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SYMMETRIES IN STEINHAUS TRIANGLES AND IN GENERALIZED PASCAL TRIANGLES[1]

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Abstract

We give explicit formulae for obtaining the binary sequences which produce Steinhaus triangles and generalized Pascal triangles with rotational and dihedral symmetries.

1. Introduction

Let $\mathbb{F}_2$ be the field of order 2 and $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{F}_2^n$ a binary sequence of length n. The *derivate* of $\mathbf{x}$ is the sequence $\partial\mathbf{x} = (x_0+x_1, x_1+x_2, \ldots, x_{n-2}+x_{n-1})$. We define $\partial^0\mathbf{x} = \mathbf{x}$, $\partial^1\mathbf{x} = \partial\mathbf{x}$ and, for $2 \le i \le n-1$, $\partial^i\mathbf{x} = \partial\partial^{i-1}\mathbf{x}$. The *Steinhaus triangle* of the sequence $\mathbf{x}$ is the sequence $S(\mathbf{x})$ formed by $\mathbf{x}$ and its derivatives: $S(\mathbf{x}) = (\mathbf{x}, \partial\mathbf{x}, \ldots, \partial^{n-1}\mathbf{x})$. Figure 1 shows a graphical representation of $S(\mathbf{x})$ for the sequence $\mathbf{x} = (0, 0, 1, 0, 1, 0, 0)$. The black and white circles represent ones and zeroes respectively; the first row corresponds to $\mathbf{x}$ and the following rows to the iterated derivatives. Each entry of the triangle is the binary sum of the two values immediately above it.

In 1958, H. Steinhaus [13] asked for which sequences $\mathbf{x} = (x_0, \ldots, x_{n-1})$ the triangle $S(\mathbf{x})$ is balanced, that is, $S(\mathbf{x})$ has as many zeroes as ones. He observed that no sequence of length $n \equiv 1, 2 \pmod 4$ produces a balanced triangle, so the problem was to decide if they exist for lengths $n \equiv 0, 3 \pmod 4$. H. Harborth [9] answered the

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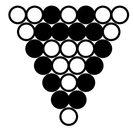

Figure 1: Steinhaus triangle $S(\mathbf{x})$ for the sequence $\mathbf{x} = (0, 0, 1, 0, 1, 0, 0)$

question in the affirmative by constructing examples of such sequences. S. Eliahou et al. studied binary sequences generating balanced triangles with some additional condition: sequences of length n, all of whose initial segments of length $n-4t$ for $0 \leq t \leq n/4$ generate balanced triangles [5], symmetric and antisymmetric sequences [6], and sequences with zero sum [7]. F.M. Malyshev and E.V. Kutyreva [11] estimated the number of Steinhaus triangles (which they call Boolean Pascal triangles) of sufficiently large size n containing a given number $\omega \leq kn$ $(k > 0)$ of ones. More recently, J. Chappelon [4] considered a generalization by J. C. Molluzzo [12] to sequences with entries in $\mathbb{Z}_m$, with the condition that every element in $\mathbb{Z}_m$ has the same multiplicity in the triangle. Here, we focus on the symmetry of the graphical representation of Steinhaus triangles.

Let $\mathbf{u} = (u_0, \ldots, u_\ell)$ and $\mathbf{v} = (v_0, \ldots, v_\ell)$ be two binary sequences in $\mathbb{F}_2^{\ell+1}$ with $u_0 = v_0$. The *general binary Pascal triangle*, or *Pascal triangle* for short, $P(\mathbf{u}, \mathbf{v})$, is the double indexed sequence $\mathbf{z}(r, c)$ defined by the initial conditions

$$\mathbf{z}(r, 0) = u_r, \quad \mathbf{z}(r, r) = v_r, \quad (0 \leq r \leq \ell), \tag{1}$$

and the recurrence

$$\mathbf{z}(r, c) = \mathbf{z}(r-1, c-1) + \mathbf{z}(r-1, c), \qquad (2 \leq r \leq \ell,\ 1 \leq c \leq r-1). \tag{2}$$

The Pascal triangle $P(\mathbf{u}, \mathbf{v})$ is similar to the ordinary Pascal triangle, but the left and right sides are not filled with ones, but with the given values $u_0, \ldots, u_\ell$ on the left side and $v_0, \ldots, v_\ell$ on the right side. Recurrence (2) is the usual recurrence of binomial numbers, but here the initial conditions are those in (1) and the sum is done in $\mathbb{F}_2$. Figure 2 shows a graphic representation of a Pascal triangle.

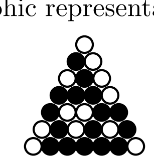

Figure 2: The Pascal triangle $P(\mathbf{u}, \mathbf{v})$ for the sequences $\mathbf{u} = (0, 1, 0, 1, 1, 0, 0)$ and $\mathbf{v} = (0, 0, 0, 0, 1, 1, 0)$

The particular case $u_0 = \cdots = u_\ell = v_0 = \cdots = v_\ell = 1$ is the ordinary Pascal triangle modulo 2, which is known to be related to the Sierpiński sieve [8, 14]. H. Harborth and G. Hurlbert [10] showed that for every natural number n there exists a natural number ℓ and binary sequences of $\mathbf{u}$ and $\mathbf{v}$ of length $\ell + 1$ such that the Pascal triangle $P(\mathbf{u}, \mathbf{v})$ has exactly n ones. Moreover, they determine the minimum possible value of ℓ. As for Steinhaus triangles, here we are interested in the symmetry of Pascal triangles.

Both Steinhaus and Pascal triangles appear in the context of cellular automata, particularly in a bi-dimensional cellular automaton following the local rule represented in Figure 3, see [1, 2, 3].

Figure 3: Local rule of a cellular automaton that generates Steinhaus and Pascal triangles

In this context, A. Barbé [3] has studied symmetries in Steinhaus and Pascal triangles (which he called *binary difference pattern* and Δ-*binary difference pattern*, respectively) as patterns in such a bi-dimensional cellular automaton. A Steinhaus or Pascal triangle is said to have *rotational symmetry* if its graphical representation is invariant under rotations of 120 and 240 degrees, and it is said to have *dihedral symmetry* if it has rotational symmetry and the graphical representation is invariant by axial symmetry with respect to the height of the triangle. Besides enumeration results counting the number of Steinhaus and Pascal triangles with rotational and dihedral symmetries, for example, he characterizes by matrix properties the sequences which produce Steinhaus and Pascal triangles with rotational and dihedral symmetry. Our goal here is to give formulae for explicitly obtaining such sequences.

In Sections 2 and 3, we give formulae for obtaining the sequences $\mathbf{x} \in \mathbb{F}_2^n$ such that $S(\mathbf{x})$ has rotational and dihedral symmetry, respectively.

If $\mathbf{u} = (u_0, \ldots, u_\ell)$ and $\mathbf{v} = (v_0, \ldots, v_\ell)$, and the Pascal triangle $P(\mathbf{u}, \mathbf{v})$ has rotational symmetry, then obviously $v_i = u_{\ell-i}$ for $0 \leq i \leq \ell$, so the triangle is determined by $\mathbf{u}$. In Sections 4 and 5 we give formulae for obtaining the sequences $\mathbf{u}$ such that the corresponding Pascal triangle has rotational and dihedral symmetry, respectively.

Finally, Section 6 deals with the possibility of changing $\mathbb{F}_2$ to an arbitrary abelian group throughout the discussion.

2. Rotational Symmetry in Steinhaus Triangles

Consider the Steinhaus triangle $S(\mathbf{x})$ of the sequence $\mathbf{x} = (x_0, \ldots, x_{n-1})$. The coordinates of $\partial^r\mathbf{x}$ will be indexed from 0 to $n-1-r$ and denoted by $\mathbf{x}(r,c)$, that is, $\partial^r\mathbf{x} = (\mathbf{x}(r,0), \mathbf{x}(r,1), \ldots, \mathbf{x}(r,n-1-r))$. In particular, $\mathbf{x}(0,i) = x_i$ for $0 \le i \le n-1$. It is known (and easily proved by induction) that the entry $\mathbf{x}(r,c)$ of the triangle is

$$\mathbf{x}(r,c) = \sum_{i=0}^{r} \binom{r}{i} x_{c+i}. \tag{3}$$

The set $S(n) = \{S(\mathbf{x}) : \mathbf{x} \in \mathbb{F}_2^n\}$ is a $\mathbb{F}_2$-vector space of dimension n. Let $SR(n)$ be the set of Steinhaus triangles of size n with rotational symmetry, that is, Steinhaus triangles whose graphical representations are invariant under rotations of 120 and 240 degrees. In terms of coordinates, a Steinhaus triange $S(\mathbf{x})$ is rotationally symmetric if

$$\mathbf{x}(r,c) = \mathbf{x}(c, n-r-c-1) \quad \text{or, equivalently,} \quad \mathbf{x}(r,c) = \mathbf{x}(n-r-c-1, r),$$

for $0 \le r \le n-1, \quad 0 \le c \le n-r-1$. In a natural way $SR(n)$ is a vector subspace of $S(n)$. Define $\epsilon_3(n) = 1$ if $n \equiv 1 \pmod 3$ and 0 otherwise; A. Barbé ([3], Property 7) shows that the dimension of $SR(n)$ is $d(n) = \dim SR(n) = \lfloor n/3 \rfloor + \epsilon_3(n)$. We shall show that the $d(n)$ central coordinates in $\mathbf{x} = (x_0, \ldots, x_{n-1})$ can be given arbitrary values determining a Steinhaus triangle with rotational symmetry. Note that the d central coordinates are

$$\begin{array}{ll} x_q, x_{q+1}, \ldots, x_{2q}, & \text{if } n = 3q+1; \\ x_{q+1}, x_{q+2}, \ldots, x_{2q}, & \text{if } n = 3q+2; \\ x_q, x_{q+1}, \ldots, x_{2q-1}, & \text{if } n = 3q. \end{array} \tag{4}$$

