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NORM EUCLIDEAN QUATERNIONIC ORDERS

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Abstract

We determine the norm Euclidean orders in a positive definite quaternion algebra over $\mathbb{Q}$.

1. Introduction

Lagrange (1770) proved the four square theorem via Euler's four square identity and a descent argument. Hurwitz [4] gave a quaternionic proof using the order $\Lambda(2)$ with $\mathbb{Z}$-basis: $1, i, j, \frac{1}{2}(1+i+j+k)$. Here $i^2 = j^2 = -1$ and $ij = -ji = k$, the standard basis of the quaternions. The key property of $\Lambda(2)$ is that it is norm Euclidean, namely, given $\alpha, \beta \in \Lambda(2)$ with $\beta \neq 0$, there exist $q, r \in \Lambda(2)$ such that $\alpha = \beta q + r$ and $N(r) < N(\beta)$. Liouville (1856) showed there are exactly seven positive definite quaternion norm forms (that is, 2-fold Pfister forms) $x^2 + ay^2 + bz^2 + abw^2$, with a, b positive integers, that represent all positive integers. These are $(1, a, b, ab) = (1, 1, 1, 1), (1, 1, 2, 2), (1, 1, 3, 3), (1, 2, 2, 4), (1, 2, 3, 6), (1, 2, 4, 8)$ and $(1, 2, 5, 10)$. See volume III of Dickson [1] for more details.

Recently Deutsch constructed norm Euclidean orders to prove the universality of all but the last. Here we show there are, up to equivalence, exactly three norm Euclidean orders in a positive definite quaternion algebra over $\mathbb{Q}$. This includes one in $(\frac{-2,-5}{\mathbb{Q}})$.

2. Quaternionic Orders

Q will denote a positive definite quaternion algebra over $\mathbb{Q}$. Write $Q = (\frac{a,b}{\mathbb{Q}})$. Here $a, b < 0$ are rational and Q has a basis $e_1 = 1$, e_2, e_3, e_4 with $e_2^2 = a$, $e_3^2 = b$ and $e_2 e_3 = -e_3 e_2 = e_4$. For $\alpha = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4$ the conjugate is $\bar{\alpha} = x_1 e_1 - x_2 e_2 - x_3 e_3 - x_4 e_4$, the trace of α is $\mathrm{tr}(\alpha) = \alpha + \bar{\alpha}$ and the norm is

$N(\alpha) = \alpha\bar{\alpha} = x_1^2 + ax_2^2 + bx_3^2 + abx_4^2$. An *order* in Q is a finitely generated $\mathbb{Z}$-module $A \subset Q$ such that A is a ring and $\mathbb{Q} \otimes_{\mathbb{Z}} A = Q$.

We recall the basic properties of orders.

Lemma 1. *Let A be an order in Q.*

1. $A \cap \mathbb{Q} \subset \mathbb{Z}$.

2. *A is closed under conjugation.*

3. $N(A) \subset \mathbb{Z}$.

Proof. (1) If $p/q \in A$ for relatively prime integers p and q then all $p^n/q^n \in A$, contrary to A being finitely generated over $\mathbb{Z}$.

(2), (3) An element $\alpha \in A$ is root of $p(x) = x^2 - \mathrm{tr}(\alpha)x + N(\alpha) \in \mathbb{Q}[x]$. If $p(x)$ factors over $\mathbb{Q}$ then $\alpha \in A \cap \mathbb{Q} \subset \mathbb{Z}$ so that $\alpha = \bar{\alpha} \in A$ and $N(\alpha) \in \mathbb{Z}$. If $p(x)$ is irreducible over $\mathbb{Q}$ then, as α is integral over $\mathbb{Z}$, we have $\mathrm{tr}(\alpha)$, $N(\alpha) \in \mathbb{Z}$. And $\bar{\alpha} = \mathrm{tr}(\alpha) - \alpha \in A$. □

Let Q_1 and Q_2 be quaternion algebras over $\mathbb{Q}$ and let $A \subset Q_1, B \subset Q_2$ be orders. We call A and B *isomorphic* if they are isomorphic as rings. An isomorphism $\varphi : A \to B$ fixes $\mathbb{Z}$ so φ is a $\mathbb{Z}$-module map that preserves conjugation. In particular, $N(\varphi(\alpha)) = N(\alpha)$ so that if A is norm Euclidean then so is B. Further, if q_1 is a norm form for A and q_2 is a norm form for B then q_1 and q_2 are $\mathbb{Z}$-isometric.

Conversely, if q_1 and q_2 are $\mathbb{Z}$-isometric then the isometry extends to a $\mathbb{Q}$-isometry and so an isomorphism $Q_1 \to Q_2$. This restricts to an isomorphism $A \to B$. For details see [6] or [5].

The following is well-known but we have been unable to find the result in the literature. $\Lambda(n)$ denotes the maximal order of reduced discriminant n, see [3] for further information. The same paper gives the basic facts about the genus and class of an order that we need.

