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COMBINATORICS OF INTEGER PARTITIONS IN ARITHMETIC PROGRESSION

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Abstract

The partitions of a positive integer n in which the parts are in arithmetic progression possess interesting combinatorial properties that distinguish them from other classes of partitions. We exhibit the properties by analyzing partitions with respect to a fixed length of the arithmetic progressions. We also address an open question concerning the number of integers k for which there is a k-partition of n with parts in arithmetic progression.

1. Introduction

The partitions of an integer $n > 0$ in which the parts are in arithmetic progression possess a simple but interesting structure. In addition, the system of sets of all such partitions, with a fixed length, is endowed with a consecutive enumeration pattern.

Let $\lambda = (n_1, n_2, \ldots, n_k)$ denote a nondecreasing partition of n; that is, λ satisfies $n_1 + \cdots + n_k = n,\ 1 \leq n_1 \leq \cdots \leq n_k \leq n$. Let $AP(n, k)$ denote the set of k-partitions of n with the parts in arithmetic progression (AP), and let $AP(n) = \bigcup_k AP(n, k)$.

A few authors have considered the problem of enumerating the set $AP(n)$ with respect to a fixed common difference (see [4, 5]). Their results are generally complicated. A simpler exposition is given in [6] whereby the enumeration was executed according to a fixed length of the AP's. However, all of the above references failed to bring out the intrinsic structure of these partitions.

This paper fills the gap. It turns out that the combinatorial properties of the partitions are most readily exhibited by analyzing the set $AP(n, k)$ for a given $k > 0$. Our starting point is the main result of [6] on the number $ap(n, k) = |AP(n, k)|$, stated below.

Theorem 1. (Munagi-Shonhiwa) (i) *Let n be a positive integer and $k > 0$ an even number such that $ap(n, k) > 0$. Then*

$$ap(n,k) = \left\lfloor \frac{n + k(k-2)}{k(k-1)} \right\rfloor, \quad \text{if } \ k|n \tag{1}$$

$$ap(n,k) = \left\lfloor \frac{2n + k(k-3)}{2k(k-1)} \right\rfloor, \quad \text{if } \ k \nmid n. \tag{2}$$

(ii) *Let n be a positve integer and k an odd number such that $ap(n, k) > 0$, $k > 1$. Then*

$$ap(n,k) = \left\lfloor \frac{2n + k(k-3)}{k(k-1)} \right\rfloor. \tag{3}$$

Theorem 1 is established in [6] by solving a linear Diophantine equation.

The purpose of this paper is twofold. Firstly, we give three new derivations of Theorem 1. The first two proofs are constructive, while the third proof is both constructive and bijective. The combinatorial structures of the partitions are revealed in the course of the proofs (Sections 2 and 3). Secondly (Section 4), we consider an open question posed in [6], namely to specify the number of *AP divisors*.

An integer $k > 0$ is called an AP divisor of n if $AP(n, k) \neq \emptyset$. A divisor of n is an AP divisor since $(\frac{n}{k}, \ldots, \frac{n}{k}) \in AP(n, k)$, but not conversely in general. For example, $(9, 16, 23, 30) \in AP(78, 4)$ but $4 \nmid 78$. Let $Div(n)$ and $APDiv(n)$, respectively, denote the set of divisors and AP divisors of n. Then $Div(n) \subseteq APDiv(n)$. We give a generating function for the cardinality of $APDiv(n)$.

We will need the following summation formula for $(a, a + d, \ldots, a + (k-1)d) \in AP(n, k)$:

$$a + (a+d) + \cdots + (a + (k-1)d) = ka + \binom{k}{2} d = n, \quad d \geq 0, \ 1 \leq k \leq n. \tag{4}$$

2. Two Proofs of Theorem 1

The following lemma is fundamental to the proofs.

Lemma 2. *Let $AP(n, k) \neq \emptyset$ and let d be the common difference of a fixed element.*

(i) If k is an odd integer, then $n \equiv 0 \pmod{k}$.

(ii) *If k is an even integer, then $n \equiv 0 \pmod{\frac{k}{2}}$.*

Furthermore, we have,

(iii) If k is even and $k|n$, then d is even.

(iv) If $k \nmid n$, then d is odd.

(v) If $k \nmid n$, then $n \equiv \binom{k}{2} \pmod{k}$.

