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A NOTE ON STIRLING SERIES

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Abstract

We study sums $S = S(d,n,k) = \sum_{j\ge 1} \frac{\left[{j \atop d}\right]}{j^k\binom{n+j}{j} j!}$ with $d \in \mathbb{N} = \{1,2,\dots\}$ and $n,k \in \mathbb{N}_0 = \{0,1,2,\dots\}$ and relate them to (finite) multiple zeta functions. As a byproduct of our results we obtain asymptotic expansions of $\zeta(d+1) - H_n^{(d+1)}$ as n tends to infinity. Furthermore, we relate sums S to Nielsen's polylogarithm.

1. Introduction

The unsigned Stirling numbers of the first kind, also called Stirling cycle numbers, are defined by the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad n \ge 1, \quad \text{with} \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \delta_{n,0}, \quad n \ge 0,$$

where $\delta_{i,j}$ denotes the Kronecker delta function. Throughout this work we use Knuth's notation $\left[{n \atop k}\right]$. It is well-known that Stirling numbers of the first kind are closely related to harmonic numbers, i.e. $\left[{n \atop 2}\right] = (n-1)!H_{n-1}$, $\left[{n \atop 3}\right] = (n-1)!(H_{n-1}^2 - H_n^{(2)})/2$, where for $s, n \in \mathbb{N}$ the values $H_n^{(s)} = \sum_{\ell=1}^{n} 1/\ell^s$ denote n-th harmonic numbers of order s, $H_n = H_n^{(1)}$. Furthermore, it is known (e.g., see Adamchik [1]) that Stirling numbers of the first kind are expressible in terms of (finite) multiple zeta functions defined by

$$\zeta_N(a_1,\dots,a_\ell) = \sum_{N \ge n_1 > n_2 > \dots > n_\ell \ge 1} \frac{1}{n_1^{a_1} n_2^{a_2} \dots n_\ell^{a_\ell}},$$
$$\zeta(a_1,\dots,a_\ell) = \sum_{n_1 > n_2 > \dots > n_\ell \ge 1} \frac{1}{n_1^{a_1} n_2^{a_2} \dots n_\ell^{a_\ell}},$$

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by the following formula:

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)!\zeta_{n-1}(\underbrace{1,\ldots,1}_{k-1}) = (n-1)!\cdot\zeta_{n-1}(\{1\}_{k-1}).$$

We use the shorthand notations $\zeta(\cup_{i=1}^{r}\{a_i\}) = \zeta(a_1,\ldots,a_r)$, and $\zeta(\cup_{i=1}^{r}\{a\}) = \zeta(\{a\}_r)$. Note that for $n, s \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ we have $\zeta_n(s) = H_n^{(s)}$. We are interested in evaluations of sums $S = \sum_{j\geq 1} \frac{\left[{j \atop d}\right]}{j^k\binom{n+j}{j}j!}$ with $d \in \mathbb{N} = \{1,2,\ldots\}$ and $n, k \in \mathbb{N}_0$. We assume that n and k are choosen in such a way that $n + k \geq 1$ in order to ensure that the sum converges. Special instances of this family of sums have been studied by Adamchik [1], and also by Choi and Srivastava [6] (see, e.g., page 252).

2. Evaluation of Sum S

We obtain the following result.

Theorem 1. *The sum $S = S(d,n,k)$ with $d \in \mathbb{N} = \{1,2,\ldots\}$ and $n, k \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ can be evaluated in terms of harmonic numbers and (finite) multiple zeta functions,*

$$S = \sum_{m=2}^{k+1}(-1)^{k+1-m}\zeta(m,\{1\}_{d-1}) \sum_{\substack{m_1,\ldots,m_{k+1-m}\geq 0 \\ \sum_{i=1}^{k+1-m} i\cdot m_i = k+1-m}} \prod_{r=1}^{k+1-m} \frac{(H_n^{(r)})^{m_r}}{r^{m_r} m_r!}$$
$$+ (-1)^k \sum_{h=1}^{k} \sum_{1\leq \ell_1<\ell_2<\cdots<\ell_{h-1}<k} \zeta_n(\ell_1, \ell_2-\ell_1, \ldots, \ell_{h-1}-\ell_{h-2}, d+k-l_{h-1}),$$

subject to $\ell_0 := 0$. We have the short equivalent expression

$$S = (-1)^k\zeta_n^*(\{1\}_{k-1}, d+1) + \sum_{m=2}^{k+1}(-1)^{k+1-m}\zeta(m,\{1\}_{d-1})\zeta_n^*(\{1\}_{k+1-m}).$$