Proposition 2. *If A is a norm Euclidean order in Q then A is isomorphic to $\Lambda(p)$ for one of $p = 2, 3, 5, 7$ or 13.*

Proof. We first check that A is maximal. Suppose B is an order with $A \subset B \subset Q$. Pick $b \in B$. Then there exists a positive integer d such that $db \in A$, as $\mathbb{Q} \otimes A = Q$.There exist $q, r \in A$ such that $db = dq + r$ with $N(r) < N(d)$. But $N(d(b - q)) = N(d)N(b - q) \geq N(d)$ unless $b - q = 0$. Hence $b \in A$ and $A = B$.

Next we check the fourth condition of [3] Theorem 2. Let $\alpha \in A$ with $N(\alpha) = bc$ where $(b, c) = 1$. Greatest common divisors exist in A since it is Euclidean. Set $\beta = (\alpha, b)$. Then $\alpha = \beta\gamma$ for some $\gamma \in A$. $N(\beta)$ divides $N(\alpha) = bc$ and $N(b) = b^2$.

Hence $N(\beta)$ divides b. Further,

$$\begin{aligned}
\beta &= \alpha s + bt \quad \text{for some } s, t \in A \\
\bar{\beta} &= \bar{s}\bar{\alpha} + b\bar{t} \\
\beta\bar{\beta} &= N(\alpha)N(s) + [as\bar{t} + b\bar{s}\bar{\alpha} + bt\bar{t}]b \\
N(\beta) &= bcN(s) + \delta b,
\end{aligned}$$

where $\delta \in A$. Thus $N(\beta)/b = cN(s) + \delta \in \mathbb{Q} \cap A = \mathbb{Z}$ and so b divides $N(\beta)$. As Q is positive definite, $N(\beta) = b$.

Then [3] Theorem 2 gives that the genus of A is the class of A. The result follows from the discussion on p. 232 of [3]. □

Now $\Lambda(2)$ is the Hurwitz algebra and so norm Euclidean. $\Lambda(3)$ is isomorphic (the norm forms are equivalent) to the algebra $H_{1,3,3}$ constructed by Deutsch [1] and so it is also norm Euclidean. $\Lambda(5)$ will be shown to be norm Euclidean in the next section. To test $\Lambda(7)$ and $\Lambda(13)$ we will use the following.

Lemma 3. *Let A be an order in Q. A is norm Euclidean if and only if for all $h \in Q$ there exists a $q \in A$ such that $N(h - q) < 1$.*

Proof. ($\Longrightarrow$) Let $h \in Q = \mathbb{Q} \otimes_{\mathbb{Z}} A$. Clear denominators to get $n \in \mathbb{Z}$ such that $nh \in A$. We have $nh = nq + r$ for some $q, r \in A$ with $N(r) < N(n) = n^2$. Then $N(h - q) = N(r/n) < 1$.

($\Longleftarrow$) Let $\alpha, \beta \in A$ with $\beta \neq 0$. Set $h = \beta^{-1}\alpha \in Q$. We have by assumption $q \in A$ with $N(h - q) < 1$. Set $r_0 = h - q$. Then $\alpha = \beta q + \beta r_0$ and $N(\beta r_0) < N(\beta)$. □

For each order we give the norm form, complete squares and deduce a $\mathbb{Z}$-basis $v_1 = 1$, v_2, v_3, v_4. We then test 3. We use the notation throughout of $q = \sum a_i v_i$, with each $a_i \in \mathbb{Z}$ so that $q \in A$. We also write $q = \sum q_i e_i$, where the e_i form the usual basis of Q.

For $\Lambda(7) \subset Q = (\frac{-1,-7}{\mathbb{Q}})$ the norm form, representation by completing squares, and additional basis elements are, in order

$$\begin{gathered}
x^2 + y^2 + 2z^2 + 2w^2 + xz + yw, \\
(x + \frac{1}{2}z)^2 + (y + \frac{1}{2}w)^2 + 7(\frac{1}{2}z)^2 + 7(\frac{1}{2}w)^2, \qquad (1) \\
v_2 = e_2 \quad v_3 = \frac{1}{2}(1 + e_3) \quad v_4 = \frac{1}{2}(e_2 + e_4).
\end{gathered}$$

Take $h = \frac{1}{4}(1 + e_2 + e_3 + e_4)$. For $q \in \Lambda(7)$ each $q_i \in \frac{1}{2}\mathbb{Z}$. Hence $N(h - q) \geq (\frac{1}{4})^2 + (\frac{1}{4})^2 + 7(\frac{1}{4})^2 + 7(\frac{1}{4})^2 = 1$. So $\Lambda(7)$ is not norm Euclidean.

For $\Lambda(13) \subset Q = (\frac{-7,-13}{\mathbb{Q}})$ the norm form, representation by completing squares, and additional basis elements are, in order

$$\begin{aligned}
&x^2+2y^2+2z^2+4w^2+xy+yz+xw+yw+2zw,\\
&(x+\frac{1}{2}y+\frac{1}{2}w)^2+7(\frac{1}{2}y+\frac{1}{7}z+\frac{1}{14}w)^2+13(\frac{1}{2}w)^2\\
&\quad+91(\frac{1}{7}z+\frac{1}{14}w)^2,\\
&v_2=\frac{1}{2}(1+e_2)\quad v_3=\frac{1}{7}(e_2+e_4)\quad v_4=\frac{1}{14}(7+e_2+7e_3+e_4).
\end{aligned} \tag{2}$$

