POLYNOMIALS RELATED TO HARMONIC NUMBERS AND EVALUATION OF HARMONIC NUMBER SERIES I[1]

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Abstract

In this paper we focus on two new families of polynomials which are connected with exponential polynomials $\phi_n(x)$ and geometric polynomials $\mathcal{F}_n(x)$. We discuss their generalizations and show that these new families of polynomials and their generalizations are useful to obtain closed forms of some series related to harmonic numbers.

1. Introduction

In this work we are interested in two new families of polynomials, namely *harmonic-exponential* polynomials and *harmonic-geometric* polynomials. We introduce these polynomials and discuss several interesting generalizations of them with the help of Theorem 1.

Suppose we are given an entire function f and a function g, analytic in a region containing the annulus $K = \{z : r < |z| < R\}$ where $0 < r < R$. Hence these functions have the following series expansions,

$$f(x) = \sum_{n=0}^{\infty} p_n x^n \text{ and } g(x) = \sum_{n=-\infty}^{\infty} q_n x^n.$$

Now we are ready to state Boyadzhiev's theorem.

Theorem 1. ([7]) *Let the functions f and g be described as above. If the series*

$$\sum_{n=-\infty}^{\infty} q_n f(n) x^n$$

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converges absolutely on K, then

$$\sum_{n=-\infty}^{\infty} q_n f(n) x^n = \sum_{m=0}^{\infty} p_m \sum_{k=0}^{m} \left\{ {m \atop k} \right\} x^k g^{(k)}(x) \tag{1}$$

holds for all $x \in K$.

If we consider the function g in Theorem 1 as an analytic function on the disk $K = \{z : |z| < R\}$, then the formula (1) turns out to be

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} f(n) x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k g^{(k)}(x). \tag{2}$$

We show that families of polynomials and their generalizations presented in this paper are considerably useful to obtain closed forms of some series related to harmonic numbers. For instance we obtain the following closed forms:

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} k H_k \right) x^n = \frac{x(1 - \ln(1-x))}{(1-x)^3}, \tag{3}$$

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} k^2 H_k \right) x^n = \frac{x(1 + 2x - (1+x)\ln(1-x))}{(1-x)^4}, \tag{4}$$

$$\sum_{n=1}^{\infty} \left(\frac{n^2}{2!} + n + 1 \right) H_n x^n = \frac{3x - \left(-2 + x - x^2\right)\ln(1-x)}{2(1-x)^2}. \tag{5}$$

Also for hyperharmonic series, one of our results is

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} k H_k^{(\alpha)} \right) x^n = \frac{x(1 - \alpha \ln(1-x))}{(1-x)^{\alpha+2}} \tag{6}$$

where α is a nonnegative integer.

In the rest of this section we will introduce some important notions.

1.1. Stirling Numbers of the First and Second Kind

Stirling numbers of the first kind denoted by $\left[{n \atop k} \right]$ and Stirling numbers of the second kind denoted by $\left\{ {n \atop k} \right\}$ are defined by means of

$$(x)_n = x(x-1)\ldots(x-n+1) = \sum_{k=0}^{n} \left[{n \atop k} \right] x^k \tag{7}$$

and

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k \tag{8}$$

respectively (see [1, 17]). These numbers are quite common in combinatorics; see, e.g., [4, 5, 12, 20].

We note that for $n \geq k \geq 1$ the following identity holds;

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}. \tag{9}$$

There is a certain generalization of these numbers, namely r-Stirling numbers (see [9]), which are similar to the weighted Stirling numbers [10, 11]. Combinatorial meanings, recurrence relations, generating functions and several properties of these numbers are given in [9]. The concepts of r-geometric polynomials, r-exponential polynomials and their harmonic versions concerning the r-Stirling numbers are given in [16].

1.2. Exponential Polynomials and Numbers

Exponential polynomials (or single variable Bell polynomials) [2, 20], $\phi_n(x)$, are defined by

$$\phi_n(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k. \tag{10}$$

Grunert expressed these polynomials in terms of Stirling numbers of the second kind and obtained some fundamental formulas [18]. Besides the work of Grunert [18], some other well-known studies on these polynomials may be found in [2], [6], and [23]. We refer the reader to [8] for comprehensive information on exponential polynomials.

