NEW SEQUENCES THAT CONVERGE TO A GENERALIZATION OF EULER'S CONSTANT

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Abstract

The purpose of the paper is to give some sequences that converge quickly to a generalization of Euler's constant, i.e., the limit of the sequence

$$\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right)_{n\in\mathbb{N}},$$

where $a\in(0,+\infty)$.

1. Introduction

Euler's constant, being one of the most important constants in mathematics, was investigated by many mathematicians. Usually denoted by γ, this constant is the limit of the sequence $(D_n)_{n\in\mathbb{N}}$ defined by $D_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n$, for each $n\in\mathbb{N}$. It is well-known that $\lim\limits_{n\to\infty} n(D_n-\gamma)=\frac{1}{2}$ (see [1], [2], [3], [5, pp. 73–75], [7], [13, Problem 18, pp. 38, 197], [14], [21], [23], [24], [25], [26]).

In order to increase the slow rate of convergence of the sequence $(D_n)_{n\in\mathbb{N}}$ to γ, D. W. DeTemple considered in [4] the sequence $(R_n)_{n\in\mathbb{N}}$ defined by $R_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln\left(n+\frac{1}{2}\right)$, for each $n\in\mathbb{N}$, and he proved that $\frac{1}{24(n+1)^2}<R_n-\gamma<\frac{1}{24n^2}$, for each $n\in\mathbb{N}$.

L. Tóth used in [22] the sequence $(T_n)_{n\in\mathbb{N}}$ defined by $T_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln\left(n+\frac{1}{2}+\frac{1}{24n}\right)$, for each $n\in\mathbb{N}$, and T. Negoi proved in [12] that $\frac{1}{48(n+1)^3}<\gamma-T_n<\frac{1}{48n^3}$, for each $n\in\mathbb{N}$.

Let $a\in(0,+\infty)$. We consider the sequence $(y_n(a))_{n\in\mathbb{N}}$ defined by

$$y_n(a)=\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a},$$

for each $n\in\mathbb{N}$. The sequence $(y_n(a))_{n\in\mathbb{N}}$ is convergent (see, for example, [6, p. 453]; see also [15], [16], [17], [18], [19], [20] and some of the references therein) and its limit, denoted by $\gamma(a)$, is a generalization of Euler's constant. Clearly, $\gamma(1) = \gamma$. Numerous results regarding the generalization of Euler's constant $\gamma(a)$ we have obtained in [15], [16], [17], [18], [19] and [20].

We mention the following representation of $\gamma(a)$ ([19, Theorem 2.2.4, p. 78]):

$$\gamma(a) = y_n - \frac{1}{2(a+n-1)} + \sum_{k=1}^{m} \frac{B_{2k}}{2k(a+n-1)^{2k}} - (2m+1)! \int_n^\infty \frac{P_{2m+1}(x)}{(a+x-1)^{2m+2}}\,\mathrm{d}x,$$

for each $n \in \mathbb{N}$, any $m \in \mathbb{N}$, where B_{2k} is the Bernoulli number of index $2k$ and $P_{2m+1}(x) = (-1)^{m-1} \sum_{k=1}^{\infty} \frac{2\sin(2k\pi x)}{(2k\pi)^{2m+1}}$, obtained by applying the Euler-Maclaurin summation formula ([6, p. 524], [5, p. 86]). If we take $a = 1$ in the above-mentioned representation, then we obtain a result presented, for example, in [6, pp. 527, 528], [5, pp. 88, 89].

Recent results regarding Euler's constant have been obtained by C. Mortici in [9], [10], [11].

Also, we remind the following lemma (C. Mortici [8, Lemma]), which is a consequence of the the Stolz-Cesaro Theorem, the case $\frac{0}{0}$.

Lemma 1. *Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of real numbers and $x^* = \lim_{n\to\infty} x_n$. We suppose that there exists $\alpha \in \mathbb{R}$, $\alpha > 1$, such that*

$$\lim_{n\to\infty} n^\alpha (x_n - x_{n+1}) = l \in \overline{\mathbb{R}}.$$

Then there exists the limit

$$\lim_{n\to\infty} n^{\alpha-1}(x_n - x^*) = \frac{l}{\alpha - 1}.$$

In Section 2 we present classes of sequences with the argument of the logarithmic term modified and that converge quickly to $\gamma(a)$.

