ON BASES WITH A T-ORDER

Quan-Hui Yang[1]
School of Mathematical Sciences, Nanjing Normal University, Nanjing, China
yangquanhui01@163.com

Feng-Juan Chen
School of Mathematical Sciences, Nanjing Normal University, Nanjing, China and
Department of Mathematics, Suzhou University, Suzhou, China
cfjsz@126.com

Abstract

For any set $T = \{t_1, t_2, \ldots, t_n\} \subseteq \mathbb{N}$, a basis A is said to have a T-order s if every sufficiently large integer is the sum of $s - t_1$ or $s - t_2$ or $\ldots$ or $s - t_n$ elements taken from A (allowing repetitions), where s is the least integer with this property. We write $\mathrm{ord}^{(T)}(A) = s$. In this paper, we characterize those bases A which have a T-order.

1. Introduction

A set A of nonnegative integers is said to be an asymptotic basis of order r if every sufficiently large integer can be expressed as a sum of at most r elements taken from A (allowing repetitions) and r is the least integer with this property. We write $\mathrm{ord}(A) = r$. A basis A is said to have an exact order r' if every sufficiently large integer is the sum of exactly r' elements taken from A (again, allowing repetitions) and r' is the least integer with this property. In this case we write $\mathrm{ord}^*(A) = r'$. In [6], Erdős and Graham characterized those bases A which have an exact order. They proved the following result: a basis $A = \{a_1, a_2, \ldots\}$ has an exact order if and only if $\gcd\{a_{k+1} - a_k : k = 1, 2, \ldots\} = 1$. For related research, one may refer to [1-5].

In this note, we introduce the concept of T-order as follows and generalize the result by Erdős and Graham[6].

Definition 1. For any set $T = \{t_1, t_2, \ldots, t_n\} \subseteq \mathbb{N}$, a basis A is said to have a T-order if there exists an integer s such that every sufficiently large integer is the sum of $s - t_1$ or $s - t_2$ or $\ldots$ or $s - t_n$ integers taken from A (allowing repetitions). We indicate the least such s by $\mathrm{ord}^{(T)}(A) = s$.

[1] Corresponding author

Definition 2. For any set $T = \{t_1, t_2, \ldots, t_n\} \subseteq \mathbb{N}$, a basis A is said to have an exact T-order if A has a T-order, and A does not have a T'-order for any set $T' \subsetneq T$.

By the definition, it's obvious that $\mathrm{ord}^*(A) = \mathrm{ord}^{(T)}(A)$ when $T = \{0\}$. It is easy to find examples of bases A which do not have a T-order. For example, if $A = \{x > 0 : x \equiv 1 \pmod 3\}$ and $T = \{0, 1\}$, then A doesn't have a T-order. By the definition, we know that a basis A has a T-order for any set T when it has an exact order. Meanwhile, if $0 \in T$, then $\mathrm{ord}(A) \leqslant \mathrm{ord}^{(T)}(A) \leqslant \mathrm{ord}^*(A)$. It is clear that if $0 \in T$, $0 \in A$ and $\mathrm{ord}(A) = r$, then $\mathrm{ord}^{(T)}(A) = \mathrm{ord}^*(A) = r$. However, it is not difficult to construct examples of bases A such that

$$\mathrm{ord}^{(T)}(A) > \mathrm{ord}(A) \text{ or } \mathrm{ord}^*(A) > \mathrm{ord}^{(T)}(A).$$

For example, if

$$T = \{0, 1\}$$

and

$$A_1 = \bigcup_{k=0}^{\infty} \{x : 3^{2k} + 1 \leqslant x \leqslant 3^{2k+1}\},$$

then

$$\mathrm{ord}(A_1) = 3 \quad \text{and} \quad \mathrm{ord}^{(T)}(A_1) = 4.$$

If

$$T = \{0, 1\} \quad \text{and} \quad A_2 = \{x > 0 : x \equiv 2 \pmod 6 \text{ or } x \equiv 3 \pmod 6\},$$

then

$$\mathrm{ord}^{(T)}(A_2) = 3 \quad \text{and} \quad \mathrm{ord}^*(A_2) = 5.$$

In this paper, we characterize those bases A which have a T-order.