The first few exponential polynomials are:

$$\begin{aligned} \phi_0(x) &= 1,\ \phi_1(x) = x,\ \phi_2(x) = x + x^2, \\ \phi_3(x) &= x + 3x^2 + x^3,\ \phi_4(x) = x + 7x^2 + 6x^3 + x^4. \end{aligned} \tag{11}$$

The well-known exponential numbers (or Bell numbers) are obtained by setting $x = 1$ in $\phi_n(x)$ i.e.,

$$\phi_n := \phi_n(1) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}, \tag{12}$$

(see [3, 12, 13]). The first few exponential numbers are:

$$\phi_0 = 1,\ \phi_1 = 1,\ \phi_2 = 2,\ \phi_3 = 5,\ \phi_4 = 15. \tag{13}$$

1.3. Geometric Polynomials and Numbers

Geometric polynomials are defined in [21, 22] as follows:

$$\mathcal{F}_n(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! x^k. \tag{14}$$

We use $\mathcal{F}_n$, one of the most common notations for these polynomials in the honor of Guido Fubini [24]. These polynomials are also called as Fubini polynomials [7] or ordered Bell polynomials [22].

The first few geometric polynomials are:

$$\begin{aligned} \mathcal{F}_0(x) &= 1,\ \mathcal{F}_1(x) = x,\ \mathcal{F}_2(x) = x + 2x^2, \\ \mathcal{F}_3(x) &= x + 6x^2 + 6x^3,\ \mathcal{F}_4(x) = x + 14x^2 + 36x^3 + 24x^4. \end{aligned} \tag{15}$$

In particular, setting $x = 1$ in (14) we get the n-th geometric number (or ordered Bell number) $\mathcal{F}_n$ as:

$$\mathcal{F}_n := \mathcal{F}_n(1) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! \tag{16}$$

(see [7, 22]).

The first few geometric numbers are:

$$\mathcal{F}_0 = 1,\ \mathcal{F}_1 = 1,\ \mathcal{F}_2 = 3,\ \mathcal{F}_3 = 13,\ \mathcal{F}_4 = 75. \tag{17}$$

Boyadzhiev [7] introduced “the general geometric polynomials” as:

$$\mathcal{F}_{n,r}(x) = \frac{1}{\Gamma(r)} \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \Gamma(k+r)\, x^k, \tag{18}$$

where $\mathrm{Re}(r) > 0$. In the third section we will deal with the general geometric polynomials.

Exponential and geometric polynomials are connected by the relation

$$\mathcal{F}_n(x) = \int_0^\infty \phi_n(x\lambda)\, e^{-\lambda} d\lambda \tag{19}$$

(see [7]).

In [15] the authors obtained some fundemental properties of exponential and geometric polynomials and numbers using Euler-Seidel matrices.

1.4. Harmonic and Hyperharmonic Numbers

The n-th harmonic number is the n-th partial sum of the harmonic series:

$$H_n := \sum_{k=1}^{n} \frac{1}{k}, \tag{20}$$

where $H_0 = 0$.

For an integer $\alpha > 1$, let

$$H_n^{(\alpha)} := \sum_{k=1}^{n} H_k^{(\alpha-1)} \tag{21}$$

with $H_n^{(1)} := H_n$ being the n-th hyperharmonic number of order α [4, 13].

These numbers can be expressed in terms of binomial coefficients and ordinary harmonic numbers as:

$$H_n^{(\alpha)} = \binom{n+\alpha-1}{\alpha-1}(H_{n+\alpha-1} - H_{\alpha-1}) \tag{22}$$

(see [13, 19]).

The well-known generating functions of the harmonic and hyperharmonic numbers are given by

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x} \tag{23}$$

and

$$\sum_{n=1}^{\infty} H_n^{(\alpha)} x^n = -\frac{\ln(1-x)}{(1-x)^{\alpha}}, \tag{24}$$

respectively [14].

The following relations connect harmonic and hyperharmonic numbers with the Stirling and r-Stirling numbers of the first kind:

$$\begin{bmatrix} k+1 \\ 2 \end{bmatrix} = k! H_k, \tag{25}$$

and

$$k! H_k^{(r)} = \begin{bmatrix} n+r \\ r+1 \end{bmatrix}_r \tag{26}$$

(see [4]).

2. Transformation of Harmonic Numbers

In this section we study the series related to harmonic numbers by using the transformation formula (2).

We set g in (2) as the generating function of harmonic numbers, which is given by equation (23). After rearranging the kth derivative of the right-hand side of (23) we obtain the following nice result:

Proposition 2. *We have*

$$\frac{d^k}{dx^k}\left\{-\frac{\ln(1-x)}{1-x}\right\} = \frac{k!\,(H_k - \ln(1-x))}{(1-x)^{k+1}}. \tag{27}$$

Proof. Follows by induction on k. □

From (27) we have

$$g^{(k)}(x) = \frac{k!\,(H_k - \ln(1-x))}{(1-x)^{k+1}} \tag{28}$$

and

$$g^{(k)}(0) = k!H_k. \tag{29}$$

Now we are ready to state a transformation formula for the series related to harmonic numbers.