Remark 2. The second expression for the sum S is given according to a variant of finite multiple zeta functions, $\zeta_N^*(a_1,\ldots,a_k)$, which recently attracted some interest [2, 12, 9, 7], where the summation indices satisfy $N \geq n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$ in contrast to $N \geq n_1 > n_2 > \cdots > n_k > 1$, as in the usual definition (1),

$$\zeta_N^*(a_1,\dots,a_k) = \sum_{N\geq n_1\geq n_2\geq\cdots\geq n_k\geq 1} \frac{1}{n_1^{a_1}n_2^{a_2}\dots n_k^{a_k}}.$$

The form stated above is due to the conversion formula below applied to $\zeta_n^*(\{1\}_{k-1}, d+1)$,

$$\zeta_N^*(a_1,\dots,a_k) = \sum_{h=1}^{k} \sum_{\substack{1\leq \ell_1<\ell_2<\cdots<\ell_{h-1}<k \\ \ell_0=0}} \zeta_N\Bigg(\sum_{i_1=1}^{\ell_1} a_{i_1}, \sum_{i_2=\ell_1+1}^{\ell_2} a_{i_2}, \dots, \sum_{i_h=\ell_{h-1}+1}^{k} a_{i_h}\Bigg).$$

Note that the first term $h = 1$ should be interpreted as $\zeta_N(\sum_{i_1=\ell_0+1}^k a_{i_1})$, subject to $\ell_0 = 0$. The notation $\zeta_N^*(a_1,\dots,a_k)$ is chosen in analogy with Aoki and Ohno [2] where infinite counterparts of $\zeta_N^*(a_1,\dots,a_k)$ have been treated; see also Ohno [12].

Remark 3. The sum $\zeta(m, \{1\}_{d-1})$ can be completely transformed into single zeta values. By results of Borwein, Bradley and Broadhoarst [3]

$$\zeta(2, \{1\}_d) = \zeta(d+2)$$

$$\zeta(3, \{1\}_d) = \frac{d+2}{2}\zeta(d+3) - \frac{1}{2}\sum_{\ell=1}^{d}\zeta(\ell+1)\zeta(d+2-\ell).$$

Furthermore, in the general case of $\zeta(m+2, \{1\}_d) = \zeta(d+2, \{1\}_m)$ one obtains products of up to $\min\{m+1, d+1\}$ zeta values, according to the generating function, see [3],

$$\sum_{m,n\geq 0} \zeta(m+2, \{1\}_n)x^{m+1}y^{n+1} = 1 - \exp\Bigg(\sum_{k\geq 2} \frac{x^k + y^k - (x+y)^k}{k}\zeta(k)\Bigg). \qquad (1)$$

Below we state three specific evaluations of the sum S for special choices of d, n, k.

Corollary 4. *For $k = 0$ and arbitrary $n, d \in \mathbb{N}$ we get*

$$S(d, n, 0) = \sum_{j\geq 1} \frac{\left[j \atop d \right]}{\binom{n+j}{j} j!} = \frac{1}{n^d}.$$

For $k = 1$ and arbitrary $n, d \in \mathbb{N}$ we get

$$S(d,n,1) = \sum_{j\geq 1} \frac{\left[{j \atop d}\right]}{j\binom{n+j}{j} j!} = \zeta(2,\{1\}_{d-1}) - \zeta_n(d+1) = \zeta(d+1) - H_n^{(d+1)}, \qquad (2)$$

