CONGRUENCES FOR OVERPARTITION K-TUPLES

Shi-Chao Chen[1]

*Institute of Contemporary Mathematics, School of Mathematics and Information
Science, Henan University, Kaifeng, 475001, P.R.China*
schen@henu.edu.cn

Abstract

An *overpartition* of the nonnegative integer n is a non-increasing sequence of natural
numbers whose sum is n in which the first occurrence of a number may be overlined.
Let $k \geq 1$ be an integer. An overpartition k-tuple of a positive integer n is a k-tuple
of overpartitions wherein all listed parts sum to n. Let $\overline{p}_k(n)$ be the number of
overpartition k-tuples of n. In this paper, we will give a short proof of Keister,
Sellers and Vary's theorem on congruences for $\overline{p}_k(n)$ modulo powers of 2. We also
obtain some congruences for $\overline{p}_k(n)$ modulo prime ℓ and integer $2k$.

1. Introduction

An *overpartition* of the nonnegative integer n is a non-increasing sequence of natural
numbers whose sum is n in which the first occurrence of a number may be overlined
(see [3]). Let $\overline{p}(n)$ be the number of overpartitions of an integer n. For convenience,
define $\overline{p}(0) = 1$. For example, the overpartitions of 4 are

$$
\begin{aligned}
&4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1, \\
&\overline{4}, \quad \overline{3}+\overline{1}, \quad \overline{3}+1, \quad 3+\overline{1}, \quad \overline{2}+2, \quad \overline{2}+\overline{1}+1, \\
&\overline{2}+1, \quad 2+1+\overline{1}, \quad \overline{1}+1+1+1.
\end{aligned}
$$

Thus $\overline{p}(4) = 14$. The generating function for $\overline{p}(n)$ is

$$
\overline{P}(q) := \sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots .
$$

An *overpartition k-tuple* of a positive integer n is a k-tuple of overpartitions
wherein all listed parts sum to n (see [6,9]). Let $\overline{p}_k(n)$ be the number of overpar-

tition k-tuples of n. The generating function for $\overline{p}_k(n)$ is

$$\overline{P}_k(q) := \sum_{n=0}^{\infty} \overline{p}_k(n) q^n = \prod_{n=1}^{\infty} \left(\frac{1 + q^n}{1 - q^n} \right)^k. \tag{1}$$

A number of mathematicians have used the overpartition function to interpret or to prove combinatorial identities arising from basic hypergeometric series. Many arithmetic properties are also obtained. For more information about overpartitions, see [2,3,4,5,7]. Recently, Keister, Sellers and Vary [6] studied some arithmetic properties of the number of overpartition k-tuples $\overline{p}_k$. They obtained some congruences for $\overline{p}_k$ modulo powers of 2 and modulo a prime. In this paper, we will give a short proof of a theorem in [6] and determine the relation between $\overline{p}_{\ell^s}$ and $\overline{p}$ modulo a prime ℓ. For any positive integer k, we also prove a congruence for $\overline{p}_k$ modulo $2k$.

2. Congruences for $\overline{p}_k$ Modulo Powers of 2

In this section we will prove the following Theorem 1. This theorem is proved in [6] by induction arguments. Here we give a short proof by an application of the binomial theorem which seems more clear and natural.

Theorem 1 *Let $m \geq 0$ be an integer and $r \geq 1$ be an odd integer. If $k = 2^m r$, then for all $n \geq 1$,*

$$\overline{p}_k(n) \equiv \begin{cases} 2^{m+1} \pmod{2^{m+2}} & \text{if } n \text{ is a square or twice a square,} \\ 0 \pmod{2^{m+2}} & \text{otherwise.} \end{cases}$$

To begin, we prove the following preliminary lemma. Let p be a prime. We denote by $\nu_p(m)$ the exponent of the highest power of p dividing integer m. Clearly, $\nu_p(\frac{a}{b}) = \nu_p(a) - \nu_p(b)$ if $\frac{a}{b} \in \mathbb{Q}$ and $\nu_p(ab) = \nu_p(a) + \nu_p(b)$ if $a, b \in \mathbb{Z}$.