Proposition 3. *For an entire function f the following transformation formula holds.*

$$\begin{aligned}\sum_{n=0}^{\infty} H_n f(n)\, x^n \quad = \quad & \frac{1}{1-x}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\sum_{k=0}^{n}\left\{ {n \atop k} \right\} k!H_k\left(\frac{x}{1-x}\right)^k \\ & -\frac{\ln(1-x)}{1-x}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\sum_{k=0}^{n}\left\{ {n \atop k} \right\} k!\left(\frac{x}{1-x}\right)^k.\end{aligned} \tag{30}$$

Proof. Employing (28), (29) in (2) gives the statement. $\square$

Geometric polynomials $\mathcal{F}_n(x)$ appear in the second part of the right-hand side of Equation (30). The first part of the right-hand side contains a new family of polynomials. We will denote them by $\mathcal{F}_n^h(x)$ and call them as *harmonic-geometric* polynomials because of their factor H_k. Hence the harmonic-geometric polynomials are

$$\mathcal{F}_n^h(x) := \sum_{k=0}^{n}\left\{ {n \atop k} \right\} k!H_k x^k. \tag{31}$$

The first few harmonic-geometric polynomials are:

$$\begin{array}{|l|}\hline \mathcal{F}_0^h(x) = 0 \\ \hline \mathcal{F}_1^h(x) = x \\ \hline \mathcal{F}_2^h(x) = x + 3x^2 \\ \hline \mathcal{F}_3^h(x) = x + 9x^2 + 11x^3 \\ \hline \mathcal{F}_4^h(x) = x + 21x^2 + 66x^3 + 50x^4 \\ \hline \mathcal{F}_5^h(x) = x + 45x^2 + 275x^3 + 500x^4 + 274x^5 \\ \hline \end{array}. \tag{32}$$

Using these notation we reformulate Equation (30) as follows:

$$\sum_{n=0}^{\infty} H_n f(n)\, x^n = \frac{1}{1-x}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left\{\mathcal{F}_n^h\left(\frac{x}{1-x}\right) - \mathcal{F}_n\left(\frac{x}{1-x}\right)\ln(1-x)\right\}. \tag{33}$$

Formula (33) enables us to calculate closed forms of some series related to harmonic numbers.

Corollary 4. *For any nonnegative integer m the following equality holds:*

$$\sum_{n=1}^{\infty} n^m H_n x^n = \frac{1}{1-x}\left\{\mathcal{F}_m^h\left(\frac{x}{1-x}\right) - \mathcal{F}_m\left(\frac{x}{1-x}\right)\ln(1-x)\right\}. \tag{34}$$

Proof. It follows directly by setting $f(x) = x^m$ in (33). □

Remark 5. Equation (34) is a generalization of the generating function of harmonic numbers, since the case $m = 0$ gives equation (23). Besides this ordinary case, thanks to formula (34), we obtain generating functions of several interesting series related to harmonic numbers. For instance, the case $m = 1$ in (34) gives

$$\sum_{n=1}^{\infty} n H_n x^n = \frac{x(1-\ln(1-x))}{(1-x)^2}, \tag{35}$$

and the case $m = 2$ gives

$$\sum_{n=1}^{\infty} n^2 H_n x^n = \frac{x(1+2x-(1+x)\ln(1-x))}{(1-x)^3}, \tag{36}$$

and so on.

Now we extend our results to multiple sums.

Proposition 6. *We have*

$$\sum_{n=1}^{\infty}\left(\sum_{r=0}^{n}\binom{n+s-r}{s} r^m H_r\right)x^n = \sum_{n=1}^{\infty}\left(\sum_{0\le i\le i_1\le\cdots\le i_s\le n} i^m H_i\right)x^n$$
$$= \frac{1}{(1-x)^{s+2}}\left[\mathcal{F}_m^h\left(\frac{x}{1-x}\right) - \mathcal{F}_m\left(\frac{x}{1-x}\right)\ln(1-x)\right], \tag{37}$$

where m and s are nonnegative integers.