In terms of coding theory, $SR(n)$ is a binary linear code of length $n(n+1)/2$ and dimension $d(n)$, and the $d(n)$ central bits in $\mathbf{x}$ can be taken as information bits in $S(\mathbf{x})$, while the remaining $\mathbf{x}(r,c)$ are the redundancy bits.

Consider first the smallest values of n, see Figure 4. For $n = 1$, there exist two triangles in $SR(1)$, which are $S((0))$ and $S((1))$; both are rotationally symmetric, and the value of x_0 determines the triangle $S((x_0))$. For $n = 2$, there exists one rotationally symmetric triangle in $SR(2)$, which is $S((0,0))$, and no coordinate can be chosen. For $n \ge 3$, there exist two rotationally symmetric triangles in $SR(3)$, which are $S((0,0,0))$ and $S((0,1,0))$, and the central coordinate x_1 determines the triangle $S(0, x_1, 0)$.

For $n \ge 4$, we have the following theorem.

Theorem 1 *Let $n \ge 4$ be an integer, and $d = \dim SR(n)$. For each vector $\mathbf{x} \in \mathbb{F}_2^n$, let $\hat{\mathbf{x}}$ be the vector formed by the d central coordinates of $\mathbf{x}$. Then, the mapping $f\colon SR(n) \to \mathbb{F}_2^d$ defined by $S(\mathbf{x}) \mapsto \hat{\mathbf{x}}$ is an isomorphism.*

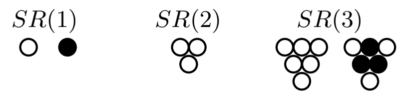

Figure 4: Rotationally symmetric Steinhaus triangles of size $n \leq 3$

Proof. The mapping f is clearly linear and both vector spaces have the same dimension. Then, it suffices to prove that f is exhaustive. The proof depends on the congruence class of n modulo 3.
`Case` $n = 3q + 1$. See Figure 5.

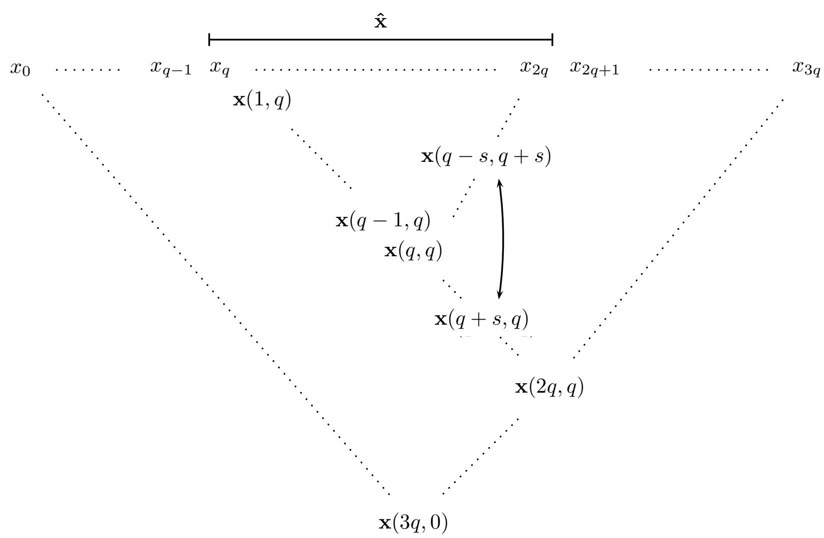

Figure 5: Case $n = 3q + 1$

Let $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{F}_2^n$ be such that $S(\mathbf{x}) \in SR(n)$. In this case $\hat{\mathbf{x}} = (x_q, \ldots, x_{2q})$. Because of formula (3), in $S(\mathbf{x})$, all the entries in the triangle of vertices $x_q = \mathbf{x}(0, q)$, $x_{2q} = \mathbf{x}(0, 2q)$ and $\mathbf{x}(q, q)$ are linearly determined by $\hat{\mathbf{x}} = (x_q, \ldots, x_{2q})$, and, in particular, those in the lines from $x_q = \mathbf{x}(0, q)$ to $\mathbf{x}(q, q)$ and from $x_{2q} = \mathbf{x}(0, 2q)$ to $\mathbf{x}(q, q)$. Because of the rotational symmetry, the values in the line from $x_{2q} = \mathbf{x}(0, 2q)$ to $\mathbf{x}(q, q)$ are the same as those in the line from $\mathbf{x}(2q, q)$ to $\mathbf{x}(q, q)$, that is,

$$\mathbf{x}(q + s, q) = \mathbf{x}(q - s, q + s), \qquad (0 \leq s \leq q).$$

Thus, we have all the entries in column q, that is, $\mathbf{x}(0, q), \ldots, \mathbf{x}(2q, q)$, as a linear combination of $x_q, \ldots, x_{2q}$. Column q determines linearly all the values in the triangle of vertices $x_q = \mathbf{x}(0,q)$, $\mathbf{x}(2q,q)$ and $x_{3q} = \mathbf{x}(0,3q)$. In particular, for $x_{2q+1}, \ldots, x_{3q}$ there exist $a_{i,e} \in \mathbb{F}_2$ such that

$$x_{2q+e} = \sum_{i=0}^{q} a_{i,e} x_{q+i}, \quad (1 \le e \le q). \tag{5}$$

The values of $x_{3q} = \mathbf{x}(0,3q)$, $\mathbf{x}(1,3q-1), \ldots, \mathbf{x}(q-1,2q+1)$ are also determinated by the values in column q. Because of rotational symmetry, we have $x_0 = \mathbf{x}(0,0) = \mathbf{x}(0,3q)$, $\ldots$, $x_{q-1} = \mathbf{x}(0,q-1) = \mathbf{x}(q-1,2q+1)$. Therefore, for some $b_{i,e} \in \mathbb{F}_2$, we have

$$x_e = \sum_{i=0}^{q} b_{i,e} x_{q+i}, \quad (0 \le e \le q-1). \tag{6}$$

Now, the set of $S(\mathbf{x})$ with $\mathbf{x} = (x_0, \ldots, x_{n-1})$ satisfying (5) and (6) is a vector space containing $SR(n)$ and of dimension $d = q+1 = \dim SR(n)$. Hence, it is $SR(n)$. Therefore, given $\hat{\mathbf{x}}$, formulae (5) and (6) allow us to calculate the vector $\mathbf{x} = (x_0, \ldots, x_{n-1})$ such that $f(S(\mathbf{x})) = \hat{\mathbf{x}}$.

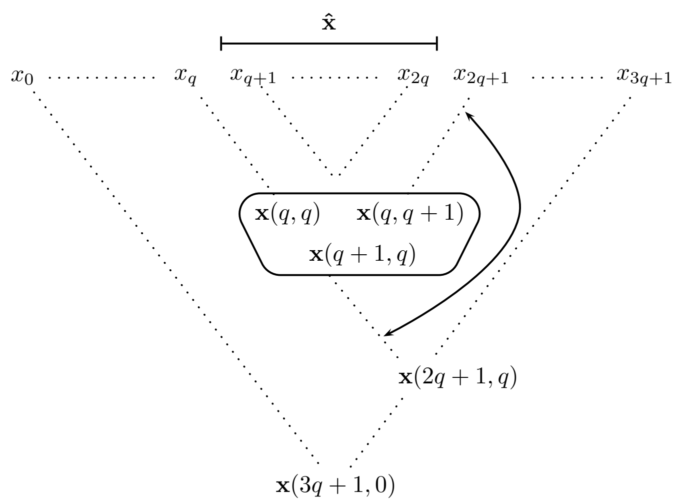

Figure 6: Case $n = 3q+2$

`Case` $n = 3q+2$. See Figure 6.

