A LEFT WEIGHTED CATALAN EXTENSION[1]

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Abstract

There are many representations of the Catalan numbers. In this article, we will examine the ballot problem and extend it beyond the standard p-Catalan extension to a left weighted extension. This extension reduces naturally to the p-Catalan numbers (including the standard 2-Catalan numbers). We will then give a variety of examples of the extension for other representations of the Catalan numbers to demonstrate its general applicability.

1. Introduction

The Catalan numbers enumerate a wide variety of combinatorial structures. The generalized p-Catalan numbers (where $p = 2$ reduces to the Catalan numbers) count corresponding generalized structures in which some parameters have been adjusted. Specifically, in structures such as the ballot problem, the p-Catalan version changes the weight of the "opening" elements (the positive ballots) to $p - 1$, while the "closing" elements (the negative ballots) remain the same. In this paper, we will give a representation that allows the "opening" weights to be non-identical, and count the number of structures which fit within a specific "weight set" that specifies allowable weights (including multiplicity) for the opening elements. We will make frequent references to Stanley's list of Catalan representations from Enumerative Combinatorics II [10, 11]. These can be found in Exercise 6.19, where each of the representations discussed is given as a part of the exercise. Throughout, we will simply refer to the part of the exercise, such as **(r)** for exercise 6.19.r, which details the ballot problem.

2. The Ballot Problem

The ballot problem examines the number of ways a series of $2n$ votes can occur such that a given candidate never trails and the voting ends in a tie. This is represented by looking at arrangements of n copies each of 1 and -1 (which we will represent as just $-$), whose partial sums are always non-negative, and whose total sum is zero, as shown in Figure 1.

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[1]The material presented here is from my dissertation [6].

$$111 - -- \qquad 11 - 1 - - \qquad 11 - -1- \qquad 1 - 11 - - \qquad 1 - 1 - 1-$$

Figure 1: Ballot problem for $n = 3$

From this representation, it is easy to find natural bijections to most other representations of the Catalan numbers. For example, the Dyck paths are given by reading each 1 as a vertical step up and each $-$ as a horizontal step to the right. The formula for the Catalan numbers is well established to be $C_n = \dfrac{1}{n+1}\binom{2n}{n}$ [10].

A common extension to the Catalan numbers is the p-Catalan numbers [3, 4], where $p > 1$. Under this extension, the usual Catalan numbers are considered the 2-Catalan numbers. For the ballot problem, instead of n positive votes of 1 and n negative votes of -1, we have n positive votes of $p - 1$ and $n(p - 1)$ negative votes of -1. (We retain the partial sum and total sum requirements on the sequences.) For example, for $p = 3$, $n = 2$, the three possible sequences are given by Figure 2. Note that this extension is one-sided: that is, it changes the weights on one side

$$22 - - - - \qquad 2 - 2 - -- \qquad 2 - -2 - -$$

Figure 2: 3-Catalan ballot problem for $n = 2$

(the increases) but not on the other (the decreases). The formula for the p-Catalan numbers is $C_{n,p} = \dfrac{1}{pn+1}\binom{pn+1}{n}$ [3, 4].

2.1. One-Sided Generalization

Now consider the case where we allow the weights on the positive side to vary instead of being equal. For example, fix a weight set $\mathbb{B}$, as a multiset over $\mathbb{N}$ such that the sum of all the elements in $\mathbb{B}$ is equal to n. Note that $\mathbb{B}$ can also be considered a partition of n. Now we can ask how many arrangements exist with the elements of $\mathbb{B}$ as our positive numbers and n copies of -1 that have non-negative partial sums and a total sum of zero. Restricting the multi-set $\mathbb{B}$ to $\{1^n\}$ returns us to the Catalan number case detailed above, and a weight set of $\{(p-1)^n\}$ is the p-Catalan number case.

Theorem 1. *Assume* $\mathbb{B} = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}$, *with* $\sum a_i = m$, $\sum i a_i = n$. *The number of valid ballot arrangements with weighted positive votes given by the multiset* $\mathbb{B}$ *over* $\mathbb{N}$ *is given by* $\binom{n+1+m}{n+1, a_1, a_2, \ldots, a_n} \frac{1}{n+1+m}$.

Proof. Define $W(\mathbb{B})$ to be the set of words created by taking $n + 1$ copies of -1 and the elements of $\mathbb{B}$, and arranging them in any order. There are $\binom{n+m+1}{n+1, a_1, a_2, \ldots, a_n}$ elements of $W(\mathbb{B})$. Now, we set up an equivalence relation among these words, such that any two words are equivalent if one is the cyclic shift of the other.

Given any $v = (v_1 v_2 \ldots v_{n+m+1}) \in W(\mathbb{B})$, will we find a member of its equivalence class which has non-negative partial sums excepting the final element (which must be -1). Let $z = \min\{i : \sum_{j=1}^{i} v_j < 0 \text{ and } \sum_{j=i+1}^{k} v_j > 0 \text{ for all } k > i\}$. The set consists of all points where the partial sum of the letters in the word to the left is negative and all partial sums of letters to the right are non-negative, and v_z is the first such point. Thus, the elements after v_z form the maximal "tail" which has non-negative partial sums. If $z = n + m + 1$, let $v' = v$. Otherwise, we move the tail to the front of the sequence, forming $v' = (v_z \, v_{z+1} \ldots v_{m+n+1} \, v_1 \ldots v_{z-1})$. Now, partial sums of v' will be non-negative up to v'_{m+n}, and the last element of v' will be -1. Note the following two properties:

- v' is in the same equivalence class as v: Since v' is simply a cyclic shift of v by z, they are in the same equivalence class.

- v' is unique for each equivalence class: Assume, to the contrary, that we have two elements v and w that belong to the same equivalence class and $v' \neq w'$. Let x be the cyclic shift of v' to obtain w', i.e., $w' = (v_x \, v_{x+1} \ldots v_{m+n+1} \, v_1 \ldots v_{x-1})$. Now, by our formation of v', we know that $\sum_{i=1}^{x-1} v_i$ is non-negative and we know that $\sum_{i=1}^{m+n+1} w_i = -1$. But this means that $\sum_{j=1}^{m+n+1-x} w_j = \sum_{i=x+1}^{m+n+1} v_i < 0$, which violates the restrictions on w', giving us a contradiction.