For $n = 0$ and arbitrary $d, k \in \mathbb{N}$ we get

$$S(d,0,k) = \zeta(k+1,\{1\}_{d-1}).$$

In order to prove the results above we proceed as follows. Since

$$\frac{1}{\binom{n+j}{j}} = \frac{n!}{(n+j)^{\underline{n}}} = \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}\frac{(-1)^{\ell-1}}{j+\ell},$$

we obtain

$$S = \sum_{j\geq 1} \frac{\left[{j \atop d}\right]}{j^k\binom{n+j}{j} j!} = \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}(-1)^{\ell-1}\sum_{j\geq 1}\frac{\left[{j \atop d}\right]}{j!j^k(j+\ell)}.$$

We use partial fraction decomposition and obtain

$$\frac{1}{j^k(j+\ell)} = \sum_{m=2}^{k}\frac{(-1)^{k-m}}{j^m\ell^{k+1-m}} + \frac{(-1)^{k+1}}{\ell^k}\Big(\frac{1}{j} - \frac{1}{j+\ell}\Big).$$

Consequently, by using the partial fraction decomposition above and the representation of Stirling numbers by finite multiple zeta functions, we get

$$\begin{aligned} S &= \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}(-1)^{\ell-1}\sum_{m=2}^{k+1}\frac{(-1)^{k+1-m}}{\ell^{k+2-m}}\sum_{j\geq 1}\frac{\zeta_{j-1}(\{1\}_{d-1})}{j^m} \\ &\quad + \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}(-1)^{\ell-1}\frac{(-1)^k}{\ell^{k+1}}\sum_{j\geq 1}\zeta_{j-1}(\{1\}_{d-1})\Big(\frac{1}{j} - \frac{1}{j+\ell}\Big) = S_1 + S_2. \end{aligned}$$

By definition of the multiple zeta function we get

$$\begin{aligned} S_1 &= \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}(-1)^{\ell-1}\sum_{m=2}^{k+1}\frac{(-1)^{k+1-m}}{\ell^{k+2-m}}\zeta(m,\{1\}_{d-1}) \\ &= \sum_{m=2}^{k+1}(-1)^{k+1-m}\zeta(m,\{1\}_{d-1})\sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}\frac{(-1)^{\ell-1}}{\ell^{k+2-m}}. \end{aligned}$$

We rewrite the inner sum as

$$\sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}\frac{(-1)^{\ell-1}}{\ell^{k+2-m}} = \sum_{\ell=1}^{n}\binom{n}{\ell}\frac{(-1)^{\ell-1}}{\ell^{k+1-m}}.$$

This sum can be evaluated by using the following result of Flajolet and Sedgewick [8]:

$$\sum_{\ell=1}^{n}\binom{n}{\ell}\frac{(-1)^{\ell-1}}{\ell^{m}} = \sum_{\sum_{i=1}^{m} i\cdot m_i = m}\prod_{r=1}^{m}\frac{\left(H_n^{(r)}\right)^{m_r}}{r^{m_r} m_r!}.$$

We recall that $H_n^{(s)} = \sum_{\ell=1}^{n} 1/\ell^s$ denotes the n-th harmonic number of order s; in other words we have $H_n^{(s)} = \zeta_n(s)$, according to our previous definition of finite multiple zeta functions (1). Furthermore, it is well-known that $\sum_{\ell=1}^{n}\binom{n}{\ell}\frac{(-1)^{\ell-1}}{\ell^m} = \zeta_n^*(\{1\}_m)$, which can immediately be deduced by repeated usage of the formula $\binom{n}{k} = \sum_{\ell=k}^{n}\binom{\ell-1}{k-1}$. The multiple zeta function $\zeta(m,\{1\}_d)$ is evaluated using a result of Borwein, Bradley and Broadhoarst [3] (see Remark 3). Consequently, we can write sum S_1 as a finite sum involving higher order harmonic numbers and products of zeta functions and obtain the first part of our result. For the simplification of the inner sum

$$S_2 = \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}(-1)^{\ell-1}\frac{(-1)^k}{\ell^{k+1}}\sum_{j\geq 1}\zeta_{j-1}(\{1\}_{d-1})\Big(\frac{1}{j}-\frac{1}{j+\ell}\Big),$$

we use the notation $T_{m,\ell} = \sum_{j\geq 1}\zeta_{j-1}(\{1\}_m)\big(\frac{1}{j}-\frac{1}{j+\ell}\big)$. Subsequently, we interchange summation (compare with [11]). First we start with the simple case $m=1$ and calculate $T_{1,\ell}$, since it is most instructive.