Proof. By multiplying the right-hand side of (34) with $\frac{1}{(1-x)^{s+1}}$ and the left-hand side of (34) with its Newton binomial series, and considering the equation

$$\sum_{r=0}^{n}\binom{n+s-r}{s} r^m H_r = \sum_{0\le i\le i_1\le\cdots\le i_s\le n} i^m H_i\,, \tag{38}$$

we obtain the statement. □

Corollary 7. *We have*

$$\sum_{n=0}^{\infty} H_n^{(s)} x^n = \frac{-\ln(1-x)}{(1-x)^s}.$$

Proof. Setting $m = 0$ in (37) and considering $\mathcal{F}_0^h(x) = 0$ and $\mathcal{F}_0(x) = 1$ gives the desired result. $\square$

$$\sum_{n=1}^{\infty} \left(1^m H_1 + 2^m H_2 + \cdots + n^m H_n\right) x^n \tag{39}$$
$$= \frac{1}{(1-x)^2}\left[\mathcal{F}_m^h\left(\frac{x}{1-x}\right) - \mathcal{F}_m\left(\frac{x}{1-x}\right)\ln(1-x)\right].$$

In the light of (39) we get the following sums:

$m = 1$ gives

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} kH_k\right)x^n = \frac{x\left(1-\ln(1-x)\right)}{(1-x)^3}, \tag{40}$$

$m = 2$ gives

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} k^2 H_k\right)x^n = \frac{x\left(1+2x-(1+x)\ln(1-x)\right)}{(1-x)^4}, \tag{41}$$

and so on.

Remark 8. All these formulas and equations which we have obtained until now reveal that harmonic-geometric polynomials have strong relation with the series of harmonic numbers. We could state the generating functions of some series related to harmonic numbers in terms of harmonic-geometric polynomials, as in the equations (34) and (37).

Remark 9. Most of the results in this section are obtained by setting $f(x) = x^m$ in the transformation formula (33). It is possible to obtain more general results by setting $f(x)$ in (33) as an arbitrary polynomial of order m as:

$$f(x) = p_m x^m + p_{m-1}x^{m-1} + \cdots + p_1 x + p_0 \tag{42}$$

where $p_0, p_1, \cdots, p_{m-1}, p_m$ are any complex numbers. Hence we get the following equation which is more general than (34):

$$\sum_{n=0}^{\infty}\left(p_m n^m + p_{m-1}n^{m-1} + \cdots + p_1 n + p_0\right) H_n x^n \tag{43}$$
$$= \frac{1}{1-x}\sum_{k=0}^{m} p_k \left\{\mathcal{F}_k^h\left(\frac{x}{1-x}\right) - \mathcal{F}_k\left(\frac{x}{1-x}\right)\ln(1-x)\right\}.$$

Specializing coefficients of f gives more closed forms of harmonic number series. Each polynomial creates another sum. For instance by setting $p_k = 1$ for each $k = 0, 1, \ldots, m$ in (42) we get

$$\sum_{n=0}^{\infty} \left(n^m + n^{m-1} + \cdots + n + 1\right) H_n x^n \tag{44}$$
$$= \frac{1}{1-x} \sum_{k=0}^{m} \left\{ \mathcal{F}_k^h \left(\frac{x}{1-x} \right) - \mathcal{F}_k \left(\frac{x}{1-x} \right) \ln(1-x) \right\}.$$

This formula leads the following sums:

The case $m = 1$ in (44) gives

$$\sum_{n=1}^{\infty} (n+1) H_n x^n = \frac{x - \ln(1-x)}{(1-x)^2}. \tag{45}$$

The case $m = 2$ in (44) gives

$$\sum_{n=1}^{\infty} \left(n^2 + n + 1\right) H_n x^n = \frac{x^2 + 2x - \left(1 + x^2\right) \ln(1-x)}{(1-x)^3}, \tag{46}$$

and so on.

By setting $p_k = k$ for each $k = 0, 1, \ldots, m$ in (42) we get

$$\sum_{n=1}^{\infty} \left(m n^m + (m-1) n^{m-1} + \cdots + n\right) H_n x^n \tag{47}$$
$$= \frac{1}{1-x} \sum_{k=1}^{m} k \left\{ \mathcal{F}_k^h \left(\frac{x}{1-x} \right) - \mathcal{F}_k \left(\frac{x}{1-x} \right) \ln(1-x) \right\}.$$

We can give some examples of special cases of (47) as well. For example the case $m = 1$ in (47) gives the sum (35). The case $m = 2$ in (47) gives

$$\sum_{n=1}^{\infty} n (2n+1) H_n x^n = \frac{3x(1+x) - x(x+3)\ln(1-x)}{(1-x)^3}, \tag{48}$$

and so on.

By setting $p_k = \frac{1}{k!}$ for each $k = 0, 1, \ldots, m$ we get the following general formula:

$$\sum_{n=1}^{\infty} \left(\frac{n^m}{m!} + \frac{n^{m-1}}{(m-1)!} + \cdots + n + 1 \right) H_n x^n \tag{49}$$
$$= \frac{1}{1-x} \sum_{k=1}^{m} \frac{1}{k!} \left\{ \mathcal{F}_k^h \left(\frac{x}{1-x} \right) - \mathcal{F}_k \left(\frac{x}{1-x} \right) \ln(1-x) \right\}.$$

Choosing $m = 1$ in this formula we turn back to (45). The case $m = 2$ gives

$$\sum_{n=1}^{\infty}\left(\frac{n^2}{2!}+n+1\right)H_n x^n = \frac{3x-\left(-2+x-x^2\right)\ln(1-x)}{2(1-x)^2}. \tag{50}$$

For the other values of m we get the closed forms of these kind of interesting harmonic number series.