If we remove the final -1 from v' we see that we have a unique element of $\mathbb{B}$. Since there are $n + m + 1$ numbers in each sequence, there are $n + m + 1$ members of each equivalence class, so we have that the number of valid arrangements of $\mathbb{B}$ is $\binom{n+m+1}{n+1, a_1, a_2, \ldots, a_n} \frac{1}{n + m + 1}$.   □

The sequences v' that we found and the equivalence relation on the v is a version of Lukasiewicz words, as seen in Stanley [10, Example 6.6.7].

We refer to this generalization as the "left weighted Catalan numbers" since we are only allowing the positive weights to vary, and fixing the negative weights at negative one. (The term "Left" comes from the parenthesis representation of the Catalan numbers, where the left parentheses were weighted; this was the original representation under investigation for use in the Inversion problem given later.)

Central to this problem is the weight set $\mathbb{B}$, which determines the structure we are looking at, and also acts as a partition of $[n] = \{1, 2, \ldots, n\}$. Given a partition, $\mathbb{B}$, we get the formula in Theorem 1 for the number of ballot sequences we have for that partition. If we sum this over all partitions $\mathbb{B}$ of $[n]$, we get the Large Schröder numbers. We can see this by considering Schröder's second problem, stated by Stanley as plane trees with n leaves and no vertex of degree one [10, Example 6.2.8] and mapping these trees to the ballot structure we have given (see the

bijection between **(r)** and **(d)** later in this article.) Thus, we have found an inter-
mediate step between the Catalan numbers (and the p-Catalan numbers), which we
are generalizing, and the Large Schröder numbers, which is the sum of our formula
over all $\mathbb{B}$.

3. Other Representations

In this section, we extend other representations of the Catalan numbers, based on
a fixed weight set $\mathbb{B}$.

The representations are organized by structure type. For each Catalan version,
we will list the base 2-Catalan case, then give a description of the left weighted
Catalan case. For most items, we will just give a sketch of the bijection, rather
than a full proof. Finally, we will show five of the structures from each Catalan
representation that belong to the weight set $\mathbb{B} = \{1^1, 2^2\}$. We start with a complete
list of the weighted ballots for this $\mathbb{B}$, shown in Figure 3; the first five (bolded)
elements are those we will be repeating for each subsequent Catalan example.

Where we refer to a proof for the standard Catalan case, we are referring to the
one implied or given by Stanley, except as noted otherwise. In our extension, we will
use the following notation: $\mathbb{B} = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}$ is our weight set, $m = \sum_{i=1}^{n} a_i$
is the cardinality of $\mathbb{B}$ and $n = \sum_{i=1}^{n} i a_i$. As previously mentioned, $\mathbb{B}$ also functions
as a partition of $[n]$.

221 $-----$	**2** $-$ **2** $--$ **1** $--$	**2** $--$ **21** $---$	**2** $--$ **1** $-$ **2** $--$
1 $-$ **2** $-$ **2** $---$	$12--2----$	$1-22-----$	$12---2--$
$12-2-----$	$122------$	$2--12---$	$2--2--1-$
$2--2-1--$	$2-1--2--$	$2-1-2---$	$2-12----$
$2-2----1-$	$2-2-1---$	$2-21----$	$21---2--$
$21--2---$	$21-2----$	$212-----$	$22----1-$
$22---1--$	$22--1---$	$22-1----$	$1-2--2--$

Figure 3: Weighted ballots (Catalan case **(r)**) $\mathbb{B} = \{1^1, 2^2\}$

4. Trees

(c): The set of binary trees with n nodes.

Generalization (Figure 4): We look at binary trees with n nodes, such that the tree
is partitioned into "straight line" groups of nodes, such that the number of nodes

in the groups is given by $\mathbb{B}$ (*i.e.*, there are a_i groups of size i, for all i). To make a straight line group of nodes, take a node, called the base, and note whether it is a left or right child. Then take the same child of the base, and that child of the child, etc., until we have i nodes in a straight line. (Define the root of the tree to be a left child.) This reduces to the Catalan case where each node is its own group, *i.e.*, we always use the trivial partition of singletons.
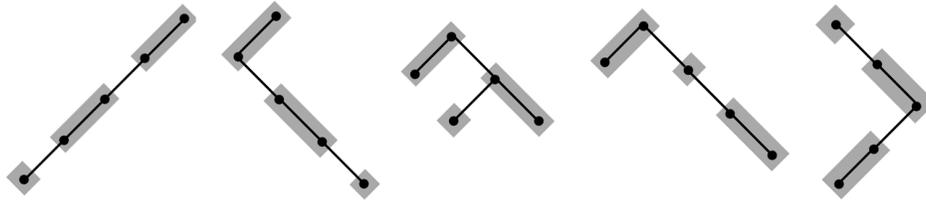

Figure 4: Weighted Calatan examples for **(c)**

We can create a bijection to **(r)** by starting with the root, counting the number of nodes in its straight line group as the first positive number. Then we examine the children of this group, in pre-order, and record a $-$ for each missing child. Once a new group is found, repeat the process on the new group. (As usual, omit the final -1). This will give us a sequence of positive weights counting the nodes in groups, and the list of -1 can never exceed the number of nodes, since each node only adds one additional possibility for -1 (until the final missing child, which we omit).

(d): Plane trees with n internal nodes, all of degree 2. (Each node has 0 or 2 children.) In the p-Catalan case, this becomes plane trees with n internal nodes, all of degree p. (Each node has 0 or p children.)

Generalization (Figure 5): Trees of type $(n + 1, 0, a_1, a_2, \ldots a_n)$. (Recall that this means that the tree has $n + 1$ leaves, 0 nodes with 1 child, a_1 nodes with two children etc.)
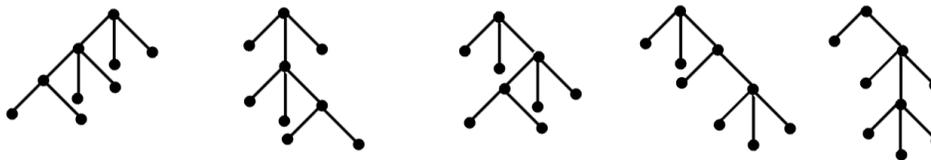

Figure 5: Weighted Calatan examples for **(d)**

Bijects to (**r**): Each positive ballot corresponds to an internal node in the tree of degree equal to the ballot's weight plus one, and each negative ballot (plus an additional one) corresponds to a leaf. The positive ballots increase the number of active branches of the tree by their weight, and the negative ballots decrease the number of them by one each. The tree is read in pre-order (parent before leaves) ordering to recover the ballot sequence.

