A PROOF OF CATALAN'S CONVOLUTION FORMULA

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Abstract

A new proof is given for the k-fold convolution of the Catalan numbers. This is done by enumerating a certain class of polygonal dissections called k-in-n dissections.

1. Introduction

The Catalan numbers are defined as follows.

Definition 1. For any $n \geq 0$,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

For $n < 0$, $C_n = 0$.

It is well known that for any $n \geq 0$,

$$\sum_{i \geq 0} C_i C_{n-i} = C_{n+1}. \tag{1}$$

In other words, the two-fold convolution of Catalan numbers is itself equal to a Catalan number. In 1887, Catalan proved the following k-fold convolution formula.

Theorem 2. [2] *Let $1 \leq k \leq n$. Then*

$$\sum_{i_1+\ldots+i_k=n} C_{i_1-1} \cdots C_{i_k-1} = \frac{k}{2n-k}\binom{2n-k}{n}. \tag{2}$$

Larcombe and French [4] present an account of Catalan's original proof of Theorem 2 (see also [5] for a discussion of the convolution and a related new identity). This proof applies the Lagrange inversion formula to the functional equation

$$y = 2 - 4x/y, \tag{3}$$

resulting in an expression for y^{-k} as a power series. Comparing this to a direct solution of (3) produces an equation satisfied by the k-th power of the generating function $\sum_{n=0}^{\infty} C_n x^n$, and this equation is shown to be equivalent to (2).

Tedford [6] exhibits several interpretations of the Catalan k-fold convolution. In particular, the left-hand side of (2) is equal to the number of lattice paths with up and right unit steps from $(0,0)$ to the point $(n, n-k)$, which stay below the main diagonal. This fact can be used to give another known proof of Theorem 2 (also presented in [4, Appendix]). The reflection principle shows that for any $a > b$, the number of such lattice paths from $(0,0)$ to (a,b) is $\frac{a-b}{a+b}\binom{a+b}{a}$ (see, for example, [3, p. 70]). Setting $a = n$ and $b = n-k$ then gives the result of the theorem.

One of the most natural interpretations of the Catalan numbers is as the number of triangulations of a polygon. In this note we use this interpretation to give a new proof of Theorem 2. We arrive at this proof using Theorem 5, which enumerates a class of polygonal dissections called k-in-n dissections. The proof presented here is unique in that it uses only the interpretation of the Catalan numbers as the number of polygon triangulations, and furthermore does not use generating function techniques.

2. The k-in-n Dissections

Definition 3. Let $n \geq 3$ and let $0 \leq k \leq n-3$.

1. A *k-dissection* of an n-gon is a partition of the n-gon into $k+1$ parts by k noncrossing diagonals.

2. A *triangulation* of an n-gon is an $(n-3)$-dissection.

3. For $k \geq 4$, an *k-in-n dissection* is an $(n-k)$-dissection of an n-gon into one k-gon and $n-k$ triangles (see Figure 1). A *3-in-n* dissection is a triangulation with one of its $n-2$ triangles marked.

4. Let $f_k(n)$ be the number of k-in-n dissections.

It is well known that for $n \geq 3$ the number of triangulations of an n-gon is C_{n-2}.

Lemma 4. *Let $3 \leq k \leq n$. Then*

$$(n-k)f_k(n) = n \sum_{i=2}^{n-k+1} C_{i-1} f_k(n-i+1). \tag{4}$$

Proof. The left-hand side of (4) is the number of k-in-n dissections, with one of the $n-k$ diagonals marked. These can also be chosen as follows. Choose one vertex

v out of the n vertices, then choose $2 \leq i \leq n-k+1$. Form the diagonal from v to a vertex which is a distance i from v (proceeding, say, counterclockwise along the edges of the n-gon). Mark this diagonal. Now choose a triangulation of the resulting $(i+1)$-gon and a k-in-$((n-i)+1)$ dissection of the resulting $((n-i)+1)$-gon. Each such choice results in a unique k-in-n dissection with one of the diagonals marked. □

As will be shown, Lemma 4 together with Lemma 8 below can be used to enumerate the k-in-n dissections.

Theorem 5. *Let $3 \leq k \leq n$. The number of k-in-n dissections is*

$$f_k(n) = \binom{2n-k-1}{n-1}.$$

Note 6. There is a bijection between k-in-n dissections and k-crossing partitions of $\{1, \ldots n\}$, as defined in [1]. Thus Theorem 5 is equivalent to [1, Theorem 1].

The following lemmas will be used in the proof of Theorem 5.