Proof. The assertions are easy consequences of Equation (4). We comment only on (iv) and (v). If $k \nmid n$, then by (i) and (ii), k is even and $\frac{k}{2}|n$. Thus if n is odd, both $k/2$ and $2a + (k-1)d$, that is, $k/2$ and d, must be odd. If n is even, then $k/2$ is even and d is odd. In fact (iii) forbids d from being even. So $k|(n - \frac{k}{2})$ since $n - \frac{k}{2} = ka + \binom{k}{2}d - \frac{k}{2} = ka + k\frac{(k-1)d-1}{2}$. Lastly, (v) follows from the relation $n - \frac{k}{2} - k\frac{k}{2} = n - \binom{k}{2}$. □

Let $\lambda, \lambda' \in AP(n,k)$ have common differences d, d', respectively, where $\lambda = (a_1, \dots, a_k), \lambda' = (a'_1, \dots, a'_k)$. Assume $\lambda \neq \lambda'$. If $a_1 = a'_1$, then $d = d'$, which implies $\lambda = \lambda'$, a contradiction (since the fixed numbers a_1, d, k determine a unique AP). This implies a sorting property.

There is a natural total order, $\prec$, in the set $AP(n,k)$, defined as follows:

$$(a_1, \dots, a_k) \prec (b_1, \dots, b_k) \iff a_1 < b_1. \tag{5}$$

With this ordering, $AP(n,k)$ can be analyzed according to the sequence of common differences. In particular Lemma 2 gives:

Lemma 3. *Let $AP(n,k) \neq \emptyset$ and let the first member of $AP(n,k)$, under $\prec$, have common difference d.*

If k is odd, the sequence of common differences of members of $AP(n,k)$ is

$$d, d-1, \dots, 1, 0.$$

If k is even, the sequence of common differences of members of $AP(n,k)$ is

$$d, d-2, d-4, \dots, z,$$

where $z = 0$ or $z = 1$ according to whether d is even or odd.

Lemma 3 motivates the next definitions, where k is odd and even respectively.

$$f_O(k) \stackrel{\text{def}}{=} (\frac{k+1}{2} - 1, \frac{k+1}{2} - 2, \dots, 2, 1, 0, -1, -2, \dots, -\frac{k-1}{2} + 1, -\frac{k-1}{2}).$$

$$f_E(k) \stackrel{\text{def}}{=} (k-1, k-3, \dots, 3, 1, -1, -3, \dots, -(k-1)).$$

Note that both $f_O(k)$ and $f_E(k)$ are decreasing AP's with the terms summing to 0.

The next result follows from the fact that the term-wise sum of two AP's of the same length is again an AP. If $\pi \in AP(n,k)$ has common difference d, then $\pi + f_O(k)$ has a common difference with opposite parity to d, but the addition of $f_E(k)$ to π preserves the parity of d. The following result is now evident.

Lemma 4. *Let $\lambda \in AP(n,k)$. Then every k-partition of n with parts in arithmetic progression, has the form*

$$\begin{cases} \lambda + v f_O(k), & \text{if } k \text{ is odd} \\ \lambda + v f_E(k), & \text{if } k \text{ is even.} \end{cases}$$

where v is an integer, and $v(a_1, \ldots, a_k) = (va_1, \ldots, va_k)$.

We are ready to give two essentially different proofs of Theorem 1

2.1. First Proof

This is a direct application of Lemma 4.

(i) Let k be an even integer > 0 and denote the last partition of $AP(n,k)$, under $\prec$, by $\lambda = (a, a+d, \ldots, a+(k-1)d)$. Then from Lemma 4 $ap(n,k)$ is given by the number of different values of v such that $\lambda - v f_E(k) \in AP(n,k)$. But

$$\lambda - v f_E(k) \in AP(n,k) \iff a - v(k-1) \geq 1,\ v \geq 0 \iff 0 \leq v \leq \left\lfloor \frac{a-1}{k-1} \right\rfloor.$$

Hence,

$$ap(n,k) = \left\lfloor \frac{a-1}{k-1} \right\rfloor + 1. \tag{6}$$

By Lemma 3 and Equation 4, we have, $k|n \implies a = \frac{n}{k}$, and $k \nmid n \implies a = \left(n - \binom{k}{2}\right)/k$. Substituting the two values of a into (6), and simplifying, give (1) and (2), respectively.

(ii) Similarly, to obtain (3), we find the number of values of v such that $\lambda - v f_O(k) \in AP(n,k)$. Thus,

$$ap(n,k) = \left\lfloor \frac{2(a-1)}{k-1} \right\rfloor + 1.$$

There is one possibility for a, namely $a = \frac{n}{k}$. Hence the result.