Take $h=\frac{1}{4}(1+e_2+e_3+e_4)$. For $q\in\Lambda(13)$ we have:

$$q_1=a_1+\frac{1}{2}a_2+\frac{1}{2}a_4 \qquad q_2=\frac{1}{2}a_2+\frac{1}{7}a_3+\frac{1}{14}a_4$$
$$q_3=\frac{1}{2}a_3 \qquad q_4=\frac{1}{7}a_3+\frac{1}{14}a_4.$$

Assume $N(h-q)<1$. From $13(\frac{1}{4}-\frac{1}{2}a_3)^2<1$ we get $a_3=0$ or 1 and so $13(\frac{1}{4}-\frac{1}{2}a_3)^2=\frac{13}{16}$. Thus $(\frac{1}{4}-q_1)^2, 7(\frac{1}{4}-q_2)^2$ and $91(\frac{1}{4}-q_4)^2$ are all less than $\frac{3}{16}$. We get:

$$q_1=\frac{k}{2},\quad 0\le k\le 1 \qquad q_2=\frac{k}{14},\quad 2\le k\le 5 \qquad q_4=\frac{k}{14},\quad 3\le k\le 4.$$

A computer search shows there are only four such q_i arising from integral a_i: $(\frac{1}{2},\frac{3}{14},0,\frac{3}{14}),(\frac{1}{2},\frac{3}{14},\frac{1}{2},\frac{3}{14}),(0,\frac{2}{7},0,\frac{2}{7})$ and $(1,\frac{2}{7},\frac{1}{2},\frac{2}{7})$. In each case, $N(h-q)=1$. So $\Lambda(13)$ is not norm Euclidean.

We summarize:

Theorem 4. *The norm Euclidean orders in a positive definite quaternion algebra over $\mathbb{Q}$ are, up to isomorphism, precisely:* $\Lambda(2),\Lambda(3)$ *and* $\Lambda(5)$.

Note that the three orders of (4) are in distinct quaternion algebras.

Corollary 5. *Let Q be a positive definite quaternion algebra over $\mathbb{Q}$. Let A and B be orders in Q. If A and B are both norm Euclidean then they are isomorphic.*

3. Proof for $\Lambda(5)$

We have been unable to find a geometric proof in this case. Instead we split the verification into many cases.

For $\Lambda(5)\subset Q=(\frac{-2,-5}{\mathbb{Q}})$ the norm form, representation by completing squares, and additional basis elements are, in order

$$\begin{aligned}
&x^2+y^2+2z^2+2w^2+xy+xz+xw+yw+2zw,\\
&(x+\frac{1}{2}(y+z+w))^2+2(\frac{1}{4}(y+3z+2w))^2+5(\frac{1}{2}w)^2 \\
&\quad +10(\frac{1}{4}(-y+z))^2,\\
&v_2=\frac{1}{4}(2+e_2-e_4) \quad v_3=\frac{1}{4}(2+3e_2+e_4) \quad v_4=\frac{1}{2}(1+e_2+e_3).
\end{aligned} \tag{3}$$

We will show that $\Lambda(5)$ is norm Euclidean. Let $h=\sum h_ie_i \in Q=(\frac{-2,-5}{\mathbb{Q}})$. We want to find a $q\in A$ with $N(h-q)<1$.

Set $\bar{h}=(h_1,h_2,h_3,h_4)$. Note that:

$$e_2=v_2+v_3-1 \quad e_3=2v_4-v_2-v_3 \quad e_4=v_3-3v_2+1,$$

are all in A. Hence we may assume each $h_i\in[0,1]$. Suppose $h_1\in[\frac{1}{2},1]$ and that for $h^*=(1-h_1)+\sum_{i\geq 2}h_ie_i$ we have $q^*=\sum q_ie_i\in A$ with $N(h^*-q^*)<1$. Then, as $q_1\in\frac{1}{2}\mathbb{Z}$, we can write $q_1=k/2$ for some integer k. So $q=(1-q_1)+\sum_{i\geq 2}q_ie_i=q^*-(k-1)\in A$ and $N(h-q)=N(h^*-q^*)<1$. Thus we may assume $h_1\in[0,\frac{1}{2}]$. As $q_3\in\frac{1}{2}\mathbb{Z}$ also, the same reasoning shows we may assume $h_3\in[0,\frac{1}{2}]$. Lastly, $\frac{1}{2}(\pm e_2+e_4)\in\Lambda(5)$ implies that we may assume $e_4\in[0,\frac{1}{2}]$ also. In summary, we may assume

$$\bar{h}\in[0,\frac{1}{2}]\times[0,1]\times[0,\frac{1}{2}]\times[0,\frac{1}{2}].$$

Set $\bar{q}=(q_1,q_2,q_3,q_4)$. The following $\bar{q}$ represent elements of A:

$$\begin{aligned}
-v_2+v_4 &\to \left(0,\frac{1}{4},\frac{1}{2},\frac{1}{4}\right) & -v_2+v_3 &\to \left(0,\frac{1}{2},0,\frac{1}{2}\right)\\
v_1-v_2 &\to \left(\frac{1}{2},-\frac{1}{4},0,\frac{1}{4}\right) & 1-2v_2+v_4 &\to \left(\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\right)\\
v_3 &\to \left(\frac{1}{2},\frac{3}{4},0,\frac{1}{4}\right) & v_4 &\to \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},0\right).
\end{aligned} \tag{4}$$

Set $n(x_1,x_2,x_3,x_4)=x_1^2+2x_2^2+5x_3^2+10x_4^2$ so that $N(\sum x_ie_i)=n(\bar{x})$.