$$T_{1,\ell} = \sum_{j\geq 1} H_{j-1}\Big(\frac{1}{j}-\frac{1}{j+\ell}\Big) = \sum_{j\geq 1} H_j\Big(\frac{1}{j+1}-\frac{1}{j+1+\ell}\Big).$$

Since by definition $H_j = \sum_{h=1}^{j} 1/h$ we obtain after summation change (partial summation)

$$T_{1,\ell} = \sum_{h\geq 1}\frac{1}{h}\sum_{j\geq h}\Big(\frac{1}{j+1}-\frac{1}{j+1+\ell}\Big) = \sum_{h\geq 1}\frac{1}{h}\sum_{j=1}^{\ell}\frac{1}{j+h}.$$

By partial fraction decomposition we get

$$T_{1,\ell} = \sum_{j=1}^{\ell}\frac{1}{j}\sum_{h\geq 1}\Big(\frac{1}{h}-\frac{1}{j+h}\Big) = \sum_{j=1}^{\ell}\frac{H_j}{j} = \frac{H_\ell^2 + H_\ell^{(2)}}{2}.$$

Now we turn to the general case $T_{m,\ell}$. Shifting the index as before, and changing the order of summation leads to

$$T_{m,\ell} = \sum_{h\geq 1} \frac{\zeta_{h-1}(\{1\}_{m-1})}{h} \sum_{j\geq h} \Big(\frac{1}{j+1} - \frac{1}{j+1+\ell}\Big)$$

Consequently,

$$T_{m,\ell} = \sum_{j=1}^{\ell} \frac{1}{j} \sum_{h\geq 1} \zeta_{h-1}(\{1\}_{m-1}) \Big(\frac{1}{h} - \frac{1}{h+j}\Big) = \sum_{j=1}^{\ell} \frac{1}{j} T_{m-1,j}.$$

Hence, the value $T_{m,\ell}$ is a variant of the finite multiple zeta function $\zeta_\ell(\{1\}_{m+1})$, where the summation indices satisfy $N \geq n_1 \geq n_2 \geq \cdots \geq n_m \geq n_{m+1} \geq 1$ instead of $N \geq n_1 > n_2 > \cdots > n_m > n_{m+1} > 1$, see Remark 2, such that $T_{m,\ell} = \zeta^*_\ell(\{1\}_{m+1})$. We further obtain

$$T_{m,\ell} = \zeta^*_\ell(\{1\}_{m+1}) = \sum_{h=1}^{\ell} \binom{\ell}{h} \frac{(-1)^{h-1}}{h^{m+1}},$$

according to the well-known formula $\binom{n}{k} = \sum_{\ell=k}^{n} \binom{\ell-1}{k-1}$. Consequently, the sum S_2 simplifies to

$$\begin{aligned} S_2 \quad &= (-1)^k \sum_{\ell=1}^{n} \binom{n}{\ell} \frac{(-1)^{\ell-1}}{\ell^k} \sum_{h=1}^{\ell} \binom{\ell}{h} \frac{(-1)^{h-1}}{h^d} \\ &= (-1)^k \sum_{h=1}^{n} \frac{(-1)^{h-1}}{h^d} \sum_{\ell=h}^{n} \binom{n}{\ell} \binom{\ell}{h} \frac{(-1)^{\ell-1}}{\ell^k}, \end{aligned}$$

or equivalently

$$S_2 = (-1)^k \sum_{\ell=1}^{n} \binom{n}{\ell} \frac{(-1)^{\ell-1}}{\ell^k} \zeta^*_\ell(\{1\}_d).$$