Example 5. We illustrate the first proof with $AP(78, k)$. If $k = 4$, then $4 \nmid 78$; so the last partition is $(a, \ldots, a + 3d)$, where $a = (78 - 6)/4 = 18,\ d = 1$. Hence,

$$\begin{aligned} AP(78,4) &= \{(a_1, \ldots, a_4) = (18, 19, 20, 21) - v(3, 1, -1, -3) : v \geq 0,\ a_i > 0\} \\ &= \{(18, 19, 20, 21), (15, 18, 21, 24), (12, 17, 22, 27), \\ &\quad (9, 16, 23, 30), (6, 15, 24, 33), (13, 14, 25, 36)\}. \end{aligned}$$

On the other hand, since $6 \mid 78$ we obtain

$$\begin{aligned} AP(78,6) &= \{(a_1, \ldots, a_6) = (13, 13, 13, 13, 13, 13) - v(5, 3, 1, -1, -3, -5) : \\ &\qquad v \geq 0,\ a_i > 0\} \\ &= \{(13, 13, 13, 13, 13, 13), (8, 10, 12, 14, 16, 18), (3, 7, 11, 15, 19, 23)\}. \end{aligned}$$

2.2. Second Proof

Every set $AP(n, k)$ belongs to an ordered "vertical" system of sets of AP partitions of $N,\ N \leq n$, in the sense that $ap(n, k) = ap(N, k)$, where the first partition (under $\prec$) in the first set has first term 1. For example, $ap(22, 4) = ap(26, 4) = ap(30, 4) = 2$, and the class $\{AP(22, 4), AP(26, 4), AP(30, 4)\}$ is obtained from $AP(22, 4)$ by successive additions of $(1, 1, 1, 1)$ (see Table 1).

(i) Let an even integer $k > 0$ be given. If the first partition of $AP(n, k)$ is $(1, a_2, a_3, \ldots, a_k)$, then there is a list of first partitions

$$(1 + h, a_2 + h, \ldots, a_k + h) \in AP(n + kh, k), \quad 1 \leq h \leq k - 2. \tag{7}$$

N	elements of $AP(N, 4)$	N	elements of $AP(N, 4)$
4	$(1, 1, 1, 1)$	10	$(1, 2, 3, 4)$
8	$(2, 2, 2, 2)$	14	$(2, 3, 4, 5)$
12	$(3, 3, 3, 3)$	18	$(3, 4, 5, 6)$
16	$(1, 3, 5, 7), (4, 4, 4, 4)$	22	$(1, 4, 7, 10), (4, 5, 6, 7)$
20	$(2, 4, 6, 8), (5, 5, 5, 5)$	26	$(2, 5, 8, 11), (5, 6, 7, 8)$
24	$(3, 5, 7, 9), (6, 6, 6, 6)$	30	$(3, 6, 9, 12), (6, 7, 8, 9)$
28	$(1, 5, 9, 13), (4, 6, 8, 10),$ (7,7,7,7)	34	$(1, 6, 11, 16), (4, 7, 10, 13),$ (7,8,9,10)
$\vdots$	… ad infinitum	$\vdots$	… ad infinitum

Table 1: The second proof when $k|N$ and when $k \nmid N,\ k = 4$.

We cannot have $\lambda = (1 + (k-1), a_2 + (k-1), \ldots, a_k + (k-1))$ as a first partition because $\lambda - f_E(k) = (1, a_2 + 2, a_3 + 4, \ldots, a_k + 2(k-1))$ is the first partition, and λ is now the second partition, in $AP(n + (k-1)k, k)$.
By applying a similar argument, beginning with any j^{th} member,

$$(1, a_2, a_3, \ldots, a_k) + (j-1)f_E(k) \in AP(n,k),\ j > 0,$$

we see that this gives the period $k-1$ inside which $ap(N,k)$ is a constant:

$$ap(n,k) = ap(N,k), \quad N = n, n+k, \ldots, n + (k-2)k.$$

The sentence immediately following (7) implies that $ap(n+k(k-1), k) = ap(N,k) + 1$. By the division algorithm we have $n = qk + r$, $0 \le r < k$, which gives the infinite sequence

$$ap(c_1k + r, k) = 1,\ ap(c_2k + r, k) = 2, \ldots \iff c_j \equiv c_1 \,(\text{mod k-1}),\ j > 0. \qquad (8)$$

Thus c_1 is the value of c such that $ck + r$ is least and $AP(ck + r, k)$ is a singleton whose member has first term 1.