In the supplementary table, for each block of $\bar{h}$'s we give a $\bar{q}\in A$ from (4) and compute $n(\bar{x})$ where x_i is the maximum, over the block, of $|h_i-q_i|$. Each $n(\bar{x})$ is less than one, proving that $\Lambda(5)$ is norm Euclidean.

4. Extending Hurwitz's Proof

We return to the notation of the introduction: $(1,a,b,ab)$ denotes the form $x^2+ay^2+bz^2+abw^2$. There are exactly seven such forms, with a,b positive integers, that represent all positive integers: $(1,1,1,1),(1,1,2,2),(1,1,3,3)$, $(1,2,2,4),(1,2,3,6)$, $(1,2,4,8)$ and $(1,2,5,10)$. The first form, $(1,1,1,1)$, gives the four square theorem, proven by Hurwitz via the norm Euclidean order $\Lambda(2)$.

Deutsch [1] constructed three norm Euclidean orders: $H_{1,2,2}$ used to prove the universality of $(1,1,2,2),(1,2,2,4)$ and $(1,2,4,8)$, $H_{1,3,3}$ for $(1,1,3,3)$ and $H_{2,3,6}$ for $(1,2,3,6)$. He was unable to find a norm Euclidean order in $(\frac{-2,-5}{\mathbb{Q}})$ to prove $(1,2,5,10)$ is universal.

Now the forms $(1,1,1,1)$ and $(1,1,2,2)$ are isometric over $\mathbb{Q}$. Hence $H_{1,2,2}$ is isomorphic to the Hurwitz algebra $\Lambda(2)$. Indeed, $H_{1,2,2}$ can be found as the image of $\Lambda(2)$ under the isomorphism $(\frac{-1,-1}{\mathbb{Q}}) \to (\frac{-1,-2}{\mathbb{Q}})$ given by $1 \mapsto 1, i \mapsto e_2, j \mapsto \frac{1}{2}(e_4+e_3), k \mapsto \frac{1}{2}(e_4-e_3)$. Similarly, $(\frac{-1,-1}{\mathbb{Q}})$ is isomorphic to $(\frac{-2,-3}{\mathbb{Q}})$ and so $H_{2,3,6}$ is also isomorphic to the Hurwitz algebra. And $H_{1,3,3}$ is isomorphic to $\Lambda(3)$, as previously noted.

Lastly, we have shown that $\Lambda(5)$ is a norm Euclidean order in $(\frac{-2,-5}{\mathbb{Q}})$. Following Hurwitz's proof with this order yields: given a positive integer n, there exists $\alpha \in \Lambda(5)$ such that $N(\alpha)=n$. To prove $(1,2,5,10)$ is universal in the style of Hurwitz requires showing that for every $\alpha \in \Lambda(5)$ there exist units $u_i \in \Lambda(5)$ such that $u_1\alpha u_2 \in L(5) \equiv \mathbb{Z}-\text{span}\{1,e_2,e_3,e_4\}$. But the units of $\Lambda(5)$ are $\pm 1, \pm v_2, \pm(1-v_2)$ and a check of the cases shows that there not units u_1, u_2 such that $u_1(1+v_2)u_2 \in L(5)$.

Now, $4\Lambda(5) \subset L(5)$ which implies that $(1,2,5,10)$ represents $16n$ for any positive integer n. But we have been unable to find an extension of Euler's trick (which shows that if $2n$ is a sum of four squares then so is n). In short, we have the required norm Euclidean order of $(\frac{-2,-5}{\mathbb{Q}})$ but not an extension of Hurwitz's proof to $(1,2,5,10)$.