Let $\mathbf{x} = (x_0, \ldots, x_{n-1})$ be such that $S(\mathbf{x}) \in SR(n)$. The triangle formed by the three entries $\mathbf{x}(q,q)$, $\mathbf{x}(q,q+1)$ and $\mathbf{x}(q+1,q)$ is a triangle of size 2 rotationally symmetric because it is concentric to the triangle $S(\mathbf{x})$. Then, $\mathbf{x}(q,q) = \mathbf{x}(q,q+1) = \mathbf{x}(q+1,q) = 0$. We have the equalities

$$0 = \mathbf{x}(q,q) = \sum_{i=0}^{q} \binom{q}{i} x_{q+i} \quad \text{and} \quad 0 = \mathbf{x}(q,q+1) = \sum_{i=0}^{q} \binom{q}{i} x_{q+1+i},$$

which give the expression of x_q and x_{2q+1} as a linear combination of the coordinates of $\hat{\mathbf{x}} = (x_{q+1}, \ldots, x_{2q})$. As in the previous case, the entries in column q can be written in terms of $x_{q+1}, \ldots, x_{2q}$, and, by using rotational symmetry, $x_0, \ldots, x_q$ and $x_{2q+1}, \ldots, x_{n-1}$ are a linear combination of $x_{q+1}, \ldots, x_{2q}$. Therefore, the sequence $\hat{\mathbf{x}} = (x_{q+1}, \ldots, x_{2q})$ determines the triangle $S(\mathbf{x})$.

`Case` $n = 3q$. See Figure 7.

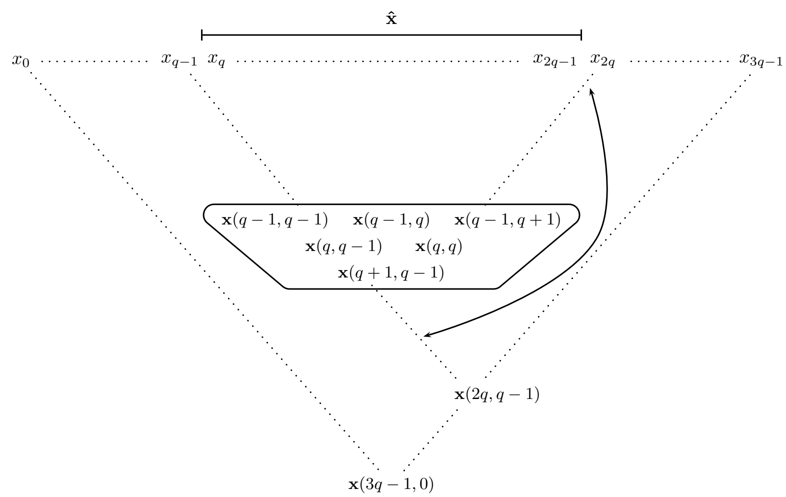

Figure 7: Case $n = 3q$

The triangle with vertices $\mathbf{x}(q-1, q-1)$, $\mathbf{x}(q-1, q+1)$ and $\mathbf{x}(q+1, q-1)$ is a triangle of size 3 rotationally symmetric because it is concentric to the triangle $S(\mathbf{x})$. Then, we have

$$0 = \mathbf{x}(q-1, q-1) = \sum_{i=0}^{q-1} \binom{q-1}{i} x_{q-1+i},$$
$$0 = \mathbf{x}(q-1, q+1) = \sum_{i=0}^{q-1} \binom{q-1}{i} x_{q+1+i}.$$

Thus, x_{q-1} and x_{2q} are a linear combination of $x_q, \ldots, x_{2q-1}$. As before, every entry can be written in terms of $x_{q-1}, \ldots, x_{2q}$. Hence, $\hat{\mathbf{x}} = (x_q, \ldots, x_{2q-1})$ determines the triangle $S(\mathbf{x})$. $\square$

Following the method of the proof, next we obtain explicit formulae for $\mathbf{x}$ in terms of the central coordinates $\hat{\mathbf{x}}$.

Assume $n = 3q + 1$. For $0 \le r \le q$, we have

$$\mathbf{x}(r, q) = \sum_{i=0}^{r} \binom{r}{i} x_{q+i}, \qquad (0 \le r \le q),$$

and, for $q+1 \le r \le 2q$, if $s = r-q$, we have

$$\mathbf{x}(r,q) = \mathbf{x}(q+s,q) = \mathbf{x}(q-s,q+s) = \sum_{j=0}^{q-s} \binom{q-s}{j} x_{q+s+j}.$$

Now, for $0 \le e \le q-1$, we have

$$\begin{aligned}
x_e &= \mathbf{x}(e, 3q-e) \\
&= \sum_{j=0}^{2q-e} \binom{2q-e}{j} \mathbf{x}(2q-j,q) \\
&= \sum_{r=e}^{2q} \binom{2q-e}{2q-r} \mathbf{x}(r,q) \\
&= \sum_{r=e}^{2q} \binom{2q-e}{r-e} \mathbf{x}(r,q) \\
&= \sum_{r=e}^{q} \binom{2q-e}{r-e} \mathbf{x}(r,q) + \sum_{r=q+1}^{2q} \binom{2q-e}{r-e} \mathbf{x}(r,q).
\end{aligned}$$

The first summand is

$$\begin{aligned}
\sum_{r=e}^{q} \binom{2q-e}{r-e} \mathbf{x}(r,q) &= \sum_{r=e}^{q} \binom{2q-e}{r-e} \sum_{i=0}^{r} \binom{r}{i} x_{q+i} \\
&= \sum_{i=0}^{q} \left(\sum_{r=e}^{q} \binom{2q-e}{r-e} \binom{r}{i} \right) x_{q+i}.
\end{aligned}$$

The second is, with $s = r-q$,

$$\begin{aligned}
\sum_{r=q+1}^{2q} \binom{2q-e}{r-e} \mathbf{x}(r,q) &= \sum_{s=1}^{q} \binom{2q-e}{q-s} \sum_{j=0}^{q-s} \binom{q-s}{j} x_{q+s+j} \\
&= \sum_{s=1}^{q} \binom{2q-e}{q-s} \sum_{i=s}^{q} \binom{q-s}{i-s} x_{q+i} \\
&= \sum_{i=1}^{q} \left(\sum_{s=1}^{q} \binom{2q-e}{q-s} \binom{q-s}{i-s} \right) x_{q+i}.
\end{aligned}$$

Putting it all together, we have

$$x_e = \sum_{i=0}^{q} \left(\sum_{r=e}^{q} \binom{2q-e}{r-e} \binom{r}{i} + \sum_{r=1}^{i} \binom{2q-e}{q-r} \binom{q-r}{i-r} \right) x_{q+i}.$$

The expressions of x_{2q+e} for $1 \le e \le q$ are obtained in an analogous way. Simple but cumbersome calculations lead to the formulae for the cases $n = 3q+2$ and $n = 3q$. Next we resume such formulae. In each case we give the free coordinates $\hat{\mathbf{x}}$ and the formulae for the remaining coordinates.

`Case` $n = 3q + 1$. $\hat{\mathbf{x}} = (x_q, \ldots, x_{2q})$.

$$\begin{aligned}
(0 \le e \le q-1) \quad x_e &= \sum_{i=0}^{q} \left(\sum_{r=e}^{q} \binom{2q-e}{r-e} \binom{r}{i} + \sum_{r=1}^{i} \binom{2q-e}{q-r} \binom{q-r}{i-r} \right) x_{q+i} \\
(1 \le e \le q) \quad x_{2q+e} &= \sum_{i=0}^{q} \left(\sum_{r=0}^{q} \binom{q+e}{r} \binom{r}{i} + \sum_{r=1}^{e} \binom{q+e}{q+r} \binom{q-r}{i-r} \right) x_{q+i}
\end{aligned}$$

`Case` $n = 3q + 2$. $\hat{\mathbf{x}} = (x_{q+1}, \ldots, x_{2q})$.

$$\begin{aligned}
(0 \le e \le q-1) \quad x_e &= \sum_{i=1}^{q} \left(\sum_{r=e}^{q-1} \binom{2q+1-e}{r-e} \left(\binom{q}{i} + \binom{r}{i} \right) \right. \\
&\quad \left. + \sum_{r=2}^{q+1} \binom{2q+1-e}{q+1-r} \left(\binom{q+1-r}{i-r} + \binom{q}{i-1} \right) \right) x_{q+i} \\
x_q &= \sum_{i=1}^{q} \binom{q}{i} x_{q+i} \\
x_{2q+1} &= \sum_{i=1}^{q} \binom{q}{i-1} x_{q+i} \\
(1 \le e \le q) \quad x_{2q+1+e} &= \sum_{i=1}^{q} \left(\sum_{r=0}^{q-1} \binom{q+1+e}{r} \left(\binom{q}{i} + \binom{r}{i} \right) \right. \\
&\quad \left. + \sum_{r=0}^{e-1} \binom{q+1+e}{q+2+r} \left(\binom{q-1-r}{i-2-r} + \binom{q}{i-1} \right) \right) x_{q+i}
\end{aligned}$$