2. Bases with a T-order

For $A = \{a_1, a_2, \ldots\}$, let $D(A) = \gcd\{a_{k+1} - a_k : k = 1, 2, \ldots\}$. It is easy to see that $D(A)$ does not depend on the order of A.

Lemma 3. If $A = \{a_1, a_2, \ldots\}$ is a basis, then $(a_k, D(A)) = 1$ for all positive integers k.

Proof. If there exists k_0 such that $(a_{k_0}, D(A)) = d > 1$, then $d \mid a_k$ for all k. Therefore any sum of elements taken from A is a multiple of d, which contradicts the condition that A is a basis. This completes the proof of Lemma 3. □

For $A = \{a_1, a_2, \ldots\}$ and a positive integer h, define hA as the h-fold sum set of A:

$$hA = \{a_{i_1} + a_{i_2} + \cdots + a_{i_h} : i_1 \leq \cdots \leq i_h\}.$$

Lemma 4. If $A = \{a_1, a_2, \ldots\}$ is a basis, then there exists a positive integer n such that $nA \cap (n + D(A))A \neq \emptyset$.

Proof. Since $D(A) = \gcd\{a_{k+1} - a_k : k = 1, 2, \ldots\}$, there is a positive integer t such that

$$\gcd\{a_{k+1} - a_k : 1 \leqslant k \leqslant t\} = D(A).$$

Thus, there exist integers $c_1, c_2, \ldots, c_t$ such that

$$\sum_{k=1}^{t} c_k(a_{k+1} - a_k) = D(A). \tag{1}$$

We define p_k and q_k by

$$p_k = \begin{cases} a_{k+1} & \text{if} \quad c_k \geqslant 0, \\ a_k & \text{if} \quad c_k < 0, \end{cases} \qquad q_k = \begin{cases} a_k & \text{if} \quad c_k \geqslant 0, \\ a_{k+1} & \text{if} \quad c_k < 0. \end{cases}$$

Then (1) can be rewritten as

$$\sum_{k=1}^{t} |c_k|(p_k - q_k) = D(A),$$

i.e.,

$$\sum_{k=1}^{t} |c_k| p_k = D(A) + \sum_{k=1}^{t} |c_k| q_k.$$

Let

$$K = \sum_{k=1}^{t} |c_k| p_k q_k.$$

Since

$$K = \sum_{k=1}^{t} \sum_{i=1}^{|c_k| p_k} q_k \in (\sum_{k=1}^{t} |c_k| p_k)A$$

and

$$K = \sum_{k=1}^{t} \sum_{j=1}^{|c_k| q_k} p_k \in (\sum_{k=1}^{t} |c_k| q_k)A,$$

we have $K \in nA \cap (n + D(A))A$, where $n = \sum_{k=1}^{t} |c_k| q_k$. This completes the proof of Lemma 4. $\square$

Theorem 5. For any set $T = \{t_1, t_2, \ldots, t_n\} \subseteq \mathbb{N}$, a basis $A = \{a_1, a_2, \ldots\}$ has a T-order if and only if $t_1, t_2, \ldots, t_n$ contains a complete system of incongruent residues modulo $D(A)$.

Proof. (Necessity). Suppose that $\mathrm{ord}^{(T)}(A) = s$. Since $D(A) = \gcd\{a_{k+1} - a_k : k = 1, 2, \ldots\}$, we have $a_{k+1} \equiv a_k \pmod{D(A)}$ for all k. Therefore, any sum of $s - t_i$ elements of A is congruent to $(s - t_i)a_1$ modulo $D(A)$ for $i = 1, 2, \cdots, n$. If $t_1, t_2, \ldots, t_n$ does not contain a complete system of incongruent residues modulo $D(A)$, then $(s-t_1)a_1, (s-t_2)a_1, \ldots, (s-t_n)a_1$ does not contain a complete system of incongruent residues modulo $D(A)$ either. It contradicts $\mathrm{ord}^{(T)}(A) = s$.