2. Sequences That Converge to $\gamma(a)$

Theorem 2. *Let $a \in (0, +\infty)$. We specify that $\gamma(a)$ is the limit of the sequence $(y_n(a))_{n\in\mathbb{N}}$ from Introduction.*

(i) We consider the sequence $(\alpha_{n,2}(a))_{n\in\mathbb{N}}$ defined by

$$\begin{aligned}\alpha_{n,2}(a) = {} & \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \frac{1}{2(a+n-1)} + \frac{1}{12(a+n-1)^2} \\ & - \ln\left(\frac{a+n-1}{a} + \frac{1}{120a(a+n-1)^3}\right),\end{aligned}$$

for each $n \in \mathbb{N}$. *Then*

$$\lim_{n\to\infty} n^6(\gamma(a) - \alpha_{n,2}(a)) = \frac{1}{252}.$$

(*ii*) *We consider the sequence* $(\beta_{n,2}(a))_{n\in\mathbb{N}}$ *defined by*

$$\beta_{n,2}(a) = \alpha_{n,2}(a) + \frac{1}{252(a+n-1)^6},$$

for each $n \in \mathbb{N}$. *Then*

$$\lim_{n\to\infty} n^8(\beta_{n,2}(a) - \gamma(a)) = \frac{121}{28800}.$$

Proof. (*i*) We have

$$\begin{aligned}
&\alpha_{n+1,2}(a) - \alpha_{n,2}(a)\\
&= \frac{1}{2(a+n)} + \frac{1}{2(a+n-1)} + \frac{1}{12(a+n)^2} - \frac{1}{12(a+n-1)^2}\\
&\quad - \ln\left(a+n+\frac{1}{120(a+n)^3}\right) + \ln\left(a+n-1+\frac{1}{120(a+n-1)^3}\right)\\
&= \frac{1}{2(a+n)} + \frac{1}{2(a+n)\left(1-\frac{1}{a+n}\right)} + \frac{1}{12(a+n)^2} - \frac{1}{12(a+n)^2\left(1-\frac{1}{a+n}\right)^2}\\
&\quad - \ln\left(1+\frac{1}{120(a+n)^4}\right) + \ln\left(1-\frac{1}{a+n}+\frac{1}{120(a+n)^4\left(1-\frac{1}{a+n}\right)^3}\right),
\end{aligned}$$

for each $n \in \mathbb{N}$. Set $\varepsilon_n := \frac{1}{a+n}$, for each $n \in \mathbb{N}$. Since $\varepsilon_n \in (-1,1)$, $\frac{1}{120}\,\varepsilon_n^4 \in (-1,1]$ and $-\varepsilon_n + \frac{1}{120}\cdot\frac{\varepsilon_n^4}{(1-\varepsilon_n)^3} \in (-1,1]$, for each $n \in \mathbb{N}\setminus\{1\}$, using the series expansion ([6, pp. 171–179, p. 209]) we obtain

$$\begin{aligned}
&\alpha_{n+1,2}(a) - \alpha_{n,2}(a)\\
&= \frac{1}{2}\,\varepsilon_n + \frac{1}{2}\cdot\frac{\varepsilon_n}{1-\varepsilon_n} + \frac{1}{12}\,\varepsilon_n^2 - \frac{1}{12}\cdot\frac{\varepsilon_n^2}{(1-\varepsilon_n)^2}\\
&\quad - \ln\left(1+\frac{1}{120}\,\varepsilon_n^4\right) + \ln\left(1-\varepsilon_n+\frac{1}{120}\cdot\frac{\varepsilon_n^4}{(1-\varepsilon_n)^3}\right)\\
&= \frac{1}{42}\,\varepsilon_n^7 + \frac{1}{12}\,\varepsilon_n^8 + \frac{679}{3600}\,\varepsilon_n^9 + \frac{279}{800}\,\varepsilon_n^{10} + O(\varepsilon_n^{11}),
\end{aligned}$$