(**e**): The set of plane trees with $n + 1$ nodes.

Generalization (Figure 6): we look at plane trees with $n+1$ nodes with a partition on the edges of the tree such that the sizes of the blocks are given by $\mathbb{B}$ and in each block, all but the topmost edge connects to the leftmost child of its parent. This reduces to the Catalan case where each edge is its own block, *i.e.,* we always use the trivial partition of singletons.

In the Catalan case, the bijection to (**r**) is otained by reading the tree in pre-order, and treat each step downward as a 1 and each step back up as a -1. In the generalized case, we do the same, but read the entire block of a partition as a single positive number.
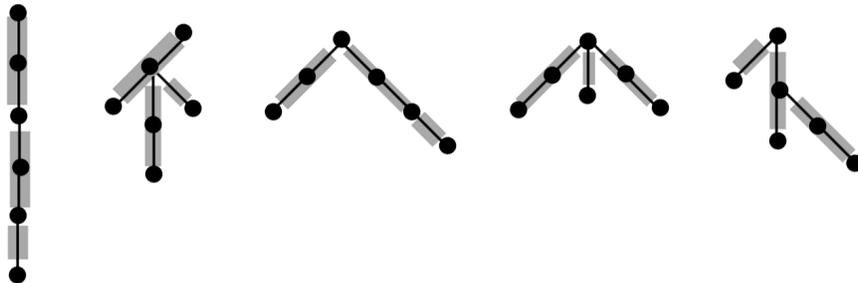

Figure 6: Weighted Calatan examples for (**e**)

(**f**): Planted trivalent trees with $2n + 2$ nodes.

Generalization (Figure 7): Planted trees with with internal node valences given by $v_2 = 0$ and for $i > 2$, v_i is the number of $i - 2$ in $\mathbb{B}$ (and the root has valence 1). In the Catalan case, the internal valences given are of size 3, corresponding to the n copies of 1 in $\mathbb{B}$.

As in the Catalan case, we find a bijection to (**d**) by cutting the root to get a tree of type $(n + 1, 0, v_3, \dots)$.
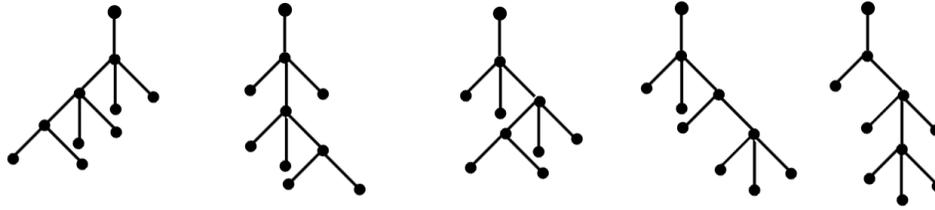
Figure 7: Weighted Calatan examples for **(f)**

(g): Plane trees with $n + 2$ nodes such that the rightmost path of each subtree of the root has even length.

Generalization (Figure 8): Plane trees as in **(e)** (with an extra group of size 1 added, to make $n + 2$ nodes) with the restrictions that the rightmost path of any subtree of the root (counting the edges from the root of the subtree along its rightmost path) is required to be of even length, and the leftmost child of the root will be its own block in the partition.
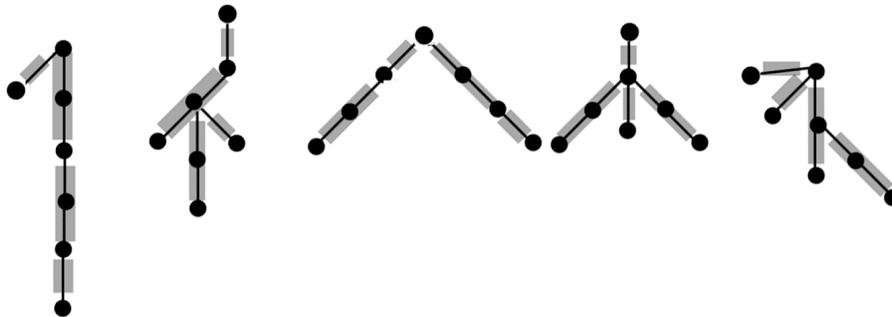

Figure 8: Weighted Calatan examples for **(g)**

The bijection to **(e)** is the same as in the Catalan case: Examine the tree and find the rightmost subtree of the root which has an odd rightmost path. Insert a new node as the leftmost child of the root and move that subtree and every subtree to its left to be a child of this new node. (For the partition, treat this new node as a singleton.) Then the rightmost path of the subtree of this new edge will be of even length. The rest of the features of **(e)** will be maintained. If there is no such subtree, insert the new edge as the leftmost subtree of the root. This process is reversible.

5. Lattice Sequences

In standard terminology, a lattice path is a path traced out on a grid of integer coordinates and each step is of unit length. For example, one definition of the Dyck paths is that they are lattice paths traced out by steps $(0,1)$ and $(1,0)$ from $(0,0)$ to (n,n) and never fall below the diagonal $x = y$. In this section, we use the term sequence, as opposed to path, to indicate that we want to distinguish the orderings of the sizes of the steps, and not just the paths they trace out. In other words, we treat the sequence $(0,2),(0,1)$ as distinct from $(0,1),(0,2)$, even though they trace out the same path. When drawing these sequences, it is helpful to think of each step as an edge, and having nodes at the end points of each step. Thus the steps given would result in two graphs of a line segment from $(0,0)$ to $(0,3)$, but the former would have a node at $(0,2)$ and the latter at $(0,1)$, both marking the end of the first step of their respective graphs.

(h): Lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y = x$. In the p-Catalan case, we instead take the steps to be $(0,p-1)$ or $(1,0)$.

Generalization (Figure 9): Lattice sequences from $(0,0)$ to (n,n) with steps $(0,k)$ or $(1,0)$, never rising above the line $y = x$, where the multiset of the values for k is given by $\mathbb{B}$.
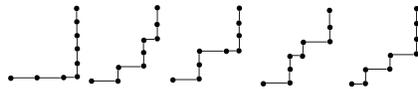

Figure 9: Weighted Calatan examples for **(h)**

As in the Catalan case, we form a bijection with **(r)** by reading each horizontal step of size k as the positive integer k, and each vertical step as -1. The restriction on the diagonal corresponds to the non-negativity of the sequences for **(r)**.