`Case` $n = 3q$. $\hat{\mathbf{x}} = (x_q, \ldots, x_{2q-1})$.

$$\begin{aligned}
(0 \le e \le q-2) \quad x_e &= \sum_{i=1}^{q} \left(\sum_{r=e}^{q} \binom{2q-e}{r-e} \left(\binom{q-1}{i} + \binom{r}{i} \right) \right. \\
&\quad \left. + \sum_{r=0}^{q-2} \binom{2q-e}{q-2-r} \left(\binom{q-2-r}{i-3-r} + \binom{q-1}{i-2} \right) \right) x_{q-1+i} \\
x_{q-1} &= \sum_{i=1}^{q-1} \binom{q-1}{i} x_{q-1+i} \\
x_{2q} &= \sum_{i=2}^{q} \binom{q-1}{i-2} x_{q-1+i} \\
(1 \le e \le q-1) \quad x_{2q+e} &= \sum_{i=1}^{q} \left(\sum_{r=0}^{q} \binom{q+1+e}{r} \left(\binom{q-1}{i} + \binom{r}{i} \right) \right. \\
&\quad \left. + \sum_{r=0}^{e-1} \binom{q+1+e}{q+2+r} \left(\binom{q-2-r}{i-3-r} + \binom{q-1}{i-2} \right) \right) x_{q-1+i}
\end{aligned}$$

n	$d(n)$	first row
1	1	(x_0)
2	0	$(0,\, 0)$
3	1	$(0,\, x_1,\, 0)$
4	2	$(x_1+x_2,\, x_1,\, x_2,\, x_1+x_2)$
5	1	$(0,\, x_2,\, x_2,\, x_2,\, 0)$
6	2	$(x_2+x_3,\, x_2,\, x_2,\, x_3,\, x_3,\, x_2+x_3)$
7	3	$(x_2+x_4,\, x_4,\, x_2,\, x_3,\, x_4,\, x_2,\, x_2+x_4)$
8	2	$(x_3+x_4,\, x_4,\, x_4,\, x_3,\, x_4,\, x_3,\, x_3,\, x_3+x_4)$
9	3	$(0,\, x_3+x_4+x_5,\, x_4,\, x_3,\, x_4,\, x_5,\, x_4,\, x_3+x_4+x_5,\, 0)$
10	4	$(x_4+x_5,\, x_4,\, x_3+x_5+x_6,\, x_3,\, x_4,\, x_5,\, x_6,\, x_3+x_4+x_6,\, x_5,\, x_4+x_5)$
11	3	$(x_4+x_6,\, x_6,\, x_4,\, x_4+x_5+x_6,\, x_4,\, x_5,\, x_6,\, x_4+x_5+x_6,\, x_6,\, x_4,$ $x_4+x_6)$
12	4	$(x_4+x_7,\, x_5+x_6+x_7,\, x_5+x_6+x_7,\, x_4+x_5+x_6,\, x_4,\, x_5,\, x_6,\, x_7,$ $x_5+x_6+x_7,\, x_4+x_5+x_6, x_4+x_5+x_6,\, x_4+x_7)$
13	5	$(x_4+x_8,\, x_5+x_7+x_8,\, x_8,\, x_5+x_6+x_8,\, x_4,\, x_5,\, x_6,\, x_7,\, x_8,$ $x_4+x_6+x_7,\, x_4,\, x_4+x_5+x_7,\, x_4+x_8)$
14	4	$(x_5+x_8,\, x_8,\, x_5+x_6+x_7,\, x_5+x_6+x_8,\, x_8,\, x_5,\, x_6,\, x_7,\, x_8,\, x_5,$ $x_5+x_7+x_8,\, x_6+x_7+x_8,\, x_5,\, x_5+x_8)$

Table 1: First rows of Steinhaus triangles of size n with rotational symmetry

Table 1 shows the application of these formulae for $1 \le n \le 14$. Figure 8 shows all triangles in $SR(7)$; they are formed by giving values to (x_2, x_3, x_4) in the 7th row of Table 1, to obtain the first rows of the triangles in $SR(7)$. Each triangle is labeled by (x_2, x_3, x_4). The first three triangles form a basis of $SR(7)$.

3. Dihedral Symmetry in Steinhaus Triangles

Let $SD(n)$ be the vector space of the dihedrally symmetric Steinhaus triangles of size n. All rotationally symmetric Steinhaus triangles of size $n \le 3$ are also dihedrally symmetric; see Figure 4. Thus, we have $SD(n) = SR(n)$ for $1 \le n \le 3$. Let $\epsilon_6(n) = 1$ if $n \equiv 1 \pmod 6$ and $\epsilon_6(n) = 0$ otherwise. It is known ([3], Corollary 2) that the dimension of $SD(n)$ is

$$\tilde{d}(n) = \dim SD(n) = \left\lfloor \frac{n+3}{6} \right\rfloor + \epsilon_6(n) = \left\lceil \frac{d(n)}{2} \right\rceil,$$

where $d(n) = \dim SR(n)$. Also, if $\mathbf{x} = (x_0, \ldots, x_{n-1})$ and $S(\mathbf{x})$ is rotationally symmetric, then $S(\mathbf{x})$ is dihedrally symmetric if, and only if, $x_i = x_{n-1-i}$ for all $i \in \{0, \ldots, n-1\}$.

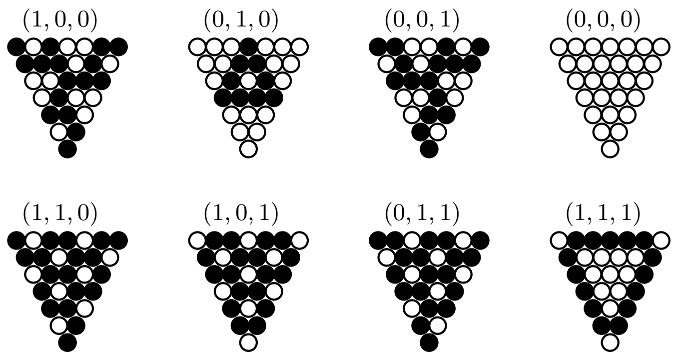

Figure 8: The eight Steinhaus triangles of size 7 with rotational symmetry

Theorem 2 *Let $n \geq 4$ be an integer, $d = \dim SR(n)$ and $\tilde{d} = \dim SD(n)$. For $\mathbf{x} \in \mathbb{F}_2^n$, let $\hat{\mathbf{x}}$ be the vector formed by the d central coordinates of $\mathbf{x}$ and $\tilde{\mathbf{x}}$ the vector formed by the first $\tilde{d}$ coordinates of $\hat{\mathbf{x}}$. Then, the mapping $f\colon SD(n) \to \mathbb{F}_2^{\tilde{d}}$ defined by $S(\mathbf{x}) \mapsto \tilde{\mathbf{x}}$ is an isomorphism.*

Proof. The mapping f is clearly linear, and both vector spaces have the same dimension. Then, it suffices to prove that f is exhaustive.

Let $\mathbf{x} = (x_0, \ldots, x_{n-1})$ be such that $S(\mathbf{x})$ is dihedrally symmetric. Since $S(\mathbf{x})$ is rotationally symmetric, the vector $\hat{\mathbf{x}}$ determines $\mathbf{x}$. Since $S(\mathbf{x})$ is dihedrally symmetric, the first $\tilde{d}$ coordinates in $\hat{\mathbf{x}}$ determine the remaining coordinates. Thus, every coordinate in $\mathbf{x}$ depends linearly on the coordinates in $\tilde{\mathbf{x}} = (x_{q_1}, \ldots, x_{q_1+\tilde{d}-1})$. For some $c_{i,j} \in \mathbb{F}_2$, we have

$$x_i = \sum_{j=0}^{\tilde{d}-1} c_{i,j} x_{q_1+j}, \qquad i \in \{0, \ldots, n-1\} \setminus \{q_1, \ldots, q_1 + \tilde{d} - 1\}. \tag{7}$$

The set of $S(\mathbf{x})$ with $\mathbf{x}$ satisfying (7) is a vector space containing $SD(n)$ and of dimension $\tilde{d} = \dim SD(n)$. Therefore, both vector spaces coincide. Thus, given $\tilde{\mathbf{x}}$, formulae (7) allows us calculate $\mathbf{x}$ such that $f(S(\mathbf{x})) = \tilde{\mathbf{x}}$. $\square$

The argument in the proof leads to the following. In each of the three cases $n = 3q+1$, $n = 3q+2$ and $n = 3q$ we must distinguish between the cases q odd and q even. As before, we give the free coordinates $\tilde{\mathbf{x}}$ and the formulae for the remaining coordinates.