for each $n \in \mathbb{N}\setminus\{1\}$. It follows that

$$\lim_{n\to\infty} n^7(\alpha_{n+1,2}(a) - \alpha_{n,2}(a)) = \frac{1}{42}.$$

Now, according to Lemma 1, we get

$$\lim_{n\to\infty} n^6(\gamma(a) - \alpha_{n,2}(a)) = \frac{1}{252}.$$

(ii) We are able to write that

$$\begin{aligned}
&\beta_{n,2}(a) - \beta_{n+1,2}(a)\\
&= \alpha_{n,2}(a) - \alpha_{n+1,2}(a) + \frac{1}{252(a+n-1)^6} - \frac{1}{252(a+n)^6}\\
&= \alpha_{n,2}(a) - \alpha_{n+1,2}(a) + \frac{1}{252(a+n)^6\left(1-\frac{1}{a+n}\right)^6} - \frac{1}{252(a+n)^6}\\
&= \alpha_{n,2}(a) - \alpha_{n+1,2}(a) + \frac{1}{252}\cdot\frac{\varepsilon_n^6}{(1-\varepsilon_n)^6} - \frac{1}{252}\,\varepsilon_n^6\\
&= \frac{121}{3600}\,\varepsilon_n^9 + \frac{121}{800}\,\varepsilon_n^{10} + O(\varepsilon_n^{11}),
\end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. It follows that

$$\lim_{n\to\infty} n^9(\beta_{n,2}(a) - \beta_{n+1,2}(a)) = \frac{121}{3600}.$$

Now, according to Lemma 1, we get

$$\lim_{n\to\infty} n^8(\beta_{n,2}(a) - \gamma(a)) = \frac{121}{28800}.$$

□

In the same manner as in the proof of Theorem 2, considering the sequence in each of the following parts, we get the indicated limit:

$\delta_{n,2}(a) = \beta_{n,2}(a) - \frac{121}{28800(a+n-1)^8}$, for each $n \in \mathbb{N}$,

$$\lim_{n\to\infty} n^{10}(\gamma(a) - \delta_{n,2}(a)) = \tfrac{1}{132};$$

$\eta_{n,2}(a) = \delta_{n,2}(a) + \frac{1}{132(a+n-1)^{10}}$, for each $n \in \mathbb{N}$,

$$\lim_{n\to\infty} n^{12}(\eta_{n,2}(a) - \gamma(a)) = \tfrac{9950309}{471744000};$$

$\theta_{n,2}(a) = \eta_{n,2}(a) - \frac{9950309}{471744000(a+n-1)^{12}}$, for each $n \in \mathbb{N}$,

$$\lim_{n\to\infty} n^{14}(\gamma(a) - \theta_{n,2}(a)) = \tfrac{1}{12};$$

$\lambda_{n,2}(a) = \theta_{n,2}(a) + \frac{1}{12(a+n-1)^{14}}$, for each $n \in \mathbb{N}$,

$$\lim_{n\to\infty} n^{16}(\lambda_{n,2}(a) - \gamma(a)) = \tfrac{6250176017}{14100480000};$$

$\mu_{n,2}(a) = \lambda_{n,2}(a) - \frac{6250176017}{14100480000(a+n-1)^{16}}$, for each $n \in \mathbb{N}$,

$$\lim_{n\to\infty} n^{18}(\gamma(a) - \mu_{n,2}(a)) = \tfrac{43867}{14364}.$$

We point out the pattern in forming the sequences from Theorem 2 and those mentioned above. For example, the general term of the sequence $(\mu_{n,2}(a))_{n\in\mathbb{N}}$ can be written in the form

$$\begin{aligned}\mu_{n,2}(a) \;=\; & \frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\frac{1}{2(a+n-1)}+\frac{B_2}{2}\cdot\frac{1}{(a+n-1)^2}\\ & -\ln\left(\frac{a+n-1}{a}-\frac{B_4}{4}\cdot\frac{1}{a(a+n-1)^3}\right)+\sum_{k=3}^{8}\frac{c_{k,2}}{(a+n-1)^{2k}},\end{aligned}$$

with

$$c_{k,2}=\begin{cases}\dfrac{B_{2k}}{2k}, & \text{if } k=2p+1, p\in\mathbb{N},\\ \dfrac{B_{2k}}{2k}-\dfrac{2}{k}\left(\dfrac{B_4}{4}\right)^{\frac{k}{2}}, & \text{if } k=2p+2, p\in\mathbb{N},\end{cases}$$

where B_{2k} is the Bernoulli number of index $2k$. Related to this remark, see also [16, Remark 3.4], [19, p. 71, Remark 2.1.3; pp. 100, 101, Remark 3.1.6].