Lemma 2 *Let $m \geq 1$ and $l \geq 0$ be an integer, and let p be a prime. Then for $1 \leq s \leq p^m - 1$, we have*

$$\nu_p\left(\binom{p^m}{s} p^{ls} \right) \geq ls + 1.$$

Proof. Clearly, $\nu_p(\frac{p^m-i}{i}) = 0$ for $1 \le i \le s-1$. We have

$$
\begin{aligned}
\nu_p\left(\binom{p^m}{s}p^{ls}\right) &= \nu_p\left(p^m \cdot \frac{p^m-1}{1} \cdot \frac{p^m-2}{2} \cdot \cdots \cdot \frac{p^m-(s-1)}{s-1} \cdot \frac{p^{ls}}{s}\right) \\
&= \nu_p(p^m) + \nu_p(\frac{p^m-1}{1}) + \nu_p(\frac{p^m-2}{2}) + \cdots + \\
&\quad + \nu_p(\frac{p^m-(s-1)}{s-1}) + \nu_p(\frac{p^{ls}}{s}) \\
&= m + ls - \nu_p(s) \\
&\ge ls + 1,
\end{aligned}
$$

where we note that $\nu_p(s) \le m-1$ for any $1 \le s \le p^m - 1$. $\square$

Remark Lemma 2 gives a natural generalization of Lemma 2 of [6]. When $l = 0$, an immediate corollary of Lemma 2 is

$$
\binom{p^m}{s} \equiv 0 \ (\mathrm{mod}\ p)
$$

for p prime, $m \ge 1$ and $1 \le s \le p^m - 1$. By the binomial theorem, this implies the following well known fact:

$$
(1 + aq^t)^{p^m} \equiv 1 + (aq^t)^{p^m} \ (\mathrm{mod}\ p) \tag{2}
$$

for any integer a.

Proof of Theorem 1. Let

$$
\theta_1(q) := \prod_{n=1}^{\infty} \frac{1-q^n}{1+q^n}.
$$

Then by (1) the generating function for $\overline{p}_k$ is

$$
\overline{P}_k(q) = \sum_{n=0}^{\infty} \overline{p}_k(n)q^n = \left(\frac{1}{\theta_1(q)}\right)^k. \tag{3}
$$

By Gauss's identity [1,Corollary 2.10], we have

$$\theta_1(q) = \sum_{-\infty}^{\infty} (-1)^n q^{n^2}$$

$$= 1 + 2\sum_{n=1}^{\infty} q^{4n^2} - 2\sum_{n=1}^{\infty} q^{(2n-1)^2}$$

$$= 1 + 2\sum_{n=1}^{\infty} q^{n^2} - 4\sum_{n=1}^{\infty} q^{(2n-1)^2}. \tag{4}$$

It is easy to see

$$\theta_1(q)^2 \equiv 1 \pmod{4}.$$

Therefore we have

$$\frac{1}{\theta_1(q)} \equiv \theta_1(q) \pmod{4}.$$

This implies that

$$\frac{1}{\theta_1(q)} = \theta_1(q) + 4f(q)$$

for some $f(q) \in \mathbb{Z}[[q]]$. In view of the binomial theorem and Lemma 2, we have for any $m \geq 0$

$$\left(\frac{1}{\theta_1(q)}\right)^{2^m} = (\theta_1(q) + 4f(q))^{2^m}$$

$$\equiv \theta_1(q)^{2^m} \pmod{2^{m+2}}$$

$$= \left(1 + 2\sum_{n=1}^{\infty} q^{n^2} - 4\sum_{n=1}^{\infty} q^{(2n-1)^2}\right)^{2^m}$$

$$\equiv \left(1 + 2\sum_{n=1}^{\infty} q^{n^2}\right)^{2^m} \pmod{2^{m+2}}$$

$$\equiv 1 + 2^{m+1}\sum_{n=1}^{\infty} q^{n^2} + 4\binom{2^m}{2}\left(\sum_{n=1}^{\infty} q^{n^2}\right)^2 \pmod{2^{m+2}}$$

$$= 1 + 2^{m+1}\sum_{n=1}^{\infty} q^{n^2} - 2^{m+1}\left(\sum_{n=1}^{\infty} q^{n^2}\right)^2 \pmod{2^{m+2}}$$

$$= 1 + 2^{m+1} \sum_{n=1}^{\infty} q^{n^2} - 2^{m+1} \left(\sum_{n=1}^{\infty} q^{2n^2} + 2 \sum_{\substack{n_1,n_2=1 \\ n_1<n_2}}^{\infty} q^{n_1^2+n_2^2} \right)$$

$$\equiv 1 + 2^{m+1} \sum_{n=1}^{\infty} q^{n^2} + 2^{m+1} \sum_{n=1}^{\infty} q^{2n^2} \pmod{2^{m+2}}. \tag{5}$$