`Case` $n = 3q + 1$. Define

$$A(i, q, e) = \sum_{r=e}^{q} \binom{2q-e}{r-e} \left(\binom{r}{i} + \binom{r}{q-i} \right) + \sum_{r=1}^{q} \binom{2q-e}{q-r} \left(\binom{q-r}{q-i} + \binom{q-r}{i} \right).$$

• $q = 2t+1$. $\quad \tilde{\mathbf{x}} = (x_q, \ldots, x_{q+t})$.

$$\begin{aligned} (0 \le e \le q-1) \quad x_e &= \sum_{i=0}^{t} A(i,q,e) x_{q+i} \\ (1 \le e \le q+t+1) \quad x_{q+t+e} &= x_{q+t+1-e} \end{aligned}$$

• $q = 2t$. $\quad \tilde{\mathbf{x}} = (x_q, \ldots, x_{q+t})$.

$$\begin{aligned} (0 \le e \le q-1) \quad x_e &= \sum_{i=0}^{t-1} A(i,q,e) x_{q+i} \\ &\quad + \left(\sum_{r=e}^{q} \binom{2q-e}{r-e}\binom{r}{t} + \sum_{r=1}^{q} \binom{2q-e}{q-r}\binom{q-r}{t-r} \right) x_{q+t} \\ (1 \le e \le q+t) \quad x_{q+t+e} &= x_{q+t-e} \end{aligned}$$

`Case` $n = 3q+2$. Define

$$\begin{aligned} A(i,q,e) &= \sum_{r=e}^{q-1} \binom{2q+1-e}{r-e} \left(\binom{q+1}{i} + \binom{r}{i} + \binom{r}{q+1-i} \right) \\ &\quad + \sum_{r=2}^{q+1} \binom{2q+1-e}{q+1-r} \left(\binom{q+1}{i} + \binom{q+1-r}{i-r} + \binom{q+1-r}{i} \right). \end{aligned}$$

• $q = 2t+1$. $\quad \tilde{\mathbf{x}} = (x_{q+1}, \ldots, x_{q+t+1})$.

$$\begin{aligned} (0 \le e \le q-1) \quad x_e &= \sum_{i=1}^{t} A(i,q,e) x_{q+i} \\ &\quad + \left(\sum_{r=e}^{q-1} \binom{2q+1-e}{r-e} \left(\binom{q}{t+1} + \binom{r}{t+1} \right) \right. \\ &\qquad \left. + \sum_{r=2}^{q+1} \binom{2q+1-e}{q+1-r} \left(\binom{q+1-r}{t+1-r} + \binom{q}{t} \right) \right) x_{q+t+1} \\ x_q &= \sum_{i=1}^{t} \binom{q+1}{i} x_{q+i} + \binom{q}{t+1} x_{q+t+1} \\ (0 \le e \le q+t) \quad x_{3q+1-e} &= x_e \end{aligned}$$

• $q = 2t$. $\quad \tilde{\mathbf{x}} = (x_{q+1}, \ldots, x_{q+t})$.

$$\begin{aligned} (0 \le e \le q-1) \quad x_e &= \sum_{i=1}^{t} A(i,q,e) x_{q+i} \\ x_q &= \sum_{i=1}^{t} \binom{q+1}{i} x_{q+i} \\ (0 \le e \le q+t) \quad x_{3q+1-e} &= x_e \end{aligned}$$

`Case` $n = 3q$. Define

$$\begin{aligned} A(i,q,e) &= \sum_{r=e}^{q} \binom{2q-e}{r-e} \left(\binom{q+1}{i+1} + \binom{r}{i+1} + \binom{r}{q-i} \right) \\ &\quad + \sum_{r=0}^{q-2} \binom{2q-e}{q-2-r} \left(\binom{q-2-r}{i-2-r} + \binom{q-2-r}{i+1} + \binom{q-1}{i-1} + \binom{q-1}{i+1} \right). \end{aligned}$$

- $q = 2t+1$. $\quad \tilde{\mathbf{x}} = (x_q, \ldots, x_{q+t})$.

$$\begin{aligned} (0 \le e \le q-2) \quad x_e &= \sum_{i=0}^{t-1} A(i,q,e) x_{q+i} \\ &\quad + \left(\sum_{r=e}^{q} \binom{2q-e}{r-e} \left(\binom{q}{t+1} + \binom{r}{t+1} \right) \right. \\ &\quad\quad \left. + \sum_{r=0}^{q-2} \binom{2q-e}{q-2-r} \left(\binom{q-2-r}{t-2-r} + \binom{q-1}{t-1} \right) \right) x_{q+t} \\ x_{q-1} &= \sum_{i=0}^{t-1} \left(\binom{q-1}{i+1} + \binom{q-1}{i-1} \right) x_{q+i} + \binom{q-1}{t+1} x_{q+t} \\ (0 \le e \le 3t+1) \quad x_{q+t+e} &= x_{q+t-e} \end{aligned}$$

- $q = 2t$. $\quad \tilde{\mathbf{x}} = (x_q, \ldots, x_{q+t-1})$.

$$\begin{aligned} (0 \le e \le q-2) \quad x_e &= \sum_{i=0}^{t-1} A(i,q,e) x_{q+i} \\ x_{q-1} &= \sum_{i=0}^{t-1} \left(\binom{q-1}{i+1} + \binom{q-1}{i-1} \right) x_{q+i} \\ (0 \le e \le 3t-1) \quad x_{q+t+e} &= x_{q+t-1-e} \end{aligned}$$

Table 2 shows the application of these formulae for $1 \le n \le 14$. For instance, for $n = 7$, the sequences $\mathbf{x}$ such that $S(\mathbf{x})$ has dihedral symmetry are the sequences of the form $\mathbf{x} = (0, x_2, x_2, x_3, x_2, x_2, 0)$ with $x_2, x_3 \in \mathbb{F}_2$. Here, $\tilde{\mathbf{x}} = (x_2, x_3)$ and $\hat{\mathbf{x}} = (x_2, x_3, x_2)$. Therefore, in Figure 8, the triangles labeled $(0, 0, 0), (0, 1, 0), (1, 0, 1)$ and $(1, 1, 1)$ form $SD(7)$.

4. Rotational Symmetry in Pascal Triangles

The results about rotational and dihedral symmetry in Pascal triangles will be deduced from the corresponding formulae in Steinhaus triangles following a technique

n	$\tilde{d}(n)$	first row
1	1	(x_0)
2	0	$(0, 0)$
3	1	$(0,\, x_1,\, 0)$
4	1	$(0,\, x_1,\, x_1,\, 0)$
5	1	$(0,\, x_2,\, x_2,\, x_2,\, 0)$
6	1	$(0,\, x_2,\, x_2,\, x_2,\, x_2,\, 0)$
7	2	$(0,\, x_2,\, x_2,\, x_3,\, x_2,\, x_2,\, 0)$
8	1	$(0,\, x_3,\, x_3,\, x_3,\, x_3,\, x_3,\, x_3,\, 0)$
9	2	$(0,\, x_4,\, x_4,\, x_3,\, x_4,\, x_3,\, x_4,\, x_4,\, 0)$
10	2	$(0,\, x_4,\, x_4,\, x_3,\, x_4,\, x_4,\, x_3,\, x_4,\, x_4,\, 0)$
11	2	$(0,\, x_4,\, x_4,\, x_5,\, x_4,\, x_5,\, x_4,\, x_5,\, x_4,\, x_4,\, 0)$
12	2	$(0,\, x_4,\, x_4,\, x_4,\, x_4,\, x_5,\, x_5,\, x_4,\, x_4,\, x_4,\, x_4,\, 0)$
13	3	$(0,\, x_4,\, x_4,\, x_4 + x_5 + x_6,\, x_4,\, x_5,\, x_6,\, x_5,\, x_4,\, x_4 + x_5 + x_6,\, x_4,\, x_4,\, 0)$
14	2	$(0,\, x_5,\, x_5,\, x_6,\, x_5,\, x_5,\, x_6,\, x_6,\, x_5,\, x_5,\, x_6,\, x_5,\, x_5,\, 0)$

Table 2: First rows of Steinhaus triangles of size n with dihedral symmetry

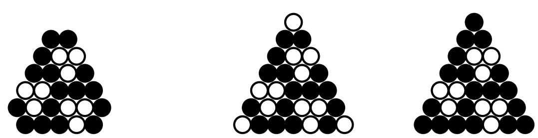

Figure 9: The rotational symmetry of the illustration on the left is equivalent to the rotational symmetry of the two Pascal triangles on the right

introduced by A. Barbé [3], consisting of associating to each Pascal triangle of size k a Steinhaus triangle of size $2k$ preserving the properties of symmetry.