(i): Paths from $(0,0)$ to $(2n,0)$ with steps $(1,-1)$ and $(1,1)$, never falling below the x-axis.

Generalization (Figure 10): Sequences from $(0,0)$ to $(2n,0)$ with steps $(1,-1)$ and (k,k), never falling below the x-axis, where the multiset of the values for k is given by $\mathbb{B}$. As in the Catalan case, this is a rotation and rescaling of **(h)**.

Figure 10: Weighted Calatan examples for **(i)**

(j): Paths from $(0,0)$ to $(2(n+1),0)$ with steps $(1,-1)$ and $(1,1)$, never falling below the x-axis, such that any maximal sequence of consecutive $(1,-1)$ steps ending on the x-axis has odd length.

Generalization (Figure 11): Paths from $(0,0)$ to $(2(n+1),0)$ with steps $(1,-1)$ and (k,k), never falling below the x-axis, such that any maximal sequence of consecutive $(1,-1)$ steps ending on the x-axis has odd length, where the multiset of the values for k is given by $\mathbb{B}$ with an additional element of size 1, which will always be the first step.
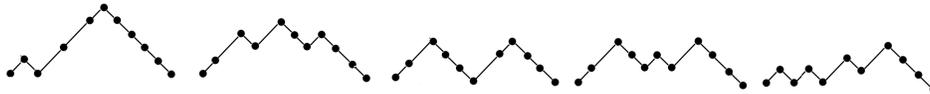
Figure 11: Weighted Calatan examples for **(j)**

As in the Catalan case, take any sequence from **(i)** and locate the rightmost descent of even length onto the x-axis. Insert a $(1,-1)$ step into this descent and a $(1,1)$ step at the beginning of the sequence. If there is no such descent, just insert a $(1,1)(1,-1)$ sequence at the beginning. This will raise the rightmost even descent to an odd one, and all previous descents (even or not) will no longer touch the x-axis. The inverse bijection is to remove the first $(1,1)$ from a given sequence and the final $(1,-1)$ from the leftmost descent that touches the x-axis.

(k): Sequences from $(0,0)$ to $(2(n+1),0)$ with steps $(1,-1)$ and $(1,1)$, never falling below the x-axis, with no peaks at height 2.

Generalization (Figure 12): The same sequences but with steps $(1,-1)$ and (k,k) where the multiset of the values for k are given by $\mathbb{B}$ plus an additional element of size 1.

Figure 12: Weighted Calatan examples for **(k)**

The bijection for this follows that of the Catalan case [5]. Start with a sequence from **(i)**, and add a single up and down step to the front. For each maximal subsequence of our sequence containing no peaks at height one, raise the subsequence by one by adding an up step at the front and a down step at the back. Now, we have added i pairs of steps, 1 for each such maximal subsequence. However, each

such subsequence must be preceded by a peak of height one. Remove the peak of height one immediately preceding each such subsequence to remove i pairs of steps, leaving us with a sequence with $n + 1$ pairs of steps and no peaks at height 2.

(l): Noncrossing pairs of sequences of $n + 1$ steps $(1, 0)$ and $(0, 1)$, which only intersect at start and end.

Generalization (Figure 13): Pairs of sequences of length $n+1$, such that the upper sequence has steps of $(0, k)$ and $(1, 0)$, the lower sequence has steps $(k, 0)$ and $(0, 1)$ and starts with an additional $(1, 0)$ step. The sequences only join at beginning and end, and the values of k (other than the initial step of the bottom sequence) are given by $\mathbb{B}$.
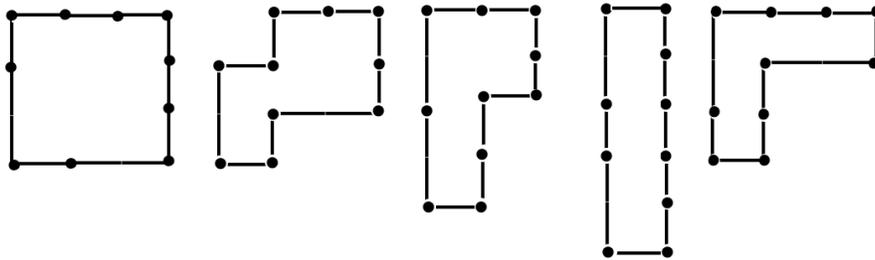

Figure 13: Weighted Calatan examples for (l)

This is in bijection to (r). Read each path, alternating from the upper to the lower, and record $(0, k)$ and $(k, 0)$ steps as the value k, and $(1,0)$ and $(0,1)$ steps as -1, ignoring the first $(1, 0)$ of the lower path and the final step. A switch between paths occurs whenever the total length of the path read so far (without the first step on the lower path) exceeds the total length of the other path read so far. The partial sums of the sequence in (r) plus one will correspond to the separation between the paths in grid steps. Until the final element, the separation will always be at least one, corresponding to a non-negative partial sum.

6. Partitions

Several of these representations distill down to pairs of partitions where the second partition has block sizes given by $\mathbb{B}$ and the first partition is some less refined partition. These are intervals on the poset of partitions of $[n]$, and the set of such intervals is the set of intervals on the partition poset of $[n]$ whose lower bounds have block sizes given by $\mathbb{B}$. Stanley [9] proved that the labels of the maximal chains

of non-crossing partitions with k blocks of size $n + 1$ are the parking functions of length n, so each interval is actually a set of subsequences of parking functions where the subsequences have the same starting and ending point in the poset, and the lower point of the intervals is being determined by the block sizes in $\mathbb{B}$.

$(\mathbf{r^*})$: We've already discussed $(\mathbf{r})$ at length above, but we wish to note a slight reinterpretation of it here. If we replace the $+1$ elements with left parentheses, and the -1 elements with right parentheses, we get a set of valid parenthesizations, where each closing parenthesis matches to exactly one open parenthesis that precedes it and visa versa.