Immediately,

$$\left(\frac{1}{\theta_1(q)} \right)^{2^{m+1}} \equiv 1 \pmod{2^{m+2}}.$$

Now we suppose that $k = 2^m r$ and $r = 2s + 1 > 0$. It follows that

$$\left(\frac{1}{\theta_1(q)} \right)^k = \left(\frac{1}{\theta_1(q)} \right)^{2^{m+1}s} \left(\frac{1}{\theta_1(q)} \right)^{2^m} \equiv \left(\frac{1}{\theta_1(q)} \right)^{2^m} \pmod{2^{m+2}}. \tag{6}$$

Theorem 1 follows from (3), (6) and (5). □

3. Congruences for $\overline{p}_k$ Modulo a Prime ℓ

Theorem 3   *Let ℓ be a prime and s be a positive integer. Then*

$$\overline{p}_{\ell^s}(n) \equiv \begin{cases} \overline{p}(m) \pmod{\ell} & \text{if } n = \ell^s m \text{ for some nonnegative integer } m, \\ 0 \pmod{\ell} & \text{otherwise.} \end{cases} \tag{7}$$

We also have that

$$\overline{p}_{\ell^s-1}(n) \equiv 0 \pmod{\ell} \tag{8}$$

for each n which is not a square modulo ℓ^s.

Proof.   By (2), it follows that

$$\frac{1}{\theta_1(q)^{\ell^s}} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^{\ell^s} \equiv \prod_{n=1}^{\infty} \left(\frac{1+q^{\ell^s i}}{1-q^{\ell^s i}} \right) = \frac{1}{\theta_1(q^{\ell^s})} \pmod{\ell}.$$

Therefore by (3) we have

$$\sum_{n=0}^{\infty} \overline{p}_{\ell^s}(n) q^n \equiv \sum_{m=0}^{\infty} \overline{p}(m) q^{\ell^s m} \pmod{\ell}.$$

Equating the coefficients, we get (7).

Observe that

$$\frac{1}{\theta_1(q)^{\ell^s-1}} = \frac{\theta_1(q)}{\theta_1(q)^{\ell^s}} \equiv \frac{1}{\theta_1(q^{\ell^s})} \cdot \theta_1(q) (\mathrm{mod} \ \ell).$$

It follows from (3) and (4) that

$$\sum_{n=0}^{\infty} \overline{p}_{\ell^s-1}(n)q^n \equiv \left(\sum_{i=0}^{\infty} \overline{p}(i)q^{\ell^s i} \right) \left(\sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \right) (\mathrm{mod} \ \ell).$$

Therefore if n is not a square modulo ℓ^s, then $\overline{p}_{\ell^s-1}(n)$ vanishes modulo ℓ. This proves (8). $\qquad\square$

Corollary 4   *Let $Q \equiv -1(\ \mathrm{mod}\ 5)$ be a prime. Then for any integer $s \geq 1$,*

$$\overline{p}_{5^s}(5^{s+1}Q^3 n) \equiv 0 \ (\mathrm{mod}\ 5)$$

for all n coprime to Q.

Proof.   Treneer [8] proved that if $Q \equiv -1(\ \mathrm{mod}\ 5)$, then

$$\overline{p}(5Q^3 n) \equiv 0 \ (\mathrm{mod}\ 5)$$

for all n coprime to Q. Therefore the corollary follows from (7). $\qquad\square$

Remarks (i) In fact, we can obtain congruence properties of $\overline{p}_k$ modulo powers of ℓ by the theory of modular forms. Treneer [8] showed that the coefficients of a weakly holomorphic modular form satisfy congruence properties modulo powers of a prime. Since the generating function for $\overline{p}_k$, i.e., $\frac{1}{\theta_1(q)^k}$, is a weakly holomorphic modular form of weight $\frac{k}{2}$ on the congruence subgroup $\Gamma_0(16)$ of $SL_2(\mathbb{Z})$, it follows from Corollary 1.3 of [8] that $\overline{p}_k$ has the following property: Suppose that $\ell \geq 5$ is a prime, k is an odd positive integer and m is a sufficiently large integer. Then for any positive integer j, there is a positive proportion of the primes $Q \equiv -1(\ \mathrm{mod}\ 16\ell^j)$ such that

$$\overline{p}_k(Q^3 \ell^m n) \equiv \ 0 \ (\ \mathrm{mod}\ \ell^j)$$

for all n coprime to $Q\ell$. A similar result holds for k even.