Let $\mathbf{u} = (u_0, \ldots, u_\ell)$ and $\mathbf{v} = (v_0, \ldots, v_\ell)$ be vectors of $\mathbb{F}_2^\ell$ with $u_0 = v_0$, and consider the Pascal triangle $P(\mathbf{u}, \mathbf{v})$. If $P(\mathbf{u}, \mathbf{v})$ has rotational symmetry, then $v_i = u_{\ell-i}$ for $0 \le i \le \ell$. It follows that: (i) the triangle is determined by $\mathbf{u}$; (ii) the three vertices of the triangle are equal: $u_0 = v_0 = u_\ell = v_\ell$, and (iii) neither the values of the vertices have influence on the remaining entries of the triangle, nor are the vertices influenced by them. It follows that we can consider Pascal triangles with the vertices removed. For instance, the rotational symmetry of the picture on the left in Figure 9 is equivalent to the rotational symmetry of two of the triangles obtained by adding to it the three vertices, all of them with the same value, as shown in the two illustrations on the right. So, in the following, we consider vertex-less Pascal triangles, though they will be still called Pascal triangles. A rotationally symmetric (vertex-less) Pascal triangle is determined by the left side, which will be indexed from bottom to top. Thus, for example, if $\mathbf{a} = (1, 0, 1, 1, 1)$, the triangle

$P(\mathbf{a})$ is that on the left in Figure 9. The length of $\mathbf{a}$ is the *size* of $P(\mathbf{a})$. Our goal is to determine explicitly which sequences $\mathbf{a}$ produce Pascal triangles with rotational and dihedral symmetry. We denote by $PR(k)$ the vector space of Pascal triangles of size k with rotational symmetry, and by $PD(k)$ the vector space of Pascal triangles of size k with dihedral symmetry.

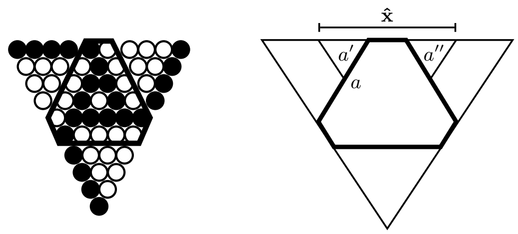

Figure 10: Left: extension of a rotationally symmetric Pascal triangle of size 5 to a rotationally Steinhaus triangle of size 10. Right: sketch of the general case

Let $P(\mathbf{a})$ be a Pascal triangle of size k rotationally symmetric. Then, by using the local rule (Figure 3), $P(\mathbf{a})$ can be extended to a unique Steinhaus triangle $S(\mathbf{x})$ of size $2k$ which has $P(\mathbf{a})$ as its inscribed and central triangle of size k; see Figure 10. The triangle $S(\mathbf{x})$ is the *extended Steinhaus triangle* of the Pascal triangle $P(\mathbf{a})$. Explicitly, if $\mathbf{a} = (a_0, \ldots, a_{k-1})$, then $\mathbf{x} = (x_0, \ldots, x_{2k-1})$ is given by

$$x_{k-i} = \sum_{j=0}^{i-1} \binom{i-1}{j} a_{k-1-j}, \quad (1 \leq i \leq k), \tag{8}$$

$$x_{k+i} = \sum_{j=0}^{i} \binom{i}{j} a_j, \quad (0 \leq i \leq k-1). \tag{9}$$

As $P(\mathbf{a})$ is rotationally symmetric, it follows that $S(\mathbf{x})$ is rotationally symmetric. Then, we have

Lemma 3 *For each integer $k \geq 3$, the mapping f which sends each Pascal triangle $P(\mathbf{a}) \in PR(k)$ to its extended Steinhaus triangle $S(\mathbf{x})$ is an isomorphism $f\colon PR(k) \to SR(2k)$.*

Proof. Clearly, the mapping f is well-defined and linear. If $f(P(\mathbf{a}))$ is filled with zeroes, then $P(\mathbf{a})$ is also filled with zeroes. Hence the mapping is injective. Finally, given an Steinhaus triangle $S(\mathbf{x}) \in SR(2k)$, the extension of its central triangle of size k is precisely $S(\mathbf{x})$. Hence the mapping is exhaustive. □

As a consequence, the dimension $\delta(k)$ of $PR(k)$ is

$$\delta(k) = \dim PR(k) = \dim SR(2k) = \left\lfloor \frac{2k}{3} \right\rfloor + \epsilon_3(2k) = 2\left\lceil \frac{k-1}{3} \right\rceil.$$

Theorem 4 *Let $k \geq 2$ be an integer and $\delta = \dim PR(k)$. For each $\mathbf{a} \in \mathbb{F}_2^k$, let $\mathbf{a}'$ be the vector formed by the first $\delta/2$ coordinates of $\mathbf{a}$ and $\mathbf{a}''$ the vector formed by the last $\delta/2$ coordinates of $\mathbf{a}$. Then, the mapping $f\colon PR(k) \to \mathbb{F}_2^\delta$ defined by $P(\mathbf{a}) \mapsto (\mathbf{a}', \mathbf{a}'')$ is an isomorphism.*

Proof. Clearly the mapping f is linear and both spaces have the same dimension δ. Thus, it suffices to show that f is exhaustive.

Let $\mathbf{a} \in \mathbb{F}_2^k$ such that $P(\mathbf{a}) \in PR(k)$. As $P(\mathbf{a})$ is rotationally symmetric, the same property holds for its extended Steinhaus triangle $S(\mathbf{x})$, which, by Theorem 1, is completely determined by the vector $\hat{\mathbf{x}}$ formed by the δ central coordinates of $\mathbf{x}$. By (8), the first half of the coordinates of $\hat{\mathbf{x}}$ depend linearly on $\mathbf{a}''$, and the second half on $\mathbf{a}'$. We conclude that $\mathbf{x}$ depends linearly on $(\mathbf{a}', \mathbf{a}'')$. As the $k - \delta$ central coordinates of $\mathbf{a}$ depend linearly on $\mathbf{x}$, we conclude that, for some $c_{i,j}, d_{i,j} \in \mathbb{F}_2$,

$$a_{\delta/2+i} = \sum_{j=0}^{\delta/2-1} c_{i,j} a_j + \sum_{j=0}^{\delta/2-1} d_{i,j} a_{k-1-j}, \qquad (0 \leq i \leq k - \delta - 1). \tag{10}$$

The set $P(\mathbf{a})$ with $\mathbf{a} \in \mathbf{F}_2^k$ satisfying (10) is a vector space of dimension δ containing $PR(k)$. Hence, it is $PR(k)$. Therefore, given $(\mathbf{a}', \mathbf{a}'')$, formulae (10) allows us to find $\mathbf{a}$ such that $f(P(\mathbf{a})) = (\mathbf{a}', \mathbf{a}'')$. $\square$

Next, we give the results obtained by following the method of the previous proof with explicit calculations. As before, the formulae depend on the remainder of the division of k by 3. In each case, we give $\mathbf{a}'$ and $\mathbf{a}''$ and the formula for obtaining the $k - \delta$ central coordinates of $\mathbf{a}$.

`Case` $k = 3s + 1$. $\quad \mathbf{a}' = (a_0, \ldots, a_{s-1}), \quad \mathbf{a}'' = (a_{k-s}, \ldots, a_{k-1})$.
Define

$$\begin{aligned}
q &= 2s \\
A(i,q,e) &= \sum_{r=e}^{q-1} \binom{2q+1-e}{r-e} \left(\binom{q}{i} + \binom{r}{i} \right) \\
&\quad + \sum_{r=2}^{q+1} \binom{2q+1-e}{q+1-r} \left(\binom{q+1-r}{i-r} + \binom{q}{i-1} \right) \\
B(i,q,e,s) &= \binom{q-e}{s}\binom{q}{i} + \sum_{t=0}^{s-1-e} \binom{q-e}{t} A(i,q,s+e+t).
\end{aligned}$$

Then,

$$\begin{aligned}
(0 \leq e \leq s) \quad a_{s+e} &= \sum_{j=0}^{s-1} \left(\sum_{i=s+1+j}^{q} \binom{i-1-s}{j} B(i,q,e,s) \right) a_j \\
&\quad + \sum_{j=0}^{s-1} \left(\sum_{i=1}^{s-j} \binom{s-i}{j} B(i,q,e,s)
\end{aligned}$$

$$+\sum_{t=s-e+1}^{q-e-j}\binom{q-e}{t}\binom{q-e-t}{j}\Bigg)\,a_{k-1-j}.$$

`Case` $k=3s+2$. $\mathbf{a}'=(a_0,\,\ldots,\,a_s)$, $\mathbf{a}''=(a_{k-s-1},\,\ldots,\,a_{k-1})$.
Define

$$\begin{aligned}
q &= 2s+1\\
A(i,q,e) &= \sum_{r=e}^{q}\binom{2q-e}{r-e}\binom{r}{i}+\sum_{r=1}^{q}\binom{2q-e}{q-r}\binom{q-r}{i-r}.
\end{aligned}$$

Then,

$$\begin{aligned}
(1\le e\le s)\quad a_{s+e} &= \sum_{j=0}^{s}\left(\sum_{t=0}^{s-e}\binom{q-e}{t}\sum_{i=s+1}^{q}\binom{i-s-1}{j}A(i,q,s+e+t)\right)a_j\\
&+\sum_{j=0}^{s}\left(\sum_{i=0}^{s}\binom{s-i}{j}\left(\binom{q-e}{s-i}+\sum_{t=0}^{s-e}\binom{q-e}{t}A(i,q,s+e+t)\right)\right)a_{k-1-j}.
\end{aligned}$$