We obtain some of these results by using operator argument in the following subsection.

2.1. The Operator (xD)

The operator (xD) is defined as:

$$(xD) f(x) = x f'(x) \tag{51}$$

where f' is the first derivative of a function f.

For any m-times differentiable function f we have

$$(xD)^m f(x) = \sum_{k=0}^{m}\left\{{m \atop k}\right\} x^k f^{(k)}(x) \tag{52}$$

(see [7]). This fact can be easily proven by induction on m with the help of (9).

We consider the generating function of the harmonic numbers in the formula (52) . With the help of Proposition 2 we have

$$\begin{aligned}(xD)^m\left(-\frac{\ln(1-x)}{1-x}\right) &= \sum_{k=0}^{m}\left\{{m \atop k}\right\} x^k \frac{k!\,(H_k-\ln(1-x))}{(1-x)^{k+1}} \\ &= \frac{1}{1-x}\left[\mathcal{F}_m^h\left(\frac{x}{1-x}\right)-\mathcal{F}_m\left(\frac{x}{1-x}\right)\ln(1-x)\right].\end{aligned}$$

On the other hand by using (51) we get

$$(xD)^m\left(\sum_{n=1}^{\infty} H_n x^n\right) = \sum_{n=1}^{\infty} n^m H_n x^n.$$

Combining these two results we obtain the formula (34).

2.2. Harmonic-Geometric Numbers

Definition 10. The *harmonic-geometric numbers* $\mathcal{F}_n^h$ are obtained by setting $x = 1$ in (31) as

$$\mathcal{F}_n^h := \mathcal{F}_n^h(1) = \sum_{k=0}^{n}\left\{{n \atop k}\right\} k! H_k. \tag{53}$$

The first few harmonic-geometric numbers are

$$\mathcal{F}_0^h = 0,\ \mathcal{F}_1^h = 1,\ \mathcal{F}_2^h = 4,\ \mathcal{F}_3^h = 21,\ \mathcal{F}_4^h = 138,\ \mathcal{F}_5^h = 1095. \tag{54}$$

Remark 11. By using (25) we can write the harmonic-geometric polynomials and numbers just in terms of Stirling numbers of the first and second kind as follows:

$$\mathcal{F}_n^h(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \left[{k+1 \atop 2} \right] x^k, \tag{55}$$

$$\mathcal{F}_n^h = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \left[{k+1 \atop 2} \right]. \tag{56}$$

2.3. Harmonic-Exponential Polynomials and Numbers

Geometric and exponential polynomials are connected to each other via equation (19). Now with this motivation we define harmonic-exponential polynomials and numbers.

Definition 12. The *harmonic-exponential polynomials* and *numbers* are, respectively, given by the following equations;

$$\phi_n^h(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\} H_k x^k \tag{57}$$

and

$$\phi_n^h := \phi_n^h(1) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} H_k. \tag{58}$$

The first few harmonic-exponential polynomials are,

$$\begin{array}{|l|} \hline \phi_0^h(x) = 0 \\ \hline \phi_1^h(x) = x \\ \hline \phi_2^h(x) = x + \frac{3}{2}x^2 \\ \hline \phi_3^h(x) = x + \frac{9}{2}x^2 + \frac{11}{6}x^3 \\ \hline \phi_4^h(x) = x + \frac{21}{2}x^2 + 11x^3 + \frac{25}{12}x^4 \\ \hline \phi_5^h(x) = x + \frac{45}{2}x^2 + \frac{275}{6}x^3 + \frac{250}{12}x^4 + \frac{137}{60}x^5 \\ \hline \end{array}. \tag{59}$$

And harmonic-exponential numbers are,

$$\phi_0^h = 0,\ \phi_1^h = 1,\ \phi_2^h = \frac{5}{2},\ \phi_3^h = \frac{22}{3},\ \phi_4^h = \frac{295}{12},\ \phi_5^h = \frac{1849}{20}. \tag{60}$$

We can extend the relation (19) for harmonic types of these polynomials as

$$\mathcal{F}_n^h(z) = \int_0^\infty \phi_n^h(z\lambda)\, e^{-\lambda} d\lambda. \tag{61}$$

3. Hyperharmonic-Geometric and Exponential Polynomials

In this section we generalize almost all of our results which we obtained in the previous section.