For Euler's constant $\gamma = 0.5772156649\ldots$ we obtain, for example:

$\alpha_{2,2}(1) = 0.5771654550\ldots$; $\alpha_{3,2}(1) = 0.5772107618\ldots$;

$\beta_{2,2}(1) = 0.5772274589\ldots$; $\beta_{3,2}(1) = 0.5772162053\ldots$;

$\delta_{2,2}(1) = 0.5772110473\ldots$; $\delta_{3,2}(1) = 0.5772155649\ldots$;

$\eta_{2,2}(1) = 0.5772184455\ldots$; $\eta_{3,2}(1) = 0.5772156932\ldots$;

$\theta_{2,2}(1) = 0.5772132959\ldots$; $\theta_{3,2}(1) = 0.5772156535\ldots$;

$\lambda_{2,2}(1) = 0.5772183822\ldots$; $\lambda_{3,2}(1) = 0.5772156709\ldots$;

$\mu_{2,2}(1) = 0.5772116186\ldots$; $\mu_{3,2}(1) = 0.5772156606\ldots$.

As can be seen, $\mu_{3,2}(1)$ is accurate to eight decimal places in approximating γ.

Theorem 3. *Let $a \in (0,+\infty)$. We specify that $\gamma(a)$ is the limit of the sequence $(y_n(a))_{n\in\mathbb{N}}$ from Introduction.*

(*i*) *We consider the sequence $(\alpha_{n,3}(a))_{n\geq 2}$ defined by*

$$\begin{aligned}\alpha_{n,3}(a) \;=\; & \frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\frac{1}{2(a+n-1)}\\ & +\frac{1}{12(a+n-1)^2}-\frac{1}{120(a+n-1)^4}\\ & -\ln\left(\frac{a+n-1}{a}-\frac{1}{252a(a+n-1)^5}\right),\end{aligned}$$

for each $n \in \mathbb{N}\setminus\{1\}$. Then

$$\lim_{n\to\infty} n^8(\alpha_{n,3}(a)-\gamma(a)) = \frac{1}{240}.$$

(*ii*) *We consider the sequence* $(\beta_{n,3}(a))_{n\geq 2}$ *defined by*

$$\beta_{n,3}(a) = \alpha_{n,3}(a) - \frac{1}{240(a+n-1)^8},$$

for each $n \in \mathbb{N} \setminus \{1\}$. *Then*

$$\lim_{n\to\infty} n^{10}(\gamma(a) - \beta_{n,3}(a)) = \frac{1}{132}.$$

(*iii*) *We consider the sequence* $(\delta_{n,3}(a))_{n\geq 2}$ *defined by*

$$\delta_{n,3}(a) = \beta_{n,3}(a) + \frac{1}{132(a+n-1)^{10}},$$

for each $n \in \mathbb{N} \setminus \{1\}$. *Then*

$$\lim_{n\to\infty} n^{12}(\delta_{n,3}(a) - \gamma(a)) = \frac{174197}{8255520}.$$