`Case` $k=3s$. $\mathbf{a}'=(a_0,\,\ldots,\,a_{s-1})$, $\mathbf{a}''=(a_{k-s},\,\ldots,\,a_{k-1})$.
Define

$$\begin{aligned}
q &= 2s\\
A(i,q,e) &= \sum_{r=e}^{q}\binom{2q-e}{r-e}\left(\binom{q-1}{i}+\binom{r}{i}\right)\\
&\quad+\sum_{r=0}^{q-2}\binom{2q-e}{q-2-r}\left(\binom{q-2-r}{i-3-r}+\binom{q-1}{i-2}\right)\\
B(i,q,e,s) &= \sum_{t=0}^{s-2-e}\binom{q-e-1}{t}A(i,q,s+e+t).
\end{aligned}$$

Then,

$$\begin{aligned}
(0\le e\le s-1)\quad a_{s+e} &= \sum_{j=0}^{s-2}\Bigg(\sum_{i=s+1}^{q}\binom{i-1-s}{j}B(i,q,e,s)\\
&\quad+\sum_{i=s+1}^{q-1}\binom{q-e-1}{s}\binom{q-1}{i}\binom{i-1-s}{j}\Bigg)a_j\\
&\quad+\left(\sum_{i=s+1}^{q}\binom{i-1-s}{s-1}B(i,q,e,s)\right)a_{s-1}\\
&\quad+\sum_{j=0}^{s-1}\Bigg(\sum_{i=1}^{s}\binom{s-i}{j}B(i,q,e,s)\\
&\qquad+\binom{q-e-1}{s}\sum_{i=1}^{s}\binom{q-1}{i}\binom{s-i}{j}\\
&\qquad+\sum_{t=s-e}^{q-e-1}\binom{q-e-1}{t}\binom{q-e-t-1}{j}\Bigg)a_{k-1-j}.
\end{aligned}$$

Table 3 shows the application of these formulae for sizes k from 3 to 14. Figure 11 shows all rotationally symmetric Pascal triangles of size 5. They are formed by giving values to $(a_0, a_1, a_3, a_4) \in \mathbb{F}_2^4$ in the 5th row of Table 3 to form the left sides of the Pascal triangles. The four triangles in the first row form a base of $PR(5)$.

k	$\delta(k)$	left row
3	2	$(a_0, 0, a_2)$
4	2	$(a_0, a_0+a_3, a_0+a_3, a_3)$
5	4	$(a_0, a_1, a_1+a_3, a_3, a_4)$
6	4	$(a_0, a_1, a_0+a_1+a_5, a_0+a_5+a_4, a_4, a_5)$
7	4	$(a_0, a_1, a_0+a_6+a_5, 0, a_0+a_1+a_6, a_5, a_6)$
8	6	$(a_0, a_1, a_2, a_0+a_7+a_6+a_5, a_0+a_1+a_2+a_7, a_5, a_6, a_7)$
9	6	$(a_0, a_1, a_2, a_1+a_7, a_1+a_2+a_7+a_6, a_1+a_7, a_6, a_7, a_8)$
10	6	$(a_0, a_1, a_2, a_2+a_7, a_0+a_1+a_9+a_7, a_0+a_2+a_9+a_8,$ $a_2+a_7, a_7, a_8, a_9)$
11	8	$(a_0, a_1, a_2, a_3, a_0+a_2+a_3+a_{10}+a_9+a_7, a_3+a_7,$ $a_0+a_1+a_3+a_{10}+a_8+a_7, a_7, a_8, a_9, a_{10})$
12	8	$(a_0, a_1, a_2, a_3, a_0+a_3+a_{11}+a_{10}+a_9,$ $a_1+a_2+a_3+a_{10}+a_9+a_8,$ $a_1+a_2+a_3+a_{10}+a_9+a_8, a_0+a_1+a_2+a_{11}+a_8,$ $a_8, a_9, a_{10}, a_{11})$
13	8	$(a_0, a_1, a_2, a_3, a_0+a_{12}+a_{11}+a_{10}+a_9, a_1+a_3+a_{11}, a_3+a_9,$ $a_1+a_{11}+a_9, a_0+a_1+a_2+a_3+a_{12}, a_9, a_{10}, a_{11}, a_{12})$
14	10	$(a_0, a_1, a_2, a_3, a_4, a_0+a_2+a_3+a_{13}+a_{12}+a_{10}+a_9,$ $a_2+a_3+a_{11}, a_2+a_{11}+a_{10},$ $a_0+a_1+a_3+a_4+a_{13}+a_{11}+a_{10}, a_9, a_{10}, a_{11}, a_{12}, a_{13})$

Table 3: Left rows of rotationally symmetric Pascal triangles of size k

5. Dihedral Symmetry in Pascal Triangles

The arguments in the previous section are easily extended to dihedral symmetry. In fact, it is easy to see that the mapping f sending each Pascal triangle to its extended Steinhaus triangle, when restricted to dihedrally symmetric Pascal triangles, also gives an isomorphism:

Lemma 5 *For each integer $k \geq 3$, the mapping f which sends each Pascal triangle $P(\mathbf{a}) \in PD(k)$ to its extended Steinhaus triangle $S(\mathbf{x})$ is an isomorphism $f\colon PD(k) \to SD(2k)$.*

As noticed by Barbé ([3], Property 15), if $\mathbf{a} = (a_0, \ldots, a_{k-1})$, the Pascal triangle $P(\mathbf{a})$ is symmetric with respect to the height of the triangle if, and only if, $a_{k-1-i} =$

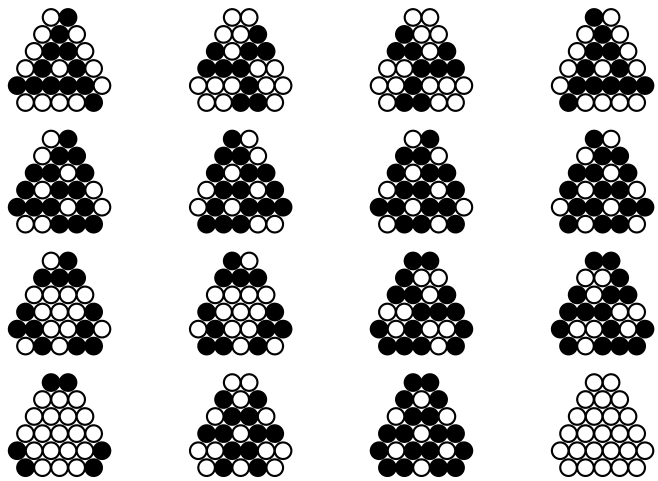

Figure 11: Rotationally symmetric Pascal triangles of size 5

a_i for $0 \le i \le k-1$. This property and Theorem 4 imply the following.

Theorem 6 *Let $k \ge 2$ be an integer and $\delta = \dim PR(k)$. For each $\mathbf{a} \in \mathbb{F}_2^k$, let $\mathbf{a}'$ be the vector formed by the first $\delta/2$ coordinates of $\mathbf{a}$. Then, the mapping $f\colon PR(k) \to \mathbb{F}_2^{\delta/2}$ defined by $P(\mathbf{a}) \mapsto \mathbf{a}'$ is an isomorphism.*

As before, we give the vector $\mathbf{a}'$ of the first $\delta/2$ coordinates and the formulae for the remaining coordinates. The functions $A(i,q,e)$ and $B(i,q,e,s)$ are defined in each case as in the rotational symmetry.
`Case` $k = 3s+1$. $\mathbf{a}' = (a_0, \ldots, a_{s-1})$, $q = 2s$.

$$\begin{aligned}
(0 \le e \le s) \quad a_{s+e} &= \sum_{j=0}^{s-1} \left(\sum_{i=s+1+j}^{q} \binom{i-1-s}{j} B(i,q,e,s) \right.\\
&\qquad + \sum_{i=1}^{s-j} \binom{s-i}{j} B(i,q,e,s)\\
&\qquad \left. + \sum_{t=s-e+1}^{q-e-j} \binom{q-e}{t} \binom{q-e-t}{j} \right) a_j\\
(1 \le e \le s) \quad a_{q+e} &= a_{s-e}
\end{aligned}$$

`Case` $k = 3s+2$. $\mathbf{a}' = (a_0, \ldots, a_s)$, $q = 2s+1$.

$$
\begin{aligned}
(1 \le e \le s) \quad a_{s+e} \quad &= \quad \sum_{j=0}^{s} \left(\sum_{t=0}^{s-e} \binom{q-e}{t} \sum_{i=s+1}^{q} \binom{i-s-1}{j} A(i,q,s+e+t) \right. \\
&\left. + \sum_{i=0}^{s} \binom{s-i}{j} \left(\binom{q-e}{s-i} + \sum_{t=0}^{s-e} \binom{q-e}{t} A(i,q,s+e+t) \right) \right) a_j \\
(0 \le e \le s) \quad a_{q+e} \quad &= \quad a_{s-e}
\end{aligned}
$$