Let us take g in (2) as

$$g(x) = \sum_{n=1}^{\infty} H_n^{(\alpha)} x^n = -\frac{\ln(1-x)}{(1-x)^{\alpha}}.$$

The next proposition gives a formula for the kth derivatives of $g(x)$.

Proposition 13. *We have*

$$\frac{d^k}{dx^k}\left\{-\frac{\ln(1-x)}{(1-x)^{\alpha}}\right\} = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \frac{1}{(1-x)^{\alpha+k}} \left(H_{k+\alpha-1} - H_{\alpha-1} - \ln(1-x)\right). \quad (62)$$

Proof. This follows by induction on k. □

Hence we have

$$g^{(k)}(x) = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \frac{1}{(1-x)^{\alpha+k}} \left(H_{k+\alpha-1} - H_{\alpha-1} - \ln(1-x)\right) \quad (63)$$

and

$$g^{(k)}(0) = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \left(H_{k+\alpha-1} - H_{\alpha-1}\right). \quad (64)$$

In the light of Equation (22) we can state (64) simply as

$$g^{(k)}(0) = k! H_k^{(\alpha)}. \quad (65)$$

Now we are ready to prove the following proposition.

Proposition 14. *Let an entire function f be given. Then we have the following transformation formula:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} H_n^{(\alpha)} f(n) x^n \\
&\quad = \frac{1}{(1-x)^{\alpha}} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{ n \atop k \right\} k! H_k^{(\alpha)} \left(\frac{x}{1-x}\right)^k \\
&\qquad - \frac{\ln(1-x)}{(1-x)^{\alpha}} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left\{ n \atop k \right\} \Gamma(k+\alpha) \left(\frac{x}{1-x}\right)^k.
\end{aligned} \quad (66)$$

Proof. Invoking (63) and (65) in (2) gives the statement. □

The second part of the right-hand side of the equation (66) contains the generalized geometric polynomials which are given by (18).

The first part of the right-hand side of the equation (66) also contains a new family of polynomials which is a generalization of (31). We refer to this new family as the *hyperharmonic-geometric polynomials* and define them by

$$\mathcal{F}^h_{n,\alpha}(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! H_k^{(\alpha)} x^k. \tag{67}$$

The first few hyperharmonic-geometric polynomials are:

$\mathcal{F}^h_{n,\alpha}(x)$	$\alpha = 2$
$n = 0$	0
$n = 1$	x
$n = 2$	$x + 5x^2$
$n = 3$	$x + 15x^2 + 26x^3$
$n = 4$	$x + 35x^2 + 156x^3 + 154x^4$
$n = 5$	$x + 75x^2 + 650x^3 + 1540x^4 + 1044x^5$

, (68)

and

$\mathcal{F}^h_{n,\alpha}(x)$	$\alpha = 3$
$n = 0$	0
$n = 1$	x
$n = 2$	$x + 7x^2$
$n = 3$	$x + 21x^2 + 47x^3$
$n = 4$	$x + 49x^2 + 282x^3 + 342x^4$
$n = 5$	$x + 105x^2 + 1175x^3 + 3420x^4 + 2754x^5$

(69)

(note that $\mathcal{F}^h_{n,1}(x) = \mathcal{F}^h_n(x)$).

With the help of these notations we can write the transformation formula (66) simply as:

$$\begin{aligned}&\sum_{n=0}^{\infty} H_n^{(\alpha)} f(n) x^n \\ &\qquad = \frac{1}{(1-x)^{\alpha}} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left[\mathcal{F}^h_{n,\alpha}\left(\frac{x}{1-x}\right) - \mathcal{F}_{n,\alpha}\left(\frac{x}{1-x}\right) \ln(1-x) \right].\end{aligned} \tag{70}$$

Now we give a simple formula as a corollary of Proposition 14.

Corollary 15. *We have*

$$\sum_{n=1}^{\infty} n^m H_n^{(\alpha)} x^n = \frac{1}{(1-x)^{\alpha}} \left[\mathcal{F}^h_{m,\alpha}\left(\frac{x}{1-x}\right) - \mathcal{F}_{m,\alpha}\left(\frac{x}{1-x}\right) \ln(1-x) \right], \tag{71}$$

where m is a nonnegative integer.

Proof. Directly seen from the taking $f(x) = x^m$ in (70). $\square$

Remark 16. Formula (71) is also a generalization of the generating function of hyperharmonic numbers since the case $m = 0$ gives (24).