(Sufficiency). Suppose that $\mathrm{ord}(A) = r$. By Lemma 4, there exist two positive integers K and n such that

$$K \in nA \cap (n + D(A))A.$$

Then, for any integer $w \geqslant 1$ we have

$$wK \in \bigcap_{k=0}^{w} (wn + kD(A))A. \tag{2}$$

Let $s = (([r/D(A)]-1)n+[r/D(A)]\, D(A))+t_n$. Now we prove that every sufficiently large integer x can be represented as the sum of $s - t_1$ or $\cdots$ or $s - t_n$ elements taken from A. Let $x_1 = x - ([r/D(A)] - 1)K$.

Case 1: $D(A) \mid x_1$. By Lemma 3, we have $(a_k, D(A)) = 1$ and $a_k \equiv a_1 (\mathrm{mod}\ D(A))$ for any integer $k \geqslant 1$. Thus

$$x_1 \in \bigcup_{D(A)|i,\ i\leqslant r} iA.$$

Setting $w = [r/D(A)] - 1$ in (2), we obtain

$$x = x_1 + ([r/D(A)] - 1)K \in (([r/D(A)] - 1)n + [r/D(A)]\, D(A))A = (s - t_n)A.$$

Case 2: $D(A) \nmid x_1$. By Lemma 3, we have $(a_1, D(A)) = 1$. Since $t_1, t_2, \ldots, t_n$ contains a complete system of incongruent residues modulo $D(A)$, we have that $(t_n - t_n)a_1, (t_n - t_{n-1})a_1, \ldots, (t_n - t_1)a_1$ also contains a complete system of incongruent residues modulo $D(A)$. Thus, there exists an integer i such that $1 \leqslant i \leqslant n$ and

$$(t_n - t_i)a_1 \equiv x_1 \pmod{D(A)}.$$

By Case 1, we have

$$x_1 - (t_n - t_i)a_1 + ([r/D(A)] - 1)K \in (([r/D(A)] - 1)n + [r/D(A)]\, D(A))A.$$

Hence for any sufficiently large integer x, there exists an integer $i\,(1 \leqslant i \leqslant n)$ such that

$$x = x_1+([r/D(A)]-1)K \in (([r/D(A)]-1)n+[r/D(A)]\, D(A)+(t_n-t_i))A = (s-t_i)A.$$

This completes the proof of Theorem 5. $\square$

Remark. By the proof Theorem 5, we have

$$\mathrm{ord}^{(T)}A \leqslant (([r/D(A)] - 1)n + [r/D(A)]\, D(A)) + t_n.$$

Corollary 6. For any set $T = \{t_1, t_2, \ldots, t_n\} \subseteq \mathbb{N}$, a basis $A = \{a_1, a_2, \ldots\}$ has an exact T-order if and only if $D(A) = n$ and $t_1, t_2, \ldots, t_n$ is a complete system of incongruent residues modulo $D(A)$.

Proof. A has an exact T-order if and only if A has a T-order and A does not have a T'-order for any set $T' \subsetneq T$. By Theorem 5, T contains a complete system of incongruent residues modulo $D(A)$ and T' does not contain a complete system of incongruent residues modulo $D(A)$. Namely $D(A) = n$ and $t_1, t_2, \ldots, t_n$ is a complete system of incongruent residues modulo $D(A)$. □

Remark. Let $T = \{0\}$, by Corollary 6, we can get the result by Erdős and Graham.

Acknowledgments We sincerely thank our supervisor Professor Yong-Gao Chen for his valuable suggestions and useful discussions.

References

[1]H. Halberstam, K. F. Roth, *Sequences,* Clarendon Press, Oxford 1966.

[2]Janczak Miroslawa, Schoen Tomasz, *Dense minimal asymptotic bases of order two,* J. Number Theory 130(3)(2010), 580-585.

[3] Schmitt, Christoph, *Uniformly thin bases of order two,* Acta Arith. 124(1)(2006), 17-26.

[4]S. Chen, W. Z. Gu , *Exact order of subsets of asymptotic bases,* J. Number Theory 41(1)(1992), 15-21.

[5]X. D. Jia, *Exact order of subsets of asymptotic bases in additive number theory,* J. Number Theory 28(2)(1988), 205-218.

[6] P. Erdős, R. L. Graham, *On bases with an exact order,* Acta Arith. 37(1980), 201-207.