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Supplementary Table

	h_1	h_2	h_3	h_4	q	maximal norm
1.	$[0,\frac{1}{4}]$	$[0,\frac{1}{4}]$	$[0,\frac{1}{4}]$	$[0,\frac{3}{16}]$	$0,0,0,0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16})=\frac{109}{128}$
			$[0,\frac{1}{8}]$	$[\frac{3}{16},\frac{1}{4}]$	$0,0,0,0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{4})=\frac{57}{64}$
			$[\frac{1}{8},\frac{1}{4}]$		$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{16})=\frac{119}{128}$
2.	$[0,\frac{1}{4}]$	$[0,\frac{1}{4}]$	$[0,\frac{1}{8}]$	$[\frac{1}{4},\frac{5}{16}]$	$\frac{1}{2},-\frac{1}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{16})=\frac{111}{128}$
				$[\frac{5}{16},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{8},\frac{3}{16})=\frac{127}{128}$
		$[0,\frac{1}{8}]$	$[\frac{1}{8},\frac{1}{6}]$	$[\frac{1}{4},\frac{1}{3}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{12})=\frac{553}{576}$
				$[\frac{1}{3},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{6},\frac{1}{6})=\frac{47}{48}$
			$[\frac{1}{6},\frac{1}{4}]$	$[\frac{1}{4},\frac{2}{5}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{3},\frac{3}{20})=\frac{697}{720}$
				$[\frac{2}{5},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{39}{40}$
		$[\frac{1}{8},\frac{1}{4}]$	$[\frac{1}{8},\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{3}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{8},\frac{3}{8},\frac{1}{12})=\frac{499}{576}$
				$[\frac{1}{3},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{3}{8},\frac{1}{4},\frac{1}{6})=\frac{269}{288}$
3.	$[0,\frac{1}{4}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{3}]$	$[0,\frac{1}{8}]$	$0,0,0,0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{3},\frac{1}{8})=\frac{259}{288}$
			$[\frac{1}{3},\frac{1}{2}]$		$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{6},\frac{1}{4})=\frac{137}{144}$
			$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{8},\frac{1}{4}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
4.	$[0,\frac{1}{4}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{3}{8}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
				$[\frac{3}{8},\frac{1}{2}]$	$\frac{1}{2},0,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
5.	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{12}{25}]$	$[0,\frac{1}{4}]$	$[0,\frac{1}{8}]$	$0,0,0,0$	$n(\frac{1}{4},\frac{12}{25},\frac{1}{4},\frac{1}{8})=\frac{19841}{20000}$
		$[\frac{12}{25},\frac{1}{2}]$	$[0,\frac{1}{5}]$		$0,0,0,0$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{5},\frac{1}{8})=\frac{147}{160}$
			$[\frac{1}{5},\frac{1}{4}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{1}{4},\frac{3}{10},\frac{1}{8})=\frac{157}{160}$
	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8})=\frac{63}{64}$
	$[0,\frac{1}{8})$		$[\frac{1}{8},\frac{1}{4}]$		$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{8})=1$
	$[\frac{1}{8},\frac{1}{4}]$	$[\frac{1}{4},\frac{3}{8}]$			$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{8},\frac{3}{8},\frac{1}{8})=\frac{61}{64}$
		$[\frac{3}{8},\frac{1}{2}]$			$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{3}{8},\frac{3}{8},\frac{1}{4},\frac{1}{8})=\frac{57}{64}$

	h_1	h_2	h_3	h_4	q	maximal norm
6.	$[0, \frac{1}{4}]$	$[\frac{1}{4}, \frac{1}{2}]$	$[0, \frac{1}{8}]$	$[\frac{1}{4}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{4}) = \frac{57}{64}$
			$[\frac{1}{8}, \frac{1}{4}]$	$[\frac{1}{4}, \frac{1}{3}]$	$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{1}{12}) = \frac{553}{576}$
				$[\frac{1}{3}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}) = \frac{7}{9}$
7.	$[0, \frac{1}{4}]$	$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{1}{4}, \frac{1}{2}]$	$[0, \frac{1}{8}]$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0$	$n(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}) = \frac{27}{32}$
				$[\frac{1}{8}, \frac{1}{4}]$	$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}) = \frac{21}{32}$
8.	$[0, \frac{1}{4}]$	$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{1}{4}, \frac{3}{8}]$	$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}) = \frac{21}{32}$
			$[\frac{1}{4}, \frac{5}{16}]$	$[\frac{3}{8}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{5}{16}, \frac{1}{8}) = \frac{213}{256}$
			$[\frac{5}{16}, \frac{1}{2}]$		$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{3}{16}, \frac{1}{4}) = \frac{253}{256}$
9.	$[\frac{1}{4}, \frac{1}{2}]$	$[0, \frac{1}{4}]$	$[0, \frac{1}{4}]$	$[0, \frac{7}{40}]$	$0, 0, 0, 0$	$n(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{7}{40}) = \frac{159}{160}$
				$[\frac{7}{40}, \frac{1}{4}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{3}{40}) = \frac{149}{160}$
10.	$[\frac{1}{4}, \frac{5}{16}]$	$[0, \frac{3}{20}]$	$[0, \frac{1}{4}]$	$[\frac{1}{4}, \frac{21}{50}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{4}, \frac{2}{5}, \frac{1}{4}, \frac{17}{100}) = \frac{123}{125}$
				$[\frac{21}{50}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{5}{16}, \frac{1}{2}, \frac{1}{4}, \frac{2}{25}) = \frac{31173}{32000}$
		$[\frac{3}{20}, \frac{1}{4}]$		$[\frac{1}{4}, \frac{1}{3}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{12}) = \frac{17}{18}$
				$[\frac{1}{3}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{5}{16}, \frac{7}{20}, \frac{1}{4}, \frac{1}{6}) = \frac{53737}{57600}$
	$[\frac{5}{16}, \frac{3}{8}]$	$[0, \frac{3}{20}]$		$[\frac{1}{4}, \frac{54}{125}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{3}{16}, \frac{2}{5}, \frac{1}{4}, \frac{91}{500}) = \frac{799117}{800000}$
				$[\frac{54}{125}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{3}{8}, \frac{1}{2}, \frac{1}{4}, \frac{17}{250}) = \frac{199873}{200000}$
		$[\frac{3}{20}, \frac{1}{4}]$		$[\frac{1}{4}, \frac{1}{3}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{3}{16}, \frac{1}{2}, \frac{1}{4}, \frac{1}{12}) = \frac{2113}{2304}$
				$[\frac{1}{3}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{3}{8}, \frac{7}{20}, \frac{1}{4}, \frac{1}{6}) = \frac{14053}{14400}$
	$[\frac{3}{8}, \frac{2}{5}]$	$[0, \frac{1}{10}]$		$[\frac{1}{4}, \frac{9}{20}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{8}, \frac{7}{20}, \frac{1}{4}, \frac{1}{5}) = \frac{1557}{1600}$
				$[\frac{9}{20}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{2}{5}, \frac{1}{2}, \frac{1}{4}, \frac{1}{20}) = \frac{399}{400}$
		$[\frac{1}{10}, \frac{1}{4}]$		$[\frac{1}{4}, \frac{3}{8}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{8}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}) = \frac{63}{64}$
				$[\frac{3}{8}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{2}{5}, \frac{2}{5}, \frac{1}{4}, \frac{1}{8}) = \frac{759}{800}$
	$[\frac{2}{5}, \frac{1}{2}]$	$[0, \frac{1}{10}]$	$[0, \frac{1}{8}]$	$[\frac{1}{4}, \frac{1}{2}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{10}, \frac{7}{20}, \frac{1}{8}, \frac{1}{4}) = \frac{1533}{1600}$
			$[\frac{1}{8}, \frac{1}{4}]$	$[\frac{1}{4}, \frac{3}{8}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{10}, \frac{7}{20}, \frac{1}{4}, \frac{1}{8}) = \frac{579}{800}$