In order to obtain the final form of S_2 for $k \in \mathbb{N}$ we combine our previous considerations as follows:

$$S_2 = (-1)^k \sum_{h_1=1}^{n} \frac{1}{h_1} \sum_{h_2=1}^{h_1} \frac{1}{h_2} \cdots \sum_{h_{k+1}=1}^{h_k} \binom{h_k}{h_{k+1}} (-1)^{h_{k+1}-1} \zeta^*_{h_{k+1}}(\{1\}_d).$$

We use the fact that $\sum_{\ell=h}^{n} \binom{n}{\ell}\binom{\ell}{h}(-1)^{\ell-1} = \delta_{h,n}(-1)^{n-1}$ and the sum S_2 simplifies to

$$S_2 = (-1)^k \zeta^*_n(\{1\}_{k-1}, d+1).$$

In the case $k = 0$ we use

$$S_2 = \sum_{h=1}^{n} \frac{(-1)^{h-1}}{h^d} \sum_{\ell=h}^{n} \binom{n}{\ell}\binom{\ell}{h}(-1)^{\ell-1} = \frac{1}{n^d}.$$

2.1. An Application: Asymptotic Expansions

Following Romik [13] we note that the limit $\lim_{n\to\infty}\sum_{j\geq 1}\frac{\left[{j \atop d}\right]}{j^k\binom{n+j}{j}j!} = 0$ provides information about the convergence of the two sums appearing in the results for $S = S_1 + S_2$, stated in Theorem 1. This is of particular interest in the special case $k = 1$ and arbitrary $n, d \in \mathbb{N}$, where we have obtained $\zeta(d+1) - H_n^{(d+1)} = \sum_{j\geq 1}\frac{\left[{j \atop d}\right]}{j\binom{n+j}{j}j!}$ (see Corollary 4).

Proposition 5. *We have the following asymptotic expansions for $n \to \infty$*

$$(-1)^k\zeta_n^*(\{1\}_{k-1}, d+1) + \sum_{m=2}^{k+1}(-1)^{k+1-m}\zeta(m, \{1\}_{d-1})\zeta_n^*(\{1\}_{k+1-m})$$
$$= \sum_{j=1}^{n}\frac{\left[{j \atop d}\right]}{j^k\binom{n+j}{j}j!} + \mathcal{O}\Big(\frac{1}{\sqrt{n}2^n}\Big).$$

In the case of $S = S(d, n, 1)$ with $k = 1$ and arbitrary $n, d \in \mathbb{N}$ we obtain in particular:

$$\zeta(d+1) - H_n^{(d+1)} = \sum_{j=1}^{n}\frac{\left[{j \atop d}\right]}{j\binom{n+j}{j}j!} + \mathcal{O}\Big(\frac{1}{\sqrt{n}2^n}\Big), \quad \textit{for} \quad n \to \infty.$$

Proof. We can split the summation range $j \geq 1$ into $1 \leq j \leq N$ and $j \geq N+1 > k$. We get

$$\sum_{j\geq N+1}\frac{\left[{j \atop d}\right]}{j^k\binom{n+j}{j}j!} < \sum_{j\geq N+1}\frac{2}{j^k\binom{n+j}{j}} < \sum_{j\geq N+1}\frac{2}{k!\binom{j}{k}\binom{n+j}{j}}$$
$$= \frac{(n+1+N)}{k!(k-1+N)\binom{n+1+N}{n}\binom{N+1}{k}}.$$

Consequently, we readily obtain, setting $N = n$ and using Stirling's formula,

$$n! = \frac{n^n}{e^n}\sqrt{2\pi n}\Big(1 + \mathcal{O}(\frac{1}{n})\Big),$$

the stated asymptotic expansions. □

3. Relation to Nielsen's Polylogarithm

Nielsen's polylogarithm $L_{k,d}(z)$ is defined by

$$L_{k,d}(z) = \frac{(-1)^{k-1+d}}{(k-1)!d!} \int_0^1 \frac{\log^{k-1}(t) \log^d(1-zt)}{t} dt.$$

By definition of the generating function of the Stirling cycle numbers

$$\sum_{n \geq k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^n}{n!} = \frac{(-1)^k \log^k(1-z)}{k!},$$

it is evident that $L_{k,d}(z) = \sum_{j \geq 1} \frac{\left[{j \atop d}\right] z^j}{j^k j!}$. Hence, we obtain the following result.