Equation (71) also makes it possible to get closed forms of some series related to hyperharmonic numbers, for instance the case $m = 1$ in (71) gives

$$\sum_{n=1}^{\infty} nH_n^{(\alpha)} x^n = \frac{x\left(1 - \alpha \ln(1-x)\right)}{(1-x)^{\alpha+1}} \tag{72}$$

where $\mathcal{F}_{1,\alpha}(x) = \alpha x$, $\mathcal{F}_{1,\alpha}^h(x) = x$.

Now we state a more general result, which extends (71) to multiple sums.

Proposition 17. *We have*

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{r=0}^{n}\binom{n+s-r}{s} r^m H_r^{(\alpha)}\right)x^n &= \sum_{n=1}^{\infty}\left(\sum_{0\le i\le i_1\le\cdots\le i_s\le n} i^m H_i^{(\alpha)}\right)x^n \\ &= \frac{1}{(1-x)^{\alpha+s+1}}\left[\mathcal{F}_{m,\alpha}^h\left(\frac{x}{1-x}\right) - \mathcal{F}_{m,\alpha}\left(\frac{x}{1-x}\right)\ln(1-x)\right] \qquad (73)\end{aligned}$$

where m and s are nonnegative integers.

Proof. The proof follows the same steps of Proposition 6 by considering equation (71). $\square$

Corollary 18. *We have*

$$\sum_{n=0}^{\infty} H_n^{(s)} x^n = -\frac{\ln(1-x)}{(1-x)^s}.$$

Proof. Letting $m = 0$ and considering $\mathcal{F}_{0,\alpha}^h(x) = 0$ and $\mathcal{F}_{0,\alpha}(x) = 1$ gives statement. $\square$

For $m = 1$ we get the following corollary.

Corollary 19. *We have*

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\sum_{r=0}^{n}\binom{n+s-r}{s} r H_r^{(\alpha)}\right)x^n \qquad (74)\\ &= \sum_{n=1}^{\infty}\left(\sum_{0\le i\le i_1\le\cdots\le i_s\le n} i H_i^{(\alpha)}\right)x^n = \frac{x\left(1-\alpha\ln(1-x)\right)}{(1-x)^{\alpha+s+2}}.\end{aligned}$$

As an example, the case $s = 0$ gives

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} kH_k^{(\alpha)}\right)x^n = \frac{x\left(1-\alpha\ln\left(1-x\right)\right)}{\left(1-x\right)^{\alpha+2}}. \tag{75}$$

Now we give an interesting formula.

Corollary 20. *We have*

$$\sum_{n=1}^{\infty}\left(1^m H_1^{(\alpha)} + 2^m H_2^{(\alpha)} + \cdots + n^m H_n^{(\alpha)}\right)x^n$$
$$= \frac{1}{\left(1-x\right)^{\alpha+1}}\left[\mathcal{F}_{m,\alpha}^{h}\left(\frac{x}{1-x}\right) - \mathcal{F}_{m,\alpha}\left(\frac{x}{1-x}\right)\ln\left(1-x\right)\right]. \tag{76}$$

Remark 21. If we set $f(x)$ as an arbitrary polynomial of order m, such as

$$f(x) = p_m x^m + p_{m-1}x^{m-1} + \cdots + p_1 x + p_0 \tag{77}$$

where $p_0, p_1, \cdots, p_{m-1}, p_m$ are any complex numbers, instead of $f(x) = x^m$ in (70) we obtain the following general formula:

$$\sum_{n=0}^{\infty}\left(p_m n^m + p_{m-1}n^{m-1} + \cdots + p_1 n + p_0\right)H_n^{(\alpha)}x^n \tag{78}$$
$$= \frac{1}{\left(1-x\right)^{\alpha}}\sum_{k=o}^{m} p_k\left\{\mathcal{F}_k^{h}\left(\frac{x}{1-x}\right) - \mathcal{F}_k\left(\frac{x}{1-x}\right)\ln\left(1-x\right)\right\}.$$

Restricting (77) one can obtain several closed forms of hyperharmonic number series in a similar fashion to what we did after Remark 9.

3.1. Some Results Using the Operator (xD)

If we set g as the generating function of hyperharmonic numbers in (52), then we get

$$\left(xD\right)^m\left(-\frac{\ln\left(1-x\right)}{\left(1-x\right)^{\alpha}}\right) = \frac{1}{\left(1-x\right)^{\alpha}}\left[\mathcal{F}_{m,\alpha}^{h}\left(\frac{x}{1-x}\right) - \mathcal{F}_{m,\alpha}\left(\frac{x}{1-x}\right)\ln\left(1-x\right)\right].$$

On the other hand, using (51) we have

$$\left(xD\right)^m\left(\sum_{n=1}^{\infty} H_n^{(\alpha)}x^n\right) = \sum_{n=1}^{\infty} H_n^{(\alpha)}n^m x^n.$$