The $+1/-1$ representation of $(\mathbf{r})$ corresponds to a non-nesting partition of $2n$ into blocks of size two, which can be seen by connecting the first $+1$ with the first -1, the second pair, the third pair, etc., up to the nth pair. However, the parenthesis representation (referred to as $(\mathbf{r^*})$ henceforth) corresponds to non-crossing partitions, since an open parenthesis matches to a close parenthesis such that the two enclose all intermediate parentheses. This correspondence of non-crossing and non-nesting is well known; these two representations are simply an easy way to demonstrate their equivalence for the rest of the representations.

Generalization (Figure 14): Instead of only having open and close parentheses, use weighted left parentheses that match up to multiple right parentheses. This gives non-crossing partitions of $[2n]$ such that $\mathbb{B}$ gives the block sizes. We represent a weighted left parenthesis by the weight, such as 31)))) being an example of an opening parenthesis set of weights 3 and 1.

$$221)))))\quad 2)2))1))\quad 2))21)))\quad 2))1)2))\quad 1)2)2)))$$

Figure 14: Weighted Calatan examples for $(\mathbf{r^*})$

$(\mathbf{o})$: In Stanley, this is described as non-intersecting arcs joining n pairs of points in the plane. Our preferred version of this representation is to think of it as partitions of $2n$ into blocks of size 2.

Generalization (Figure 15): Non-crossing partitions of $n + m$ where the elements of $\mathbb{B}$ correspond to one less than the sizes of the blocks in the partition. In terms of the original representation, this would entail nonintersecting arcs joining $2n$ points on a line where there are groups of arcs with the same left point (but no arcs with the same right point) and the group sizes would be given by $\mathbb{B}$.

This bijects with $(\mathbf{r^*})$ as noted above.

$\{1,7,8\}\{2,5,6\}\{3,4\}$ $\{1,2,8\}\{3,4,5\}\{6,7\}$ $\{1,2,3\}\{4,7,8\}\{5,6\}$
$\{1,2,3\}\{4,5\}\{6,7,8\}$ $\{1,2\}\{3,4,8\}\{5,6,7\}$

Figure 15: Weighted Calatan examples for **(o)**

(n): n nonintersecting chords joining $2n$ points on a circle.

Generalization (Figure 16): Nonintersecting chord-groups, joining $2n$ points on the circumference of a circle with the following resctrictions:

- The sizes of the chord groups are given by $\mathbb{B}$

- "Opening" points are the first point of their group when moving clockwise from angle 0, and the opening point is connected to all other points in its group.

- Points other than opening points have only a single chord, which joins them to the opening point of their chord group.

Figure 16 shows this version.
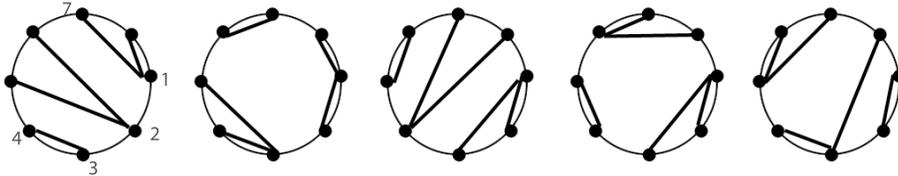
Figure 16: Weighted Calatan examples for **(n)**

An alternate version of this is to use non-intersecting "chord segments," where each segment is a_i chords in a sequence, and the chords join points evenly spaced around the circle. Here the positions of the matching opening and closing parentheses gives the points connected by the chords, and the weight of the opening chord gives the number of chords connected. See K-Catalan Structures at http://math.haifa.ac.il/toufik/enum2005.html (an extended version of [2]) for the p-Catalan case of this.

This has a trivial bijection to **(r*)**, using our notion of opening and closing parentheses. Each opening point is a left parenthesis of a given weight, and each point it is connected to are its closing parentheses, read around the circle starting from angle 0. The non-crossing of the chords corresponds to the proper matching of the parentheses in nested fashion. A similarly trivial bijection can be seen by ennumerating the $2n$ points, starting with 1 at angle 0, and then placing chords according to the partitions in **(o)**.

(**p**): Ways of drawing $n+1$ points on a line with arcs connecting them such that the arcs do not pass below the line, the arcs are noncrossing, all the arcs at a given node exit in the same direction (left or right), and the graph thus formed is a tree.

Generalization (Figure 17): As above, except that we add groupings to the arcs, such that the sizes of the groups are given by $\mathbb{B}$, and the arcs of a group are not separated by an arc from a different group.
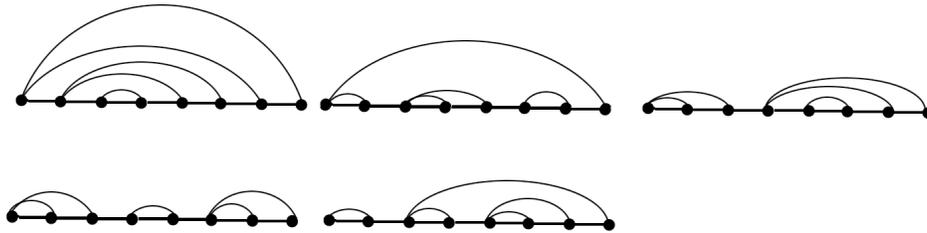

Figure 17: Weighted Calatan examples for (**p**)

Bijection to $(\mathbf{n})^2$: Cut the circle at angle 0 for (**n**) and lay it out as a line, starting with the chord coming out of the point at angle 0.

(**a**): Partitions of an $(n+2)$-gon into triangles.

Generalization (Figure 18): Partitions of an $(n+2)$-gon into polygons whose number of sides are given by $a_i + 2$ where $a_i \in \mathbb{B}$.

Following the standard bijection for the Catalan numbers to (**d**), we fix an edge as the base of the polygon, and fix a root of a tree on the midpoint of this edge. (In Figure 18 we have fixed the top edge as the base.) For a given partition of the $(n+2)$-gon, we place a node on the midpoint of each side of the polygon, then connect each node to the root with a tree edge. Recursively repeat for each newly connected edge. This makes the root have degree equal to the number of sides in the polygon containing the base in our partition minus one. The degree of each node will correspond to an element of $\mathbb{B}$ plus one, since after taking the base edge out of any polygon, it has that many remaining sides to form the children of its subtree's root node. Shapiro and Sulanke [7] showed this relationship over all partitions; this is the case specific to a chosen partition. In Figure 18, the first row shows the example polygon partitions from the set. The second row demonstrates the bijection, by superimposing the corresponding tree over the polygon partition.