Proposition 6. *The series* $S(z) = S_{d,n,k}(z) = \sum_{j \geq 1} \frac{\left[{j \atop d}\right] z^j}{j^k \binom{n+j}{j} j!}$ *can be expressed by Nielsen's polylogarithm* $L_{k,d}(z)$ *in the following way.*

$$\sum_{j \geq 1} \frac{\left[{j \atop d}\right] z^j}{j^k \binom{n+j}{j} j!} = \frac{n}{z} \int_0^z \left(1 - \frac{u}{z}\right)^{n-1} L_{k,d}(u) du.$$

Note that

$$\begin{aligned} S_{d,n,k}(z) &= \sum_{\ell=1}^{n} \ell(-1)^{\ell-1} \binom{n}{\ell} \frac{(-1)^{k-1}}{(k-1)!d!} \frac{1}{z^l} \int_0^z u^{\ell-1} \int_0^1 \frac{\log^{k-1}(t) \log^d(1-ut)}{t} dt du \\ &= \sum_{\ell=1}^{n} \ell(-1)^{\ell-1} \binom{n}{\ell} \frac{1}{z^l} \int_0^z u^{\ell-1} L_{k,d}(u) du. \end{aligned}$$

Interchanging summation and integration gives the desired result.

3.1. Generalized r-Stirling Numbers of the First Kind

In a recent work Mező [10] considered series involving so-called r-Stirling numbers of the first kind (see Broder [5]). For any positive integer $r \in \mathbb{N}$ the quantity $\left[{n \atop m}\right]_r$ denotes the number of permutations of the set $\{1, \ldots, n\}$ having m cycles such that the first r elements are in distinct cycles. These numbers obey the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}_r + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_r \quad (n > r),$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = \delta_{k,r} \quad (n = r),$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = 0 \quad (n < r).$$

For $r = 0$ and $r = 1$ these numbers coincide with the ordinary Stirling numbers of the first kind. We will consider the series

$$S^{(r)}(z) = S^{(r)}_{d,n,k,\ell}(z) = \sum_{j\geq 1} \frac{\begin{bmatrix} j+\ell+r \\ d+r \end{bmatrix}_r z^j}{j^k \binom{n+j}{j} j!},$$

which generalizes the series considered by Mező [10] (case $n = 0$) and our previously considered series S (case $\ell = r = 0$). Subsequently, we obtain representations of $S^{(r)}_{d,n,k,0}(z)$ and also of $S^{(r)}_{d,n,k,\ell}(z)$. We introduce the quantity $L^{(r)}_{n,k}(z)$, which generalizes Nielsen's polylogarithm:

$$L^{(r)}_{k,d}(z) = \frac{(-1)^{k-1+d}}{(k-1)!d!} \int_0^1 \frac{\log^{k-1}(t)\log^d(1-zt)}{(1-zt)^r t} dt.$$

Proposition 7. *The series* $S^{(r)}_{d,n,k,0}(z) = \sum_{j\geq 1} \frac{\left[j+r \atop d+r \right]_r z^j}{j^k \binom{n+j}{j} j!}$ *can be expressed by* $L^{(r)}_{k,d}(z)$ *in the following way.*

$$S^{(r)}_{d,n,k,0}(z) = \frac{n}{z} \int_0^z \left(1 - \frac{u}{z}\right)^{n-1} L^{(r)}_{k,d}(u)du.$$

The series $S^{(r)}_{d,n,k,\ell}(z)$ *can be expressed as a linear combination of the sums* $S^{(r+\ell)}_{h,n,k,0}(z)$, *with* $0 \leq h \leq d$.