It suffices to consider the case $r = 0$, which implies $c_1 = 1$ with common difference, $d = 0$.
The j^{th} term c_j of the AP $c_1, c_2, \ldots$ may satisfy $c_j = q$, or generally, $c_j \le q < c_{j+1}$. That is,

$$c_1 + (j-1)(k-1) \le q < c_1 + j(k-1) \implies j = \left\lfloor \frac{q + k - 1 - c_1}{k-1} \right\rfloor.$$

Let $j = F(n,k)$:

$$F(n,k) = \left\lfloor \frac{n + k(k-2)}{k(k-1)} \right\rfloor.$$

This gives Equation (1) at once: $k|n \implies ap(n,k) = F(n,k)$.

If $k \nmid n$, then by Lemma 2(v), $ap(n,k) = ap(N,k)$, where $N = n - \binom{k}{2}$ and $k|N$ (for example, see Table 1). Hence we obtain Equation (2): $k \nmid n \implies ap(n,k) = F(n - \binom{k}{2}, k)$.

(ii) The derivation of (3) is similar to the even case. The adjustment is to use the period $(k-1)/2$ as already indicated by the definition of $f_O(k)$. This gives (8) with $(k-1)/2$ replacing $k-1$. By Lemma 3, we must use only $c_1 = 1$, and deduce the result from

$$ap(n,k) = j = \left\lfloor \frac{2q + k - 1 - 2c_1}{k-1} \right\rfloor.$$

3. Third Proof of Theorem 1

The third proof will rely on the following observation.

> *The sum of two terms of a finite AP in symmetrical positions about the center is a constant.*

If $(a_1, \ldots, a_k) \in AP(n, k)$ has common difference d, then

$$a_j + a_{k-j+1} = 2a_1 + (k-1)d = \frac{2n}{k},\ 1 \leq j \leq \lfloor \frac{k+1}{2} \rfloor. \tag{9}$$

So $\frac{2n}{k}$ is an integer whenever $ap(n, k) > 0$. The sequence $(a_{k-j+1} - a_j)$ of absolute differences of the pairs of terms in (9) is

$$(k-1)d,\ (k-3)d, \ldots, (k-(2v-1))d, \quad v = \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{10}$$

Since a finite AP is completely determined by the first and last terms, Equation (9) implies that $(a_1, a_2, \ldots, a_k) \in AP(n, k)$ if and only if $(a_1, a_k) \in AP(\frac{2n}{k}, 2)$ with the successive implications:

$$\begin{aligned}
(a_1, a_k) \in AP\left(\frac{2n}{k}, 2\right) \quad &\Longrightarrow \quad (a_2, a_{k-1}) \in AP\left(\frac{2n}{k}, 2\right) \\
&\Longrightarrow \quad (a_3, a_{k-2}) \in AP\left(\frac{2n}{k}, 2\right) \\
&\ \ \vdots \\
&\Longrightarrow \quad (a_u, a_u) \in AP\left(\frac{2n}{k}, 2\right),
\end{aligned}$$

where $u = \lfloor k/2 \rfloor$.

Hence by (10), $AP(n, k)$ may be identified with the set of 2-partitions of $\frac{2n}{k}$ in which the differences between the pairs of parts are multiples of $(k-1)$. That is, we have a bijection:

$$\begin{aligned}
AP(n, k) &\longrightarrow \{(b, c) \in AP(\frac{2n}{k}, 2) : (k-1) | (b-c)\}, \qquad (11)\\
(a_1, a_2, \ldots, a_k) &\longmapsto (a_1, a_k).
\end{aligned}$$

Thus we have established a constructive/bijective derivation of $AP(n, k)$.

Example 6. The proof is illustrated with $AP(78, k)$. If $k = 4$, then $\frac{2n}{k} = 39$, and $AP(39, 2) = \{(1, 38), (2, 37), \ldots, (19, 20)\}$. Thus

$$\{(b, c) \in AP(39, 2) : 3|(b - c)\} = \{(3, 36), (6, 33), (9, 30), (12, 27), (15, 24), (18, 21)\},$$

which gives $ap(78, 4) = 6$ (cf. Example 5). Similarly, $k = 6$ implies $\frac{2n}{k} = 26$, and

$$\{(b, c) \in AP(26, 2) : 5|(b - c)\} = \{(3, 23), (8, 18), (13, 13)\}$$

implies $ap(78, 6) = 3$.

To complete the proof of Theorem 1 we obtain the enumeration formulas relative to the proof of this section.