	h_1	h_2	h_3	h_4	q	maximal norm
				$[\frac{3}{8},\frac{1}{2}]$	$\frac{1}{2},0,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{10},\frac{1}{10},\frac{3}{8},\frac{1}{8})=\frac{1423}{1600}$
		$[\frac{1}{10},\frac{3}{20}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{2}{5}]$	$\frac{1}{2},-\frac{1}{4},0,\frac{1}{4}$	$n(\frac{1}{10},\frac{2}{5},\frac{1}{4},\frac{3}{20})=\frac{347}{400}$
				$[\frac{2}{5},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{2},\frac{2}{5},\frac{1}{4},\frac{1}{10})=\frac{393}{400}$
		$[\frac{3}{20},\frac{1}{4}]$		$[\frac{1}{4},\frac{3}{8}]$	$\frac{1}{2},-\frac{1}{4},0,\frac{1}{4}$	$n(\frac{1}{10},\frac{1}{2},\frac{1}{4},\frac{1}{8})=\frac{783}{800}$
				$[\frac{3}{8},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{2},\frac{7}{20},\frac{1}{4},\frac{1}{8})=\frac{771}{800}$
11.	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{5}{16}]$	$[0,\frac{1}{16}]$	$0,0,0,0$	$n(\frac{1}{2},\frac{1}{4},\frac{5}{16},\frac{1}{16})=\frac{231}{256}$
			$[\frac{5}{16},\frac{1}{2}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{2},\frac{3}{16},\frac{1}{16})=\frac{199}{256}$
	$[\frac{1}{4},\frac{3}{8}]$		$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{16},\frac{1}{4}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{3}{16})=\frac{119}{128}$
	$[\frac{3}{8},\frac{1}{2}]$				$\frac{1}{2},0,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{3}{16})=\frac{103}{128}$
12.	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{3}{8}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
				$[\frac{3}{8},\frac{1}{2}]$	$\frac{1}{2},0,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
13.	$[\frac{1}{4},\frac{3}{8}]$	$[\frac{1}{4},\frac{2}{5}]$	$[0,\frac{1}{4}]$	$[0,\frac{3}{20}]$	$0,0,0,0$	$n(\frac{3}{8},\frac{2}{5},\frac{1}{4},\frac{3}{20})=\frac{1597}{1600}$
				$[\frac{3}{20},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{39}{40}$
		$[\frac{2}{5},\frac{1}{2}]$		$[0,\frac{3}{50}]$	$0,0,0,0$	$n(\frac{3}{8},\frac{1}{2},\frac{1}{4},\frac{3}{50})=\frac{7913}{8000}$
				$[\frac{3}{50},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{7}{20},\frac{1}{4},\frac{19}{100})=\frac{981}{1000}$
	$[\frac{3}{8},\frac{2}{5}]$	$[\frac{1}{4},\frac{2}{5}]$		$[0,\frac{3}{25}]$	$0,0,0,0$	$n(\frac{2}{5},\frac{2}{5},\frac{1}{4},\frac{3}{25})=\frac{1873}{2000}$
				$[\frac{3}{25},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{8},\frac{1}{2},\frac{1}{4},\frac{13}{100})=\frac{7977}{8000}$
		$[\frac{2}{5},\frac{1}{2}]$		$[0,\frac{1}{20}]$	$0,0,0,0$	$n(\frac{2}{5},\frac{1}{2},\frac{1}{4},\frac{1}{20})=\frac{399}{400}$
				$[\frac{1}{20},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{8},\frac{7}{20},\frac{1}{4},\frac{1}{5})=\frac{1557}{1600}$
	$[\frac{2}{5},\frac{1}{2}]$	$[\frac{1}{4},\frac{3}{8}]$	$[0,\frac{1}{8}]$	$[0,\frac{1}{10}]$	$0,0,0,0$	$n(\frac{1}{2},\frac{3}{8},\frac{1}{8},\frac{1}{10})=\frac{227}{320}$
				$[\frac{1}{10},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{10},\frac{1}{2},\frac{1}{8},\frac{3}{20})=\frac{1301}{1600}$
			$[\frac{1}{8},\frac{1}{4}]$	$[0,\frac{1}{8}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{10},\frac{1}{4},\frac{3}{8},\frac{1}{8})=\frac{1591}{1600}$
				$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{10},\frac{1}{2},\frac{1}{4},\frac{1}{8})=\frac{783}{800}$
		$[\frac{3}{8},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$[0,\frac{1}{8}]$	$0,0,0,0$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8})=\frac{63}{64}$