First we note that the r-Stirling numbers of the first kind have the generating function

$$\sum_{n\geq k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{z^n}{n!} = \frac{(-1)^k \log^k(1-z)}{k!(1-z)^r}.$$

We observe that

$$L_{k,d}^{(r)}(z) = \sum_{j\ge 1} \frac{\left[{j+r \atop d+r}\right]_r z^j}{j^k \binom{n+j}{j} j!} = S_{d,0,k,0}^{(r)}(z).$$

Consequently, we get

$$S_{d,n,k,0}^{(r)}(z) = \sum_{j\ge 1} \frac{\left[{j+r \atop d+r}\right]_r z^j}{j^k \binom{n+j}{j} j!} = \int_0^z \frac{n(1-\frac{u}{z})^n}{(z-u)} L_{k,d}^{(r)}(u)du.$$

Next we turn to the general case $\ell \in \mathbb{N}$. Since

$$\sum_{n\ge k} \begin{bmatrix} n+r \\ d+r \end{bmatrix}_r \frac{z^n}{n!} = \frac{(-1)^d \log^d(1-z)}{d!(1-z)^r},$$

we obtain the exponential generating function of $\left[{n+\ell+r \atop d+r}\right]_r$ by differentiating $\frac{(-1)^d \log^d(1-z)}{d!(1-z)^r}$ ℓ times with respect to z and a subsequent shift of the index:

$$\begin{aligned} \frac{\partial^\ell}{\partial z^\ell} \frac{(-1)^d \log^d(1-z)}{d!(1-z)^r} &= \sum_{n\ge d+\ell} \begin{bmatrix} n+r \\ d+r \end{bmatrix}_r \frac{z^{n-\ell}}{(n-\ell)!} \\ &= \sum_{n\ge \max\{d-\ell,0\}} \begin{bmatrix} n+\ell+r \\ d+r \end{bmatrix}_r \frac{z^n}{n!}. \end{aligned}$$

By Faà di Bruno's formula we get

$$\begin{aligned} \frac{\partial^\ell}{\partial z^\ell} \frac{(-1)^d \log^d(1-z)}{d!(1-z)^r} &= \sum_{h=0}^{\ell} \frac{d^{\underline{h}}(-1)^h \log^{d-h}(1-z)}{(1-z)^{r+\ell}} \\ &\quad \times \sum_{i=h}^{\ell} r^{\overline{\ell-i}} B_{i,h}(0!, 1!, 2!, \dots, (i-h)!), \end{aligned}$$

where $B_{i,h}(x_1, x_2, \dots, x_{i-h+1})$ denote the Bell polynomials. Consequently, we can express the sum $S_{d,n,k,\ell}^{(r)}(z)$ as a linear combination of the sums $S_{h,n,k,0}^{(r)}(z)$, with $0 \le h \le d$, which proves the stated result.

Remark 8. Note that the sums $S^{(r)}_{d,n,k,\ell}(1)=\sum_{j\ge1}\frac{\left[{j+\ell+r \atop d+r}\right]_r z^j}{j^k\binom{n+j}{j}j!}$ can in principle also be treated using our previous approach; however, the expression become much more involved, therefore we refrain from going into this matter. Furthermore, one can evaluate sums of the form $\sum_{j\ge1}\frac{\left[{j \atop d}\right]}{j^k\binom{n+j}{j}^g j!}$, with $g\in\mathbb{N}$; however, the expressions get more and more involved.

4. A Generalization of Series *S*

In the following we will briefly consider the more general series V defined by

$$V=V(a_1,\dots,a_r,n,k)=\sum_{j\ge1}\frac{\zeta_{j-1}(a_1,\dots,a_r)}{j^{k+1}\binom{n+j}{j}},$$

with $a_i\in\mathbb{N}$ for $1\le i\le r$, and $n,k\in\mathbb{N}_0=\{0,1,2,\dots\}$ such that $n+k\ge1$. We reobtain our previous Stirling cycle number series S choosing $r=d-1$ and $a_i=1$, $1\le i\le d-1$. Before we state our result for the series V we introduce one more series, namely a variant of the finite multiple zeta star function