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[2]This simple bijection, suggested by the referee, replaces a more complex one in the original submission.

Figure 18: Weighted Calatan examples for **(a)**

(pp): Noncrossing partitions of $[n]$.

$$\{1\}\{2,3\}\{4,5\} \subset \{1,2,3,4,5\} \qquad \{1,5\}\{2,3\}\{4\} \subset \{1,5\}\{2,3\}\{4\}$$
$$\{1,2\}\{3\}\{4,5\} \subset \{1,2\}\{3,4,5\} \qquad \{1,2\}\{3\}\{4,5\} \subset \{1,2\}\{3\}\{4,5\}$$
$$\{1\}\{2,5\}\{3,4\} \subset \{1\}\{2,5\}\{3,4\}$$

Figure 19: Weighted Calatan examples for **(pp)**

Generalization (Figure 19): A pair of partitions of $[n]$, A and B, such that A is non-crossing, the blocks of B are of sizes given by $\mathbb{B}$, B is a subpartition of A, and inside any block of A, the elements of any block of B are continuous, *i.e.,* there do not appear in any block of A subsequences abc where a and c are members of the same block of B and b is not. In the Catalan case, our B is the trivial partition of singletons.

Bijection to **(r*)**: Let us take the example of $12459 - 3 - 678$ for A with B being $124 - 3 - 5 - 6 - 78 - 9$. The blocks in A are groups of closing parentheses. Each partition is a group of such parentheses whose opening parentheses are consecutive. Thus, our example A tells us that the closing parentheses $1, 2, 4, 5, 9$ have their opening parentheses grouped together, the opening parenthesis for the third closing parenthesis is isolated from other opening parentheses, and the opening parentheses for the closing parentheses $6, 7, 8$ are consecutive as well. This tells us that we have $*))*)))*))))$ for our parenthesization. The B partitions are the closing parentheses for a single weighted opening parenthesis. Therefore, we know that closing parentheses $1, 2, 4$ all share the same opening parenthesis. Thus, our complete parenthesization is $311))1)))12))))$.

(**qq**): A partition A of $[n]$ such that if the numbers are arranged in order around a circle, then the convex hulls of the blocks of a given partition are pairwise disjoint.

Generalization : Pairs of partitions A and B of $[n]$ such that, if the numbers are arranged in order around a circle then the convex hulls of the blocks of a given partition are pairwise disjoint, B is a subpartition of A, and within any block of A, the subblocks of B consist of consecutive elements.

These are the same partitions as (**pp**). The circle arrangement is another way to specify that the partitions are noncrossing.

(**rr**): Noncrossing Murasaki diagrams with n vertical bars.

Generalization (Figure 20): Noncrossing Murasaki diagrams with n vertical bars such that an additional grouping of the bars is accomplished by noncrossing underlines which group the bars into discrete groups whose sizes are given by $\mathbb{B}$ and the underlines are subsets of the overlines. In the Catalan case, each bar is just its own underline group.

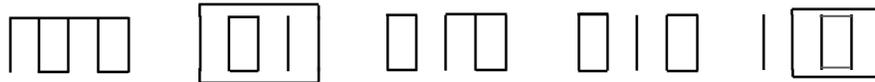This is another way to represent (**pp**).

Figure 20: Weighted Calatan examples for (**rr**)

(**uu**): Nonnesting partitions of $[n]$.

Generalization (Figure 21): As (**pp**), except that the partitions are nonnesting instead of noncrossing.

$$\{1,4\}\{2,5\}\{3\} \subset \{1,4\}\{2,5\}\{3\} \qquad \{1,2\}\{3,4\}\{5\} \subset \{1,2\}\{3,4\}\{5\}$$
$$\{1,2\}\{3,5\}\{4\} \subset \{1,2,3,5\}\{4\} \qquad \{1,2\}\{3\}\{4,5\} \subset \{1,2,3,4,5\}$$
$$\{1\}\{2,3\}\{4,5\} \subset \{1,2,3\}\{4,5\}$$

Figure 21: Weighted Calatan examples for (**uu**)

7. Permutations

(**cc**): Permutations of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that the first occurrences of each number appear in increasing order, and there is no subsequence of the form $abab$.

Generalization (Figure 22): Permutations of the multisets $\{1^{b_1+1}, 2^{b_2+1}, \ldots, k^{b_k+1}\}$ where the b_i are any arrangement of the elements of $\mathbb{B}$, the first occurrences of each number appear in increasing order, and there is no subsquence of the form *abab*.

$$(1, 2, 3, 3, 2, 2, 1, 1) \quad (1, 1, 2, 2, 2, 3, 3, 1) \quad (1, 1, 1, 2, 3, 3, 2, 2)$$
$$(1, 1, 1, 2, 2, 3, 3, 3) \quad (1, 1, 2, 2, 3, 3, 3, 2)$$

Figure 22: Weighted Calatan examples for **(cc)**

This representation is in bijection to **(r*)**, with the first occurrence of each number being an open parenthesis, and the later occurrences being the closing parentheses. The weight of the open parenthesis is given by the b_i in the multiset, and the subsequence restriction corresponds to the noncrossing condition on the parenthesis groups.

(dd): Permutations $[2n]$ such that the odd numbers appear in increasing order, the even numbers appear in increasing order, and odd a appear before $a + 1$.

Generalization (Figure 23): Permutations of the multisets $\{1, 2^{b_1}, 3, 4^{b_2}, \ldots, (2m-1), (2m)^{b_m}\}$ where the b_i are any arrangement of the elements of $\mathbb{B}$, with the restrictions above.

$$(1, 3, 5, 2, 2, 4, 4, 6) \quad (1, 2, 3, 2, 4, 5, 4, 6) \quad (1, 2, 2, 3, 5, 4, 4, 6)$$
$$(1, 2, 2, 3, 4, 5, 6, 6) \quad (1, 2, 3, 4, 5, 4, 6, 6)$$

Figure 23: Weighted Calatan examples for **(dd)**

We biject this to **(r)** by reading each odd number a as a positive integer (whose size is the b_i from the weight of $a + 1$ in the multiset), and each $a + 1$ as a -1. The restrictions on order of appearance correspond to the partial sum restrictions of **(r)**.

(ee): 321-avoiding permutations of $[n]$.