	h_1	h_2	h_3	h_4	q	maximal norm
			$[\frac{1}{8},\frac{1}{4}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{10},\frac{1}{8},\frac{3}{8},\frac{1}{8})=\frac{1441}{1600}$
			$[0,\frac{1}{4}]$	$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{10},\frac{3}{8},\frac{1}{4},\frac{1}{8})=\frac{19}{25}$
14.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{3}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{12})=\frac{17}{18}$
				$[\frac{1}{3},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{6})=\frac{139}{144}$
15.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
				$[\frac{1}{8},\frac{1}{4}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
16.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{2}{5}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{20})=\frac{73}{80}$
				$[\frac{2}{5},\frac{1}{2}]$	$\frac{1}{2},0,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{39}{40}$
17.	$[0,\frac{1}{4}]$	$[\frac{1}{2},\frac{3}{4}]$	$[0,\frac{1}{5}]$	$[0,\frac{1}{8}]$	$0,1,0,0$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{5},\frac{1}{8})=\frac{147}{160}$
			$[\frac{1}{5},\frac{1}{4}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{1}{4},\frac{3}{10},\frac{1}{8})=\frac{157}{160}$
			$[0,\frac{1}{4}]$	$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
18.	$[0,\frac{1}{4}]$	$[\frac{1}{2},\frac{3}{4}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{3}{8}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
				$[\frac{3}{8},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
19.	$[0,\frac{1}{4}]$	$[\frac{1}{2},\frac{3}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{7}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{7})=\frac{699}{784}$
				$[\frac{1}{7},\frac{1}{4}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{3}{28})=\frac{97}{98}$
20.	$[0,\frac{1}{10}]$	$[\frac{1}{2},\frac{11}{20}]$	$[\frac{1}{4},\frac{3}{8}]$	$[\frac{1}{4},\frac{3}{8}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{3}{40},\frac{3}{10},\frac{1}{4},\frac{1}{8})=\frac{1047}{1600}$
				$[\frac{3}{8},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{3}{40},\frac{1}{20},\frac{3}{8},\frac{1}{8})=\frac{87}{100}$
			$[\frac{3}{8},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{10},\frac{3}{10},\frac{1}{8},\frac{1}{4})=\frac{1429}{1600}$
		$[\frac{11}{20},\frac{3}{5}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{9}{20}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{10},\frac{7}{20},\frac{1}{4},\frac{1}{5})=\frac{387}{400}$
				$[\frac{9}{20},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{2},\frac{9}{20},\frac{1}{4},\frac{1}{20})=\frac{397}{400}$
		$[\frac{3}{5},\frac{7}{10}]$		$[\frac{1}{4},\frac{2}{5}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{10},\frac{9}{20},\frac{1}{4},\frac{3}{20})=\frac{381}{400}$
				$[\frac{2}{5},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{2},\frac{2}{5},\frac{1}{4},\frac{1}{10})=\frac{393}{400}$
		$[\frac{7}{10},\frac{3}{4}]$		$[\frac{1}{4},\frac{7}{20}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{10},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{369}{400}$
				$[\frac{7}{20},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{2},\frac{3}{10},\frac{1}{4},\frac{3}{20})=\frac{387}{400}$