$$\begin{aligned}A^*_N(a_1,\dots,a_r) &= \sum_{N\ge n_1\ge n_2\ge\dots\ge n_r\ge1}\binom{N}{n_1}\frac{(-1)^{a_1-1}}{n_1^{a_1}n_2^{a_2}\dots n_r^{a_r}}\\ &= \sum_{n_1=1}^{N}\binom{N}{n_1}\frac{(-1)^{a_1-1}}{n_1^{a_1}}\zeta^*_{n_1}(a_2,\dots,a_r),\end{aligned}$$

which can be expressed in terms of $\zeta^*_N(a_1,\dots,a_r)$ by the relation

$$A^*_N\left(a_1,\{1\}_{b_1-1},\bigcup_{i=2}^{r}\{a_i+1,\{1\}_{b_i-1}\}\right)$$
$$=\zeta^*_N\left(\bigcup_{i=1}^{r-1}\{\{1\}_{a_i-1},b_j+1\},\{1\}_{a_r-1},b_r\right),$$

which is due to Bradley [4].

Theorem 9. *The sum* $V=V(a_1,\dots,a_r,n,k)$ *with* $a_i\in\mathbb{N}$ *for* $1\le i\le r$, *and* $n,k\in\mathbb{N}_0=\{0,1,2,\dots\}$ *such that* $n+k\ge1$, *can be evaluated in terms of* *(finitely many) multiple zeta functions,*

$$V = (-1)^k \sum_{g=1}^{r} (-1)^{\sum_{f=1}^{g-1}(a_f+1)} \sum_{m=2}^{a_g} (-1)^{a_g-m} \zeta\left(m, \bigcup_{i=g+1}^{r} \{a_i\}\right) \times A_n\left(k, \bigcup_{i=1}^{g-1}\{a_i\}, a_g+1-m\right) + (-1)^{k+r+\sum_{f=1}^{r} a_f} A_n\left(k, \bigcup_{i=1}^{r}\{a_i\}, 1\right) + \sum_{m=2}^{k+1} (-1)^{k+1-m} \zeta\left(m, a_1, \dots, a_r\right) \zeta_n^*(\{1\}_{k+1-m}).$$

Proof (Sketch). The proof is analogous to the proof of Theorem 1; therefore it is only sketched. We elaborate only on the main new difficulty – the evaluation of the sum $T_{a_1,\dots,a_r;\ell} = \sum_{j\geq 1} \zeta_{j-1}(a_1,\dots,a_r)\big(\frac{1}{j} - \frac{1}{j+\ell}\big)$. Proceeding as before, i.e., interchanging summation and using partial fraction decomposition, we obtain the recurrence relation

$$T_{a_1,\dots,a_r;\ell} = \sum_{m=2}^{a_1} (-1)^{a_1-m} \zeta(m, a_2, \dots, a_r) \zeta_\ell^*(a_1+1-m) + (-1)^{a_1+1} \sum_{i=1}^{\ell} \frac{1}{i^{a_1}} T_{a_2,\dots,a_r;i}.$$

One can show that

$$T_{a_1,\dots,a_r;\ell} = \sum_{g=1}^{r} (-1)^{\sum_{f=1}^{g-1}(a_f+1)} \sum_{m=2}^{a_g} (-1)^{a_g-m} \zeta(m, \cup_{i=g+1}^{r}\{a_i\}) \times \zeta_\ell^*(\cup_{i=1}^{g-1}\{a_i\}, a_g+1-m) + (-1)^{r+\sum_{f=1}^{r} a_f} \zeta_\ell^*(\cup_{i=1}^{r}\{a_i\}, 1),$$

which implies the stated result for the series V. □

5. Historical Remark and Acknowledgement

The author H.P. has found the formula (2) empirically in 2003. He contacted several specialists about it and got feedback from Christian Krattenthaler who provided a *hypergeometric* proof for it. Eventually it turned out that it was known already [6], page 252, Equation 16. We are happy that in 2009 we could put new life into this project.

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