Proof. (*i*) We have

$$\begin{aligned}
&\alpha_{n,3}(a) - \alpha_{n+1,3}(a)\\
&= -\frac{1}{2(a+n-1)} - \frac{1}{2(a+n)} + \frac{1}{12(a+n-1)^2} - \frac{1}{12(a+n)^2}\\
&\quad -\frac{1}{120(a+n-1)^4} + \frac{1}{120(a+n)^4}\\
&\quad -\ln\left(a+n-1-\frac{1}{252(a+n-1)^5}\right) + \ln\left(a+n-\frac{1}{252(a+n)^5}\right)\\
&= -\frac{1}{2(a+n)\left(1-\frac{1}{a+n}\right)} - \frac{1}{2(a+n)} + \frac{1}{12(a+n)^2\left(1-\frac{1}{a+n}\right)^2} - \frac{1}{12(a+n)^2}\\
&\quad -\frac{1}{120(a+n)^4\left(1-\frac{1}{a+n}\right)^4} + \frac{1}{120(a+n)^4}\\
&\quad -\ln\left(1-\frac{1}{a+n}-\frac{1}{252(a+n)^6\left(1-\frac{1}{a+n}\right)^5}\right) + \ln\left(1-\frac{1}{252(a+n)^6}\right),
\end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. Set $\varepsilon_n := \frac{1}{a+n}$, for each $n \in \mathbb{N} \setminus \{1\}$. Since $\varepsilon_n \in (-1, 1)$, $-\varepsilon_n - \frac{1}{252} \cdot \frac{\varepsilon_n^6}{(1-\varepsilon_n)^5} \in (-1, 1]$ and $-\frac{1}{252}\,\varepsilon_n^6 \in (-1, 1]$, for each $n \in \mathbb{N} \setminus \{1\}$, using the series expansion ([6, pp. 171–179, p. 209]) we obtain

$$\begin{aligned}
&\alpha_{n,3}(a) - \alpha_{n+1,3}(a) \\
&= -\frac{1}{2} \cdot \frac{\varepsilon_n}{1-\varepsilon_n} - \frac{1}{2}\,\varepsilon_n + \frac{1}{12} \cdot \frac{\varepsilon_n^2}{(1-\varepsilon_n)^2} - \frac{1}{12}\,\varepsilon_n^2 - \frac{1}{120} \cdot \frac{\varepsilon_n^4}{(1-\varepsilon_n)^4} + \frac{1}{120}\,\varepsilon_n^4 \\
&\quad - \ln\left(1 - \varepsilon_n - \frac{1}{252} \cdot \frac{\varepsilon_n^6}{(1-\varepsilon_n)^5}\right) + \ln\left(1 - \frac{1}{252}\,\varepsilon_n^6\right) \\
&= \frac{1}{30}\,\varepsilon_n^9 + \frac{3}{20}\,\varepsilon_n^{10} + \frac{14}{33}\,\varepsilon_n^{11} + \frac{23}{24}\,\varepsilon_n^{12} + \frac{259573}{137592}\,\varepsilon_n^{13} + \frac{357653}{105840}\,\varepsilon_n^{14} + O(\varepsilon_n^{15}),
\end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. It follows that

$$\lim_{n\to\infty} n^9(\alpha_{n,3}(a) - \alpha_{n+1,3}(a)) = \frac{1}{30}.$$

Now, according to Lemma 1, we get

$$\lim_{n\to\infty} n^8(\alpha_{n,3}(a) - \gamma(a)) = \frac{1}{240}.$$

(ii) We are able to write that

$$\begin{aligned}
&\beta_{n+1,3}(a) - \beta_{n,3}(a) \\
&= \alpha_{n+1,3}(a) - \alpha_{n,3}(a) - \frac{1}{240(a+n)^8} + \frac{1}{240(a+n-1)^8} \\
&= \alpha_{n+1,3}(a) - \alpha_{n,3}(a) - \frac{1}{240(a+n)^8} + \frac{1}{240(a+n)^8\left(1 - \frac{1}{a+n}\right)^8} \\
&= \alpha_{n+1,3}(a) - \alpha_{n,3}(a) - \frac{1}{240}\,\varepsilon_n^8 + \frac{1}{240} \cdot \frac{\varepsilon_n^8}{(1-\varepsilon_n)^8} \\
&= \frac{5}{66}\,\varepsilon_n^{11} + \frac{5}{12}\,\varepsilon_n^{12} + \frac{972403}{687960}\,\varepsilon_n^{13} + \frac{399103}{105840}\,\varepsilon_n^{14} + O(\varepsilon_n^{15}),
\end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. It follows that