Generalization : 321 avoiding permutations of $[n]$, such that the elements of the permutation are partitioned into contiguous blocks whose sizes are given by $\mathbb{B}$.

Bijection: These are the same permutations as **(jj)** (see Figure 26 after **(jj)**); the restrictions on the blocks and the 321 avoidance matches the restrictions on them in **(jj)** without referencing the queues.

(gg): Permutations w of $[2n]$ with n cycles of length two such that the product $(1, 2, \ldots, 2n)w$ has n cycles.

Generalization (Figure 24): Permutations w of $[m + n]$ with cycle lengths given by $\mathbb{B}$ such that the product $(1, 2, \ldots, n + m)w$ has $n + 1$ cycles.

$$(1, 8, 7)(2, 6, 5)(3, 4) \qquad (1, 8, 2)(3, 5, 4)(6, 7) \qquad (1, 3, 2)(4, 8, 7)(5, 6)$$
$$(1, 3, 2)(4, 5)(6, 8, 7) \qquad (1, 2)(3, 8, 4)(5, 7, 6)$$

Figure 24: Weighted Calatan examples for **(gg)**

Bijection with **(r*)**: Take a valid parentheses arrangement, e.g., 31)2)2))))))) and label each parenthesis from 1 to $m + n$ in order of appearance, such as

$$3_1 1_2)_3 2_4)_5 2_6)_7)_8)_9)_{10})_{11})_{12}.$$

Then for each open parenthesis, write down the parentheses in that group in decreasing order of appearance as a cycle, such as

$$(12, 11, 10, 1)(9, 5, 4)(8, 7, 6)(3, 2).$$

This gives a permutation of cycles with appropriate sizes according to $\mathbb{B}$ (adding one to each element of $\mathbb{B}$). Now, our product is

$$(1, 2, \ldots, n + m)(12, 11, 10, 1)(9, 5, 4)(8, 7, 6)(3, 2),$$

which is equivalent to $(1, 3, 9)(2)(4)(5, 8)(6)(7)(10)(11)(12)$. (In Figure 24, I follow the usual convention of rotating each cycle so the smallest element is in front, which allows one to see the opening parenthesis first, followed by its closing parentheses in reverse order.)

Now, if a parenthesis c_k of one group precedes the first parenthesis of a different group, d_1, then the cycle in the product containing c_k will have the sequence c_k, d_ω where d_ω is the final parenthesis of the group that d_1 starts. In our example, $)_3$ precedes 2_4, so we see that $3, 9$ is in our product.

Furthermore, if we have the last parenthesis c_ω of a group followed by some parenthesis d_i of a different group, then the cycle in the containing c_ω will have the sequence c_ω, d_{i-1}, where d_{i-1} will be d_ω if d_i is the first parenthesis of its group (aka, d_1). In our example, $)_8$ is the last parenthesis of its group, and it is followed by $)_9$, from a different group, which means that our product will have $8, 5$.

Finally, any parenthesis c_j followed by c_{j+1} from its own group will form a singleton in the product (as we will have c_j, c_{j+1} in the first part of the product). This happens seven times in our example.

Using these facts, we can label groups of parentheses by level, so that the groups which are completely contiguous are on level 0, groups that only contain groups

of level 0 are on level 1, groups that only contain groups of level less than i are on level i. In our example, we have the groups $(3, 2)$ and $(8, 7, 6)$ on level 0, group $(9, 5, 4)$ on level 1, and group $(12, 11, 10, 1)$ on level 2.

Then we see that a level 0 group with opening parenthesis of weight f will become f singletons in the final product (every element but the last will become a singleton), and the largest element of the level 0 group will be linked into the preceding and following groups. In our example, we get $2, 6, 7$ as singletons, and the elements $3, 8$ get linked into the preceding and following groups. In the case of the 3, it is preceded by the group containing 1 and followed by the group containing 9, whereas the 8 is preceded and followed by the same group, the one containing the 5.

For a level 1 group of weight g, we see that the elements preceding the level 0 groups that it contains will be linked in a cycle with the final elements of those level 0 groups, and the final element will be linked into the preceding and following groups, and the other elements will become single cycles. If such a group contains m level 0 groups of weights $c_1 \ldots c_m$, then the entirety will contain $\sum_1^m c_i$ singletons from the level 0 groups, $g - m$ singletons from the level 1 group and m groups that are either links into the level 0 groups or singletons (depending on how many of the level 0 groups follow a different level 0 group instead of an element of our level 1 group). The final element of this group will link into the preceding and following groups, but the rest of the elements give us a number of cycles in the final product exactly equal to the weights of the opening parentheses contained within. In our example, our level 1 group is $(9, 5, 4)$ and it contains one level 0 group. Thus, the element preceding that group, 5 gets linked into the final element of that group, 8. The final element of our level one group, 9 gets linked into the preceding group, with the 3, and the following group, with the 1.

We can continue by induction on groups of higher level, seeing that each time, we will get a number of cycles in the final product equal to the sums of the weights of the opening parentheses of the contained groups, with one element linking to outside the group.

When we reach the highest level, that final link out completes a cycle, giving us the required $n + 1$ cycles in the product.

Details of the inverse map are omitted other than to note that the requirement for the number of cycles in the product enforces the non-crossing nature of the partition and the ordering of the pre-product cycles, and the sizes of the pre-product cycles gives us groups of the appropriate weight.

(ii): Permutations of $[n]$ that can be stack sorted.

Generalization (Figure 25): Permutations of $[n]$ with the elements of the permutation partitioned into contiguous blocks whose sizes are given by $\mathbb{B}$, such that the

partition is stack sortable if the elements in a block must be moved onto the stack together. (They can be removed individually.)

Bijection to **(r)** by recording the sizes of the blocks moved onto the stack as positive numbers and removals from the stack as -1.

$$(5, 4, \underline{3, 2}, 1) \quad (5, 1, \underline{3, 2}, 4) \quad (\underline{2, 1}, \underline{5, 4}, 3) \quad (\underline{2, 1}, 3, \underline{5, 4}) \quad (1, \underline{5, 2}, \underline{4, 3})$$

Figure 25: Weighted Calatan examples for **(ii)**

(jj): Permutations of $[n]$ that can be sorted on two parallel queues. (See Stanley [10].)