(ii) Congruence (8) extends Theorem 1 in [6].

4. Congruences for $\overline{p}_k$ Modulo $2k$

Theorem 5 *Let $k \geq 1$ be an integer. Then*

$$n\overline{p}_k(n) \equiv 0 \pmod{2k}.$$

In particular,

$$\overline{p}_k(n) \equiv 0 \pmod{2k}$$

for all n with $gcd(n, 2k) = 1$.

Proof. Taking the logarithmic derivative of (3), we have

$$\frac{\overline{P}_k(q)'}{\overline{P}_k(q)} = -k\frac{\theta_1(q)'}{\theta_1(q)}.$$

By (4), we have

$$\theta_1(q)' = 2\sum_{n=1}^{\infty} n^2 q^{n^2-1} - 4\sum_{n=1}^{\infty}(2n-1)^2 q^{(2n-1)^2-1}.$$

Note that

$$\overline{P}_k(q)' = \sum_{n=1}^{\infty} n\overline{p}_k(n)q^{n-1}$$

and

$$\frac{1}{\theta_1(q)} = \overline{P}(q).$$

Therefore we deduce that

$$\sum_{n=1}^{\infty} n\overline{p}_k(n)q^n = q \cdot \overline{P}_k(q)'$$

$$= -kq \cdot \theta_1(q)'\overline{P}_k(q)\overline{P}(q)$$

$$= -k\left(2\sum_{n=1}^{\infty} n^2 q^{n^2} - 4\sum_{n=1}^{\infty}(2n-1)^2 q^{(2n-1)^2}\right)\overline{P}_k(q)\overline{P}(q).$$

It follows immediately that

$$n\overline{p}_k(n) \equiv 0 \pmod{2k}. \qquad \square$$

Remark Let ℓ be a prime. Then Theorem 5 implies that

$$\sum_{n=0}^{\infty} \overline{p}_\ell(n) q^n \equiv \sum_{n=0}^{\infty} \overline{p}_\ell(\ell n) q^{\ell n} (\bmod \ell).$$

On the other hand,

$$\sum_{n=0}^{\infty} \overline{p}_\ell(n) q^n = \frac{1}{\theta_1(q)^\ell} \equiv \frac{1}{\theta_1(q^\ell)} = \sum_{n=0}^{\infty} \overline{p}(n) q^{\ell n} (\bmod \ell).$$

It follows that

$$\overline{p}_\ell(\ell n) \equiv \overline{p}(n) (\bmod \ell)$$

for all $n \geq 0$. This gives an alternative proof of (7) in the case $s = 1$.

References

[1] G.E.Andrews, *The Theory of Partitions*, Cambridge University Press (1984).

[2] K.Bringmann and J.Lovejoy, *Rank and congruences for overpartition pairs*, Int. J. of Number Theory **4** (2008), 303-322.

[3] S.Corteel and J.Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc. **356** (2004), 1623-1635.

[4] M.D.Hirschhorn and J.A.Sellers, *An infinite family of overpartition congruences modulo 12*, Integers **5** (2005), Article A20.

[5] M.D.Hirschhorn and J.A.Sellers, *Arithmetic relations for overpartitions*, J. Combin. Math. Combin. Comput. **53** (2005), 65-73.

[6] D.M.Keister, J.A.Sellers, R.G.Vary, *Some arithmetic properties of overpartition k-tuples*. Integers **9** (2009), A17, 181-190.

[7] K.Mahlburg, *The overpartition function modulo small powers of 2*, Discrete Math. **286**(3) (2004), 263-267.

[8] S.Treneer, *Congruences for the coefficients of weakly holomorphic modular forms*, Proc. London Math. Soc. **93**(3) (2006), 304-324.

[9] R.Vary, *Some overpartition k-tuple congruences*, The Penn State McNair Journal **15** (2009), to appear.