Lemma 7. *For any $n \geq 1$,*

$$\sum_{i \geq 0} i C_i C_{n-i} = \binom{2n+1}{n-1}. \tag{5}$$

Proof. Note that

$$\sum_{i \geq 0} i C_i C_{n-i} = \sum_{i \geq 0} (n-i) C_i C_{n-i}.$$

Therefore by (1),

$$\sum_{i \geq 0} i C_i C_{n-i} = \frac{1}{2} \sum_{i \geq 0} n C_i C_{n-i} = \frac{n}{2} C_{n+1} = \binom{2n+1}{n-1}.$$

□

Lemma 8. *Let $1 \leq q \leq p \leq 2q-1$. Then*

$$\sum_{i \geq 0} C_i \binom{p-1-2i}{q-1-i} = \binom{p}{q}. \tag{6}$$

Proof. We use induction on q. If $q = 1$ then $p = 1$ and both sides of (6) are equal to 1. Now suppose $q \geq 2$. If $p = q$ then both sides are equal to 1. If $p = 2q-1$ then

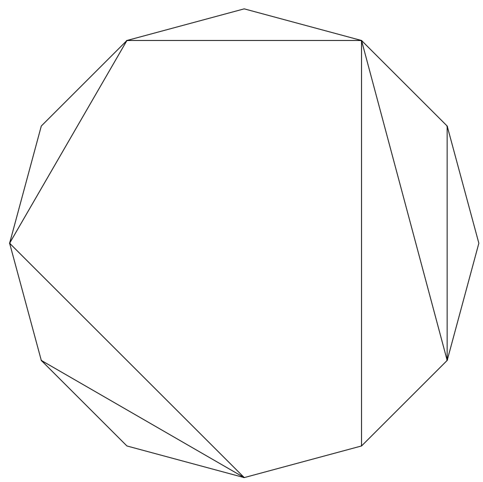

Figure 1: Example of a 5-in-12 dissection

(6) follows from (1) and (5), since

$$\begin{aligned}\sum_{i\geq 0} C_i\binom{2q-2-2i}{q-1-i} &= \sum_{i\geq 0} C_i(q-i)C_{q-1-i}\\ &= q\sum_{i\geq 0} C_iC_{q-1-i} - \sum_{i\geq 0} iC_iC_{q-1-i}\\ &= qC_q - \binom{2q-1}{q-2}\\ &= \binom{2q-1}{q}.\end{aligned}$$

Now suppose $q+1 \leq p \leq 2q-2$. Note that $q-1 \leq p-1$ and $p-1 \leq 2q-2-1 = 2(q-1)-1$. Therefore by the induction hypothesis, (6) holds for $p-1$ and $q-1$. Also $q \leq p-1$ and $p-1 \leq 2q-3 < 2q-1$, so that (6) holds for $p-1$ and q. Thus

$$\begin{aligned}\binom{p}{q} &= \binom{p-1}{q-1} + \binom{p-1}{q}\\ &= \sum_{i\geq 0}\frac{1}{i+1}\binom{2i}{i}\binom{p-2-2i}{q-2-i} + \sum_{i\geq 0}\frac{1}{i+1}\binom{2i}{i}\binom{p-2-2i}{q-1-i}\\ &= \sum_{i\geq 0}\frac{1}{i+1}\binom{2i}{i}\binom{p-1-2i}{q-1-i}.\end{aligned}$$

□

2.1. Proof of Theorem 5

Proof. Fix $k \geq 3$ and proceed by induction on n. If $n = k$ then both sides are equal to 1. Now let $n \geq k+1$. By Lemma 4 and by the induction hypothesis,

$$\begin{aligned} f_k(n) &= \frac{n}{n-k} \sum_{i=2}^{n-k-1} C_{i-1} f_k(n-i+1) \\ &= \frac{n}{n-k} \sum_{i=2}^{n-k-1} C_{i-1} \binom{2(n-i+1)-k-1}{n-i} \\ &= \frac{n}{n-k} \left(\sum_{i \geq 1} C_{i-1} \binom{2(n-i+1)-k-1}{n-i} - f_k(n) \right). \end{aligned}$$

Solving for $f_k(n)$ and applying Lemma 8, with $q = n$ and $p = 2n-k$,

$$f_k(n) = \frac{n}{2n-k} \sum_{i \geq 0} C_i \binom{2n-k-2i-1}{n-i-1} = \frac{n}{2n-k} \binom{2n-k}{n} = \binom{2n-k-1}{n-1}.$$

□

3. Proof of the Catalan Convolution Formula

The next lemma gives the relation between the number of k-in-n dissections and the Catalan convolution.

Lemma 9. *Let $3 \leq k < n$. Then*

$$k f_k(n) = n \sum_{i_1+\ldots+i_k=n} C_{i_1-1} \cdots C_{i_k-1}. \tag{7}$$

Proof. The left-hand side of (7) is the number of k-in-n dissections, with one of the vertices of the k-gon marked. These can also be chosen as follows. Choose any vertex v of the n-gon. For each vertex v, choose $i_1, \ldots, i_k$ such that $i_1 + \ldots + i_k = n$. This determines the lengths of the sides of a k-gon by starting at v and proceeding, say, counterclockwise. For example, in Figure 1, if v is the bottom vertex then the lengths are $1, 4, 2, 2, 3$. For each $1 \leq r \leq k$, there is a resulting (i_r+1)-gon sharing one edge of the k-gon. Each of these (i_r+1)-gons can be triangulated in C_{i_r-1} ways, forming a uniquely determined k-in-n dissection with one of the of the k-gons marked. □

The proof of Theorem 2 now follows from Lemma 9, since

$$\sum_{i_1+\ldots+i_k=n} C_{i_1-1}\cdots C_{i_k-1} = \frac{k}{n}f_k(n) = \frac{k}{n}\binom{2n-k-1}{n-1} = \frac{k}{2n-k}\binom{2n-k}{n}.$$

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