Generalization (Figure 26): Permutations of n such that elements of the permutation are partitioned into blocks whose sizes are given by $\mathbb{B}$, that can be sorted on two parallel queues if the elements in a block must be moved onto a queue together without any intervening removals from the queues being allowed.

Bijection to **(r)**: Whenever a group of elements is moved onto the queues, record a positive integer of the size of the group. Whenever an element is removed from a queue, record a -1.

$$(\underline{2, 3}, \underline{4, 5}, 1) \quad (1, \underline{5, 2}, \underline{3, 4}) \quad (\underline{1, 2}, \underline{4, 5}, 3) \quad (\underline{1, 2}, 3, \underline{4, 5}) \quad (\underline{1, 2}, \underline{5, 3}, 4)$$

Figure 26: Weighted Calatan examples for **(jj)**

(kk): Permutations of $[2n]$ consisting of cycles of length 2 such that the cycles form a noncrossing partition.

Generalization : Permutations of $[m + n]$ whose cycle lengths are given by $\mathbb{B}$, such that the cycles form a non-crossing partition of $[m + n]$ and each cycle is in decreasing order.

At the same time, these are both the cycles that will satisfy **(gg)** and the partitions that will satisfy **(o)** (with a specific ordering given for listing each block). (See Figure 24.)

(aaa): Linear extensions of the poset $[2] \times [n]$.

Generalization (Figure 27): Create a poset by taking the elements of the multiset $\{1, 2^{b_1}, 3, 4^{b_2}, \dots\} = \{1, 2_1, 2_2, \dots, 2_{b_1}, 3, 4_1, 4_2, \dots, 4_{b_2}, \dots\}$ where the b_i are any arrangement of the elements of $\mathbb{B}$ (as in **(dd)**) with the partial ordering $<_!$ such that if a and b are both odd or both even then $a < b \rightarrow a <_! b$. If $a = c_i$ and $b = c_j$ then $i < j \rightarrow a <_! b$. And if a is odd and $b = a + 1$ then $a <_! b$. The elements counted by our formula are the linear extensions of these posets.

Bijection: The posets correspond to the multisets of **(dd)** and the linear extensions of the posets are the restricted permutations of **(dd)**.

$$1, 3, 5, 2_1, 2_2, 4_1, 4_2, 6 \quad 1, 2_1, 3, 2_2, 4_1, 5, 4_2, 6 \quad 1, 2_1, 2_2, 3, 5, 4_1, 4_2, 6$$
$$1, 2_1, 2_2, 3, 4, 5, 6_1, 6_2 \quad 1, 2, 3, 4_1, 5, 4_2, 6_1, 6_2$$

Figure 27: Weighted Calatan examples for **(aaa)**

8. Other Equivalent Representations

(b): Binary parenthesizations of a string of $n + 1$ letters.

$$(((xx)xx)xx) \quad (x(xx(xx))x) \quad (xx((xx)xx)) \quad (xx(x(xxx))) \quad (x(x(xxx)x))$$

Figure 28: Weighted Calatan examples for **(b)**

Generalization (Figure 28): Parenthesizations of $n+1$ letters into blocks given by $\mathbb{B}$; *i.e.*, each pair of parentheses encloses b_i (where $b_i \in \mathbb{B}$) items where each item is either a letter or another parenthesization.

Bijection to **(d)**: Given such a parenthesization, we look at the outermost group of items, and the size of this group gives us the degree of the root. Then we look at the rightmost item in this grouping, and we look at each item as a group, with a letter designating a leaf, and a sub-parenthesization representing a subtree. For example, the parenthesization $((xx)xx)$ has an outermost group of size 3, giving a root of degree three. Its first child is a subtree whose root is of degree two with two leaves, and its second and third child are leaves. Contrast this with $(x(xxx))$ whose outermost group designates a root of degree two whose first child is a leaf and whose second child is a node with three children, all leaves.

(s): Sequences $1 \leq a_1 \leq \cdots \leq a_n$ with $a_i < i$.

Generalization (Figure 29): Sequences $1 \leq a_1 \leq \cdots \leq a_n$, grouped into contiguous groups of identical numbers of sizes given by $\mathbb{B}$, with each $a_i \leq i$.

$$\underline{11}\,\underline{11}\,1 \quad \underline{11}\,\underline{22}\,3 \quad \underline{11}\,\underline{22}\,2 \quad \underline{11}\,2\,\underline{33} \quad 1\,\underline{22}\,\underline{33}$$

Figure 29: Weighted Calatan examples for **(s)**

Bijection with **(h)**: a_i gives the height of the sequence at $x = i - 1$, and the groupings of the a_i give the sizes of the horizontal steps that the a_i is part of.

(ww): Standard Young Tableux of shape (n, n).

Generalization (Figure 30): The elements of $n + m$ laid out in a diagram whose first row is of width m, whose column heights are given by the weights in $\mathbb{B}$ plus one, whose rows and columns are in increasing order, and elements not in the

first row are ordered among columns such that each is greater than the elements appearing in the columns to its left. Note that this differs from a Young Tableux in that there may be gaps between elements in the later rows, and we have a stronger restriction on ordering.

123	136	145	146	135
468	258	268	257	247
57	47	37	3 8	68

Figure 30: Weighted Calatan examples for **(ww)**

Bijection to **(r)**: The first row gives the location of the positive numbers (of size equal one less than the column's height) and the other elements give the location of the -1's. The increasing column restriction is equivalent to the nonnegativity restriction on **(r)** and the fill requirement for the lower rows ensures a unique representation for each sequence.

9. Two-Sided Generalization

In the representations of our generalized Catalan numbers given above, we allowed one side to vary the weights of its elements, while requiring the other side to remain fixed. However, one can consider a two-sided ballot problem, where both positive and negative numbers can be of any weight, so long as the sum total is equal to zero, and the valid arrangements of a weight set require nonnegative partial sums. This generalization has been investigated by Curtis Coker [1], who shows that the generating function $\sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} x^{2k} (1 + x)^{2n-2k}$ gives us the number of structures with the coefficients for x^i giving the total ballot arrangements whose partitions have i blocks. Plugging in $x = 1$ in the above formula gives us $\sum_{k=1}^{n} 4^{n-k} N(n, n-k)$ where $N(n, i)$ are the Narayana numbers. This is the number of all two-sided structures whose total length is n (see Sloane's A059231 [8]).

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