(i) Let $k > 0$ be an even integer and assume that $AP(n, k) \neq \emptyset$. Then by (11), $ap(n, k)$ is given by the number of elements of $AP(\frac{2n}{k}, 2)$ in which the differences between the pairs of parts are multiples of $(k - 1)$.

The set of differences of parts of members of $AP(m, 2)$ is

$$\{m - 2,\ m - 4, \cdots,\ z + 2,\ z\},$$

where $z = 0$ or 1, depending on whether m is even or odd. Assume that m is even, which implies that $k \mid n$. Then $ap(n, k)$ is the length of the AP

$$0,\ 2(k - 1),\ 4(k - 1), \ldots, X, \quad X = 2(k - 1) \left\lfloor \frac{m - 2}{2(k - 1)} \right\rfloor,$$

where X is the largest multiple of $k - 1$ not exceeding $m - 2$. Therefore,

$$X = 0 + (ap(n, k) - 1) \cdot 2(k - 1) \text{ implies } ap(n, k) = \left\lfloor \frac{m + 2k - 4}{2(k - 1)} \right\rfloor,$$

which, on setting $m = \frac{2n}{k}$, is identical with (1).

If m is odd, then $ap(n, k)$ is the length of the AP

$$k - 1,\ 3(k - 1),\ 5(k - 1), \ldots,\ 2 \left\lfloor \frac{m - k - 1}{2(k - 1)} \right\rfloor (k - 1) + k - 1.$$

This leads to Equation (2), as one can easily verify.

(ii) For the second part, $k - 1$ is even. So m is even. In this case $ap(n, k)$ is the length of the AP

$$0,\ (k - 1),\ 2(k - 1), \ldots, \left\lfloor \frac{m - 2}{k - 1} \right\rfloor (k - 1).$$

4. The Number of AP Divisors

The number of AP divisors is clearly equal to $\tau(n)+|APDiv(n)\backslash Div(n)|$, where $\tau(n)$ is the number of positive integral divisors of n. Let $Ek(n) = APDiv(n) \setminus Div(n)$, the set of strict AP divisors.

$$|APDiv(n)| = \tau(n) + |Ek(n)|. \tag{12}$$

But we will prove

Theorem 7.
$$\sum_{n=0}^{\infty} |APDiv(n)| q^n = \sum_{k=1}^{\infty} \frac{q^k(1+q^k+q^{2k^2})}{1-q^{2k}}.$$

If $k \nmid n$, then k is an even integer and the last member of $AP(n,k)$ has common difference $d=1$ (see Lemmas 2 and 3). In other words, $k \in Ek(n) \iff AP(n,k) \neq \emptyset$ and the last member has $d=1$. Hence $|Ek(n)|$ *is the number of partitions of* n *into an even number of consecutive integers*. Passing to the conjugate partition, we find that $|Ek(n)|$ also counts partitions of n of the type $1+2+\cdots+(2m-1)+2m+2m+\cdots+2m$, where $2m$ appears $j>0$ times. That is, if λ is a partition of n into an even number $2m$ of consecutive integers, the conjugate of λ is a partition into the first $2m$ natural numbers such that only $2m$ may be repeated. The latter class of partitions translate into the generating function:

$$q^{1+2}(1+q^2+q^{2.2}+q^{3.2}+\cdots)+q^{1+2+3+4}(1+q^4+q^{2.4}+q^{3.4}+\cdots)+\cdots.$$

Thus,

$$\sum_{n=0}^{\infty} |Ek(n)| q^n = \frac{q^{\binom{3}{2}}}{1-q^2} + \frac{q^{\binom{5}{2}}}{1-q^4} + \cdots = \sum_{k=1}^{\infty} \frac{q^{\binom{2k+1}{2}}}{1-q^{2k}}.$$

On the other hand, it is known (see, for example, [7]) that $\tau(n)$ has the Lambert series generating function $\sum_{n=1}^{\infty} \frac{q^k}{1-q^k}$. Hence,

$$\sum_{n=0}^{\infty} |APDiv(n)| q^n = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} + \sum_{k=1}^{\infty} \frac{q^{\binom{2k+1}{2}}}{1-q^{2k}},$$

which gives the theorem.

The sequence of numbers of AP divisors of n, $n=1,2,\ldots$, begins as follows:

$$1,2,3,3,3,4,3,4,4,5,3,6,3,5,5,5,3,7,3,6,6,5,3,8,4,5,6,6,3,9,\ldots.$$

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