Collecting these two results again gives Equation (71), i.e.,

$$\sum_{n=1}^{\infty} H_n^{(\alpha)}n^m x^n = \frac{1}{\left(1-x\right)^{\alpha}}\left[\mathcal{F}_{m,\alpha}^{h}\left(\frac{x}{1-x}\right) - \mathcal{F}_{m,\alpha}\left(\frac{x}{1-x}\right)\ln\left(1-x\right)\right].$$

3.2. Hyperharmonic-Geometric Numbers

Definition 22. The *hyperharmonic-geometric numbers* $\mathcal{F}_{n,\alpha}^{h}$ are obtained by setting $x = 1$ in (67) as:

$$\mathcal{F}_{n,\alpha}^{h} := \mathcal{F}_{n,\alpha}^{h}\left(1\right) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! H_k^{(\alpha)}. \tag{79}$$

The first few hyperharmonic-geometric numbers are:

$\mathcal{F}_{n,\alpha}^{h}$	$\alpha = 2$	$\alpha = 3$
$n = 0$	0	0
$n = 1$	1	1
$n = 2$	6	8
$n = 3$	42	69
$n = 4$	346	674
$n = 5$	3310	7455

(80)

(note that $\mathcal{F}_{n,1}^{h} = \mathcal{F}_{n}^{h}$).

Remark 23. Using (26) , which is a relation between hyperharmonic numbers and r-Stirling numbers of the first kind, we can write the hyperharmonic-geometric polynomials and numbers in terms of Stirling numbers as:

$$\mathcal{F}_{n,r}^{h}\left(x\right) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \left[{n+r \atop r+1} \right]_r x^k, \tag{81}$$

and

$$\mathcal{F}_{n,r}^{h} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \left[{n+r \atop r+1} \right]_r \tag{82}$$

respectively. One can easily see that the relations (81) and (82) are the generalizations of the relations (55) and (56) .

For the completeness of the work let us define a generalization of the exponential polynomials.

3.3. Hyperharmonic-Exponential Polynomials and Numbers

Definition 24. The *hyperharmonic-exponential polynomials* and *numbers* are defined, respectively, as

$$\phi_{n,\alpha}^{h}\left(x\right) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\} H_k^{(\alpha)} x^k, \tag{83}$$

and

$$\phi_{n,\alpha}^{h} := \phi_{n,\alpha}^{h}\left(1\right) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} H_k^{(\alpha)}. \tag{84}$$

The first few hyperharmonic-exponential polynomials are

$\phi^h_{n,\alpha}(x)$	$\alpha = 2$
$n = 0$	0
$n = 1$	x
$n = 2$	$x + \frac{5}{2}x^2$
$n = 3$	$x + \frac{15}{2}x^2 + \frac{13}{3}x^3$
$n = 4$	$x + \frac{35}{2}x^2 + 26x^3 + \frac{77}{12}x^4$
$n = 5$	$x + \frac{75}{2}x^2 + \frac{325}{3}x^3 + \frac{385}{6}x^4 + \frac{87}{10}x^5$

(85)

and

$\phi^h_{n,\alpha}(x)$	$\alpha = 3$
$n = 0$	0
$n = 1$	x
$n = 2$	$x + \frac{7}{2}x^2$
$n = 3$	$x + \frac{21}{2}x^2 + \frac{47}{6}x^3$
$n = 4$	$x + \frac{49}{2}x^2 + 47x^3 + \frac{57}{4}x^4$
$n = 5$	$x + \frac{105}{2}x^2 + \frac{1175}{6}x^3 + \frac{285}{2}x^4 + \frac{459}{20}x^5$

. (86)

The first few hyperharmonic-exponential numbers are

$\phi^h_{n,\alpha}(x)$	$\alpha = 2$	$\alpha = 3$
$n = 0$	0	0
$n = 1$	1	1
$n = 2$	$\frac{7}{2}$	$\frac{9}{2}$
$n = 3$	$\frac{77}{6}$	$\frac{58}{3}$
$n = 4$	$\frac{611}{12}$	$\frac{347}{4}$
$n = 5$	$\frac{2197}{10}$	$\frac{24887}{60}$

. (87)

We remark that the case $\alpha = 1$ gives $\phi^h_{n,1}(x) = \phi^h_n(x)$ and $\phi^h_{n,1} = \phi^h_n$.

For these new concepts we can generalize the relation (61) as

$$\mathcal{F}^h_{n,\alpha}(x) = \int_0^\infty \phi^h_{n,\alpha}(x\lambda)\, e^{-\lambda} d\lambda. \tag{88}$$

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