$$\lim_{n\to\infty} n^{11}(\beta_{n+1,3}(a) - \beta_{n,3}(a)) = \frac{5}{66}.$$

Now, according to Lemma 1, we get

$$\lim_{n\to\infty} n^{10}(\gamma(a) - \beta_{n,3}(a)) = \frac{1}{132}.$$

(*iii*) We have

$$\begin{aligned}
&\delta_{n,3}(a) - \delta_{n+1,3}(a) \\
&= \beta_{n,3}(a) - \beta_{n+1,3}(a) + \frac{1}{132(a+n-1)^{10}} - \frac{1}{132(a+n)^{10}} \\
&= \beta_{n,3}(a) - \beta_{n+1,3}(a) + \frac{1}{132(a+n)^{10}\left(1-\frac{1}{a+n}\right)^{10}} - \frac{1}{132(a+n)^{10}} \\
&= \beta_{n,3}(a) - \beta_{n+1,3}(a) + \frac{1}{132}\cdot\frac{\varepsilon_n^{10}}{(1-\varepsilon_n)^{10}} - \frac{1}{132}\,\varepsilon_n^{10} \\
&= \frac{174197}{687960}\,\varepsilon_n^{13} + \frac{174197}{105840}\,\varepsilon_n^{14} + O(\varepsilon_n^{15}),
\end{aligned}$$

for each $n \in \mathbb{N} \setminus \{1\}$. It follows that

$$\lim_{n\to\infty} n^{13}(\delta_{n,3}(a) - \delta_{n+1,3}(a)) = \frac{174197}{687960}.$$

Now, according to Lemma 1, we get

$$\lim_{n\to\infty} n^{12}(\delta_{n,3}(a) - \gamma(a)) = \frac{174197}{8255520}.$$

□

In the same manner as in the proof of Theorem 3, considering the sequence in each of the following parts, we get the indicated limit:

$\eta_{n,3}(a) = \delta_{n,3}(a) - \frac{174197}{8255520(a+n-1)^{12}}$, for each $n \in \mathbb{N} \setminus \{1\}$,
$\lim_{n\to\infty} n^{14}(\gamma(a) - \eta_{n,3}(a)) = \frac{1}{12}$;

$\theta_{n,3}(a) = \eta_{n,3}(a) + \frac{1}{12(a+n-1)^{14}}$, for each $n \in \mathbb{N} \setminus \{1\}$,
$\lim_{n\to\infty} n^{16}(\theta_{n,3}(a) - \gamma(a)) = \frac{3617}{8160}$;

$\lambda_{n,3}(a) = \theta_{n,3}(a) - \frac{3617}{8160(a+n-1)^{16}}$, for each $n \in \mathbb{N} \setminus \{1\}$,
$\lim_{n\to\infty} n^{18}(\gamma(a) - \lambda_{n,3}(a)) = \frac{2785729949}{912171456}$;

$\mu_{n,3}(a) = \lambda_{n,3}(a) + \frac{2785729949}{912171456(a+n-1)^{18}}$, for each $n \in \mathbb{N} \setminus \{1\}$,
$\lim_{n\to\infty} n^{20}(\mu_{n,3}(a) - \gamma(a)) = \frac{174611}{6600}$.

We point out the pattern in forming the sequences from Theorem 3 and those mentioned above. For example, the general term of the sequence $(\mu_{n,3}(a))_{n\geq 2}$ can be written in the form

$$\begin{aligned}\mu_{n,3}(a) &= \frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\frac{1}{2(a+n-1)}\\ &+\frac{B_2}{2}\cdot\frac{1}{(a+n-1)^2}+\frac{B_4}{4}\cdot\frac{1}{(a+n-1)^4}\\ &-\ln\left(\frac{a+n-1}{a}-\frac{B_6}{6}\cdot\frac{1}{a(a+n-1)^5}\right)+\sum_{k=4}^{9}\frac{c_{k,3}}{(a+n-1)^{2k}},\end{aligned}$$

with

$$c_{k,3}=\begin{cases}\dfrac{B_{2k}}{2k}, & \text{if } k=3p+1, p\in\mathbb{N},\\ \dfrac{B_{2k}}{2k}, & \text{if } k=3p+2, p\in\mathbb{N},\\ \dfrac{B_{2k}}{2k}-\dfrac{3}{k}\left(\dfrac{B_6}{6}\right)^{\frac{k}{3}}, & \text{if } k=3p+3, p\in\mathbb{N},\end{cases}$$

where B_{2k} is the Bernoulli number of index $2k$. Related to this remark, see also [16, Remark 3.4], [19, p. 71, Remark 2.1.3; pp. 100, 101, Remark 3.1.6].