	h_1	h_2	h_3	h_4	q	maximal norm
	$[\frac{1}{10},\frac{1}{5}]$	$[\frac{1}{2},\frac{3}{5}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{9}{20}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{5},\frac{7}{20},\frac{1}{4},\frac{1}{5})=\frac{399}{400}$
				$[\frac{9}{20},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{2}{5},\frac{1}{2},\frac{1}{4},\frac{1}{20})=\frac{399}{400}$
		$[\frac{3}{5},\frac{3}{4}]$		$[\frac{1}{4},\frac{9}{25}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{5},\frac{1}{2},\frac{1}{4},\frac{11}{100})=\frac{1947}{2000}$
				$[\frac{9}{25},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{2}{5},\frac{2}{5},\frac{1}{4},\frac{7}{50})=\frac{1977}{2000}$
	$[\frac{1}{5},\frac{1}{4}]$	$[\frac{1}{2},\frac{3}{5}]$		$[\frac{1}{4},\frac{11}{25}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{7}{20},\frac{1}{4},\frac{19}{100})=\frac{981}{1000}$
				$[\frac{11}{25},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{3}{10},\frac{1}{2},\frac{1}{4},\frac{3}{50})=\frac{1877}{2000}$
		$[\frac{3}{5},\frac{3}{4}]$		$[\frac{1}{4},\frac{7}{20}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{39}{40}$
				$[\frac{7}{20},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{3}{10},\frac{2}{5},\frac{1}{4},\frac{3}{20})=\frac{379}{400}$
21.	$[0,\frac{1}{4}]$	$[\frac{3}{4},1]$	$[0,\frac{1}{4}]$	$[0,\frac{1}{8}]$	$0,1,0,0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
				$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
22.	$[0,\frac{1}{4}]$	$[\frac{3}{4},1]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{2}{5}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{20})=\frac{73}{80}$
				$[\frac{2}{5},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{39}{40}$
23.	$[0,\frac{1}{4}]$	$[\frac{3}{4},\frac{7}{8})$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{3}{8},\frac{1}{4},\frac{1}{8})=1$
		$[\frac{7}{8},1]$	$[\frac{1}{4},\frac{3}{8}]$		$0,1,0,0$	$n(\frac{1}{4},\frac{1}{8},\frac{3}{8},\frac{1}{8})=\frac{61}{64}$
			$[\frac{3}{8},\frac{1}{2}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8})=\frac{63}{64}$
		$[\frac{3}{4},\frac{7}{8}]$	$[\frac{1}{4},\frac{1}{3}]$	$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{8},\frac{1}{3},\frac{1}{8})=\frac{143}{144}$
			$[\frac{1}{3},\frac{1}{2}]$		$0,\frac{5}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{6},\frac{1}{8})=\frac{247}{288}$
		$[\frac{7}{8},1]$	$[\frac{1}{4},\frac{1}{2}]$		$0,\frac{5}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{3}{8},\frac{1}{4},\frac{1}{8})=\frac{13}{16}$
24.	$[0,\frac{1}{4}]$	$[\frac{3}{4},1]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{3}]$	$0,\frac{5}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{12})=\frac{17}{18}$
				$[\frac{1}{3},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{6})=\frac{139}{144}$
25.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{2},\frac{3}{4}]$	$[0,\frac{1}{8}]$	$[0,\frac{1}{16}]$	$0,1,0,0$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{16})=\frac{111}{128}$
			$[\frac{1}{8},\frac{1}{4}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{16})=\frac{119}{128}$
			$[0,\frac{1}{4}]$	$[\frac{1}{16},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16})=\frac{109}{128}$

	h_1	h_2	h_3	h_4	q	maximal norm
26.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{2},\frac{3}{4}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{3}{8}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
				$[\frac{3}{8},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
27.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{2},\frac{3}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{3}{16}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16})=\frac{109}{128}$
			$[\frac{1}{4},\frac{3}{8}]$	$[\frac{3}{16},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{16})=\frac{119}{128}$
			$[\frac{3}{8},\frac{1}{2}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{4})=\frac{57}{64}$
28.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{2},\frac{3}{4}]$	$[\frac{1}{4},\frac{1}{3}]$	$[\frac{1}{4},\frac{2}{5}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{3},\frac{3}{20})=\frac{697}{720}$
				$[\frac{2}{5},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{39}{40}$
			$[\frac{1}{3},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{3}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{1}{12})=\frac{23}{24}$
				$[\frac{1}{3},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{6},\frac{1}{6})=\frac{47}{48}$
29.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{3}{4},1]$	$[0,\frac{1}{4}]$	$[0,\frac{1}{8}]$	$0,1,0,0$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
				$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
30.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{3}{4},1]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{7}{16}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16})=\frac{109}{128}$
			$[0,\frac{1}{8}]$	$[\frac{7}{16},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{16})=\frac{111}{128}$
			$[\frac{1}{8},\frac{1}{4}]$		$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{16})=\frac{119}{128}$
31.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{3}{4},1]$	$[\frac{1}{4},\frac{5}{16}]$	$[0,\frac{1}{10}]$	$0,1,0,0$	$n(\frac{1}{2},\frac{1}{4},\frac{5}{16},\frac{1}{10})=\frac{1233}{1280}$
				$[\frac{1}{10},\frac{3}{20}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{5}{16},\frac{3}{20})=\frac{1153}{1280}$
			$[\frac{5}{16},\frac{3}{8}]$	$[0,\frac{3}{20}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{2},\frac{3}{16},\frac{3}{20})=\frac{1233}{1280}$
			$[\frac{1}{4},\frac{3}{8}]$	$[\frac{3}{20},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{10})=\frac{317}{320}$
			$[\frac{3}{8},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{8},\frac{1}{8})=\frac{51}{64}$
				$[\frac{1}{8},\frac{1}{4}]$	$0,\frac{5}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8})=\frac{63}{64}$
32.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{3}{4},1]$	$[\frac{1}{4},\frac{3}{8}]$	$[\frac{1}{4},\frac{5}{16}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{16})=\frac{119}{128}$
			$[\frac{3}{8},\frac{1}{2}]$		$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{4})=\frac{57}{64}$
			$[\frac{1}{4},\frac{1}{2}]$	$[\frac{5}{16},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16})=\frac{109}{128}$