravier, *Solitaire Clobber played on graphs*, submitted.

[4] Ivars Peterson, *Getting Clobbered*, Science News **161** (2002), `http://www.sciencenews.org/articles/20020427/mathtrek.asp`.