`Case` $k = 3s$. $\quad \mathbf{a}' = (a_0, \ldots, a_{s-1}), \quad q = 2s.$

$$
\begin{aligned}
(0 \le e \le s-1) \quad a_{s+e} \quad &= \quad \sum_{j=0}^{s-1} \left(\sum_{i=s+1}^{q} \binom{i-1-s}{j} B(i,q,e,s) \right. \\
&+ \left\lfloor \frac{q-j}{s+2} \right\rfloor \binom{q-e-1}{s} \sum_{i=s+1}^{q-1} \binom{q-1}{i} \binom{i-1-s}{j} \\
&+ \sum_{i=1}^{s} \binom{s-i}{j} \left(\binom{q-e-1}{s} \binom{q-1}{i} + B(i,q,e,s) \right) \\
&\left. + \sum_{t=s-e}^{q-e-1} \binom{q-e-1}{t} \binom{q-e-t-1}{j} \right) a_j \\
(0 \le e \le s-1) \quad a_{q+e} \quad &= \quad a_{s-1-e}
\end{aligned}
$$

Table 4 shows the left rows of dihedrally symmetric Pascal triangles of sizes k from 3 to 14. For size $k = 5$ there exist 4 triangles, which are the triangles in the last row in Figure 11.

6. Generalization to Arbitrary Abelian Groups

The construction of Steinhaus and Pascal triangles can be generalized by using an arbitrary abelian group instead of $\mathbb{F}_2$.

Let G be an abelian group and $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in G^n$. Define $\partial^0 \mathbf{x} = \mathbf{x}$, $\partial^1 \mathbf{x} = \partial \mathbf{x} = (x_0 + x_1, \ldots, x_{n-2} + x_{n-1})$, and for $2 \le i \le n-1$, $\partial^i \mathbf{x} = \partial \partial^{i-1} \mathbf{x}$. The *Steinhaus triangle* of $\mathbf{x}$ is the sequence $S(\mathbf{x}) = (\mathbf{x}, \partial \mathbf{x}, \ldots, \partial^{n-1} \mathbf{x})$. As before, we can represent $S(\mathbf{x})$ as a triangle, but now each position can take values in G. Next, we see that rotational symmetry is a strong condition on the orders of the entries of $S(\mathbf{x})$.

Proposition 7 *Let G be an abelian group, $n \ge 2$ an integer, and $\mathbf{x} \in G^n$ such that $S(\mathbf{x})$ is rotationally symmetric. Then each entry in $S(\mathbf{x})$ has order two.*

k	$\delta'(k)$	left row
3	1	$(a_0,\, 0,\, a_0)$
4	1	$(a_0,\, 0,\, 0,\, a_0)$
5	2	$(a_0,\, a_1,\, 0,\, a_1,\, a_0)$
6	2	$(a_0,\, a_1,\, a_1,\, a_1,\, a_1,\, a_0)$
7	3	$(a_0,\, a_1,\, a_1,\, 0,\, a_1,\, a_1,\, a_0)$
8	3	$(a_0,\, a_1,\, a_2,\, a_1+a_2,\, a_1+a_2,\, a_2,\, a_1,\, a_0)$
9	3	$(a_0,\, a_1,\, a_2,\, 0,\, 0,\, 0,\, a_2,\, a_1,\, a_0)$
10	3	$(a_0,\, a_1,\, a_2,\, 0,\, a_1+a_2,\, a_1+a_2,\, 0,\, a_2,\, a_1,\, a_0)$
11	4	$(a_0,\, a_1,\, a_2,\, a_3,\, a_1+a_2,\, 0,\, a_1+a_2,\, a_3,\, a_2,\, a_1,\, a_0)$
12	4	$(a_0,\, a_1,\, a_2,\, a_3,\, a_1+a_2+a_3,\, 0,\, 0,\, a_1+a_2+a_3,\, a_3,\, a_2,\, a_1,\, a_0)$
13	4	$(a_0,\, a_1,\, a_2,\, a_3,\, a_1+a_2+a_3,\, a_3,\, 0,\, a_3,\, a_1+a_2+a_3,\, a_3,\, a_2,\, a_1,\, a_0)$
14	5	$(a_0,\, a_1,\, a_2,\, a_3,\, a_4,\, a_1+a_2+a_4,\, a_3,\, a_3,\, a_1+a_2+a_4,\, a_4,\, a_3,\, a_2,\, a_1,\, a_0)$

Table 4: Left rows of Pascal triangles of size k with dihedral symmetry

Proof. Consider three entries $u = \mathbf{x}(r-1, c-1)$, $v = \mathbf{x}(r-1, c)$ and $w = \mathbf{x}(r, c)$. We have $w = u + v$. If $S(\mathbf{x})$ has rotational symmetry, then the rotation of 120 degrees produces the relation $v = w+u$. Thus, we have $w = u+v = u+(w+u) = w+u+u$. Hence $u + u = 0$. Analogously, $v + v = w + w = 0$. We conclude that each entry in $S(\mathbf{x})$ has order two. □

By Proposition 7, we can assume that G is a group such that each element has order two. Therefore, G can be given a structure of $\mathbb{F}_2$-vector space and, as an additive group, G is a direct sum of copies of $\mathbb{F}_2$. Thus, the condition of a Steinhaus triangle $S(\mathbf{x})$ with $\mathbf{x} \in G^n$ being rotationally symmetric is equivalent to the condition that each component is a rotationally symmetric Steinhaus triangle on $\mathbb{F}_2$. For instance, in order to obtain all rotationally symmetric Steinhaus triangles of size 6 on $\mathbb{F}_2 \times \mathbb{F}_2$, we must take row 6 in Table 1. Then, the first rows are obtained by giving values to $(x_2,\, x_3,\, y_2,\, y_3) \in \mathbf{F}_2^4$ in

$$((x_2 + x_3,\, y_2 + y_3),\, (x_1,\, y_2),\, (x_2,\, y_2),\, (x_3,\, y_3),\, (x_3,\, y_3),\, (x_2 + x_3,\, y_2 + y_3)).$$

Figure 12 shows the 16 Steinhaus triangles rotationally symmetric of size 6 on $\mathbb{F}_2 \times \mathbb{F}_2$. The elements in $\mathbb{F}_2 \times \mathbb{F}_2$ are represented as follows

$(0, 0)$ $(0, 1)$ $(1, 0)$ $(1, 1)$

Note that the four central triangles form a basis of the space of rotationally symmetric Steinhaus triangles of size 6 on $\mathbb{F}_2 \times \mathbb{F}_2$.

The generalization to Steinhaus triangles to abelian groups can be done by using

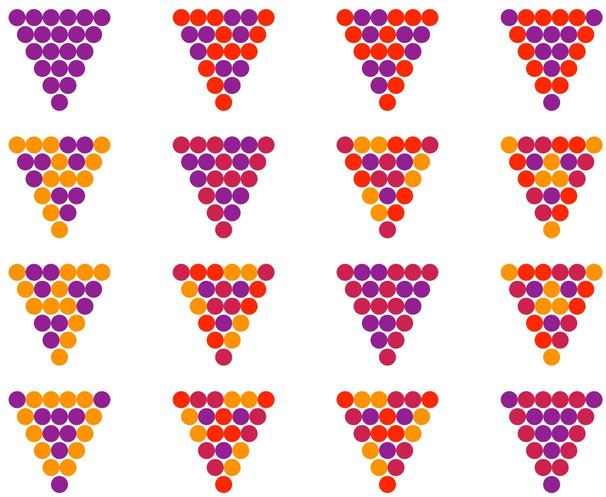

Figure 12: The 16 Steinhaus triangles rotationally symmetric of size 6 on $\mathbb{F}_2 \times \mathbb{F}_2$

alternative definitions of $\partial\mathbf{x}$. For instance, we can define

$$\partial\mathbf{x} = (x_1 - x_0, \ldots, x_{n-1} - x_{n-2}) \quad \text{or} \quad \partial\mathbf{x} = (x_0 - x_1, \ldots, x_{n-2} - x_{n-1}),$$

and then define $\partial^i\mathbf{x}$, Steinhaus triangles and rotationally symmetric Steinhaus triangles as before. Nevertheless, the above arguments can be applied in a similar way to obtain that all elements in a rotationally symmetric Steinhaus triangle have order two, and conclude that a rotationally symmetric Steinhaus triangle on an arbitrary abelian group consists in a direct sum of rotationally symmetric Steinhaus triangles on $\mathbb{F}_2$. The same is true about Steinhaus triangles with dihedral symmetry.

So far, in this section we have considered only Steinhaus triangles, but it is clear that the same considerations can be applied to Pascal triangles rotationally and dihedrally symmetric.

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