For Euler's constant $\gamma = 0.5772156649\ldots$ we obtain, for example:

$$\begin{aligned}&\alpha_{2,3}(1)=0.5772273253\ldots; && \alpha_{3,3}(1)=0.5772162000\ldots;\\ &\beta_{2,3}(1)=0.5772110492\ldots; && \beta_{3,3}(1)=0.5772155649\ldots;\\ &\delta_{2,3}(1)=0.5772184474\ldots; && \delta_{3,3}(1)=0.5772156932\ldots;\\ &\eta_{2,3}(1)=0.5772132959\ldots; && \eta_{3,3}(1)=0.5772156535\ldots;\\ &\theta_{2,3}(1)=0.5772183822\ldots; && \theta_{3,3}(1)=0.5772156709\ldots;\\ &\lambda_{2,3}(1)=0.5772116186\ldots; && \lambda_{3,3}(1)=0.5772156606\ldots;\\ &\mu_{2,3}(1)=0.5772232685\ldots; && \mu_{3,3}(1)=0.5772156685\ldots.\end{aligned}$$

As can be seen, $\lambda_{3,3}(1)$ and $\mu_{3,3}(1)$ are accurate to eight decimal places in approximating γ.

Concluding, the following remark can be made. Let $a \in (0,+\infty)$ and $q \in \mathbb{N} \setminus \{1\}$. Let $n_0 = \min\left\{ n\in\mathbb{N} \middle| a+n-1-\frac{B_{2q}}{2q}\cdot\frac{1}{(a+n-1)^{2q-1}}>0\right\}$. We consider the sequence $(\alpha_{n,q}(a))_{n\geq n_0}$ defined by

$$\begin{aligned}\alpha_{n,q}(a) &= \frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\frac{1}{2(a+n-1)}+\sum_{k=1}^{q-1}\frac{B_{2k}}{2k}\cdot\frac{1}{(a+n-1)^{2k}}\\ &-\ln\left(\frac{a+n-1}{a}-\frac{B_{2q}}{2q}\cdot\frac{1}{a(a+n-1)^{2q-1}}\right),\end{aligned}$$

for each $n \in \mathbb{N}$, $n \geq n_0$. Clearly, $\lim_{n\to\infty} \alpha_{n,q}(a) = \gamma(a)$. Based on the sequence $(\alpha_{n,q}(a))_{n\geq n_0}$, a class of sequences convergent to $\gamma(a)$ can be considered, namely

$$\{(\alpha_{n,q,r}(a))_{n\geq n_0} | r \in \mathbb{N}, r \geq q+1\},$$

where

$$\alpha_{n,q,r}(a) = \alpha_{n,q}(a) + \sum_{k=q+1}^{r} \frac{c_{k,q}}{(a+n-1)^{2k}},$$

for each $n \in \mathbb{N}$, $n \geq n_0$, with

$$c_{k,q} = \begin{cases} \dfrac{B_{2k}}{2k}, & \text{if } k \in \{qp+1, qp+2, \ldots, qp+q-1\}, p \in \mathbb{N}, \\ \dfrac{B_{2k}}{2k} - \dfrac{q}{k}\left(\dfrac{B_{2q}}{2q}\right)^{\frac{k}{q}}, & \text{if } k = qp+q, p \in \mathbb{N}. \end{cases}$$

In this section we have obtained that the sequence $(\alpha_{n,2}(a))_{n\in\mathbb{N}}$ converges to $\gamma(a)$ with order 6 and that the sequence $(\alpha_{n,3}(a))_{n\geq 2}$ converges to $\gamma(a)$ with order 8. We have also obtained that the sequence $(\alpha_{n,2,r}(a))_{n\in\mathbb{N}}$ converges to $\gamma(a)$ with order $2(r+1)$, for $r \in \{3,4,5,6,7,8\}$, and that the sequence $(\alpha_{n,3,r}(a))_{n\geq 2}$ converges to $\gamma(a)$ with order $2(r+1)$, for $r \in \{4,5,6,7,8,9\}$.

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