TWO THETA FUNCTION IDENTITIES OF RAMANUJAN AND REPRESENTATION OF A NUMBER AS A SUM OF THREE SQUARES AND AS A SUM OF THREE TRIANGULAR NUMBERS

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Abstract

Ramanujan recorded two beautiful theta function identities on p. 310 of his second notebook. Employing Ramanujan's identities, we deduce several results on the number of representations of a number as a sum of three squares and as a sum of three triangular numbers previously found by Hirschhorn and Sellers with a different approach.

1. Introduction

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2},$$

where $|ab| < 1$ and n is an integer. If we set $a = qe^{2iz}$, $b = qe^{-2iz}$, and $q = e^{\pi i \tau}$, where z is complex and $\mathrm{Im}(\tau) > 0$, then $f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta functions in its standard notation [10, p. 464]. Throughout the paper, it is assumed that $|q| < 1$.

Three special cases of $f(a, b)$ are [5, Entry 22]

$$\varphi(q) := f(q, q) = \sum_{k=-\infty}^{\infty} q^{k^2} = \frac{(-q;\ q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (-q^2; q^2)_\infty}, \tag{1}$$

$$\psi(q) := f(q, q^3) = \frac{1}{2} f(1, q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{2}$$

$$f(-q) := f(-q, -q^2) = \sum_{k=0}^{\infty} (-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty} (-1)^k q^{k(3k+1)/2} = (q; q)_\infty, \tag{3}$$

where, as customary, we define

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k),$$

and the product representations in (1)–(3) arise from Jacobi's famous triple product identity [5, Entry 19]

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

Now, if $r_3(n)$ and $t_3(n)$ denote the number of representations of n as a sum of three integer squares and as a sum of three triangular numbers, respectively, then

$$\sum_{n \geq 0} r_3(n) q^n = \left(\sum_{k=-\infty}^{\infty} q^{k^2} \right)^3 = \varphi^3(q) \tag{4}$$

and

$$\sum_{n \geq 0} t_3(n) q^n = \left(\sum_{k=0}^{\infty} q^{k(k+1)/2} \right)^3 = \psi^3(q). \tag{5}$$

In [7] and [8], M.D. Hirschhorn and J.A. Sellers found many arithmetic properties of $r_3(n)$ and $t_3(n)$ by manipulating q-series and theta functions. Their main identities are given in the following two theorems.

Theorem 1. *We have*

$$\sum_{n \geq 0} r_3(27n + 9) q^n = 5 \sum_{n \geq 0} r_3(3n + 1) q^n, \tag{6}$$

$$\sum_{n \geq 0} r_3(27n + 18) q^n = 3 \sum_{n \geq 0} r_3(3n + 2) q^n, \tag{7}$$

$$\sum_{n \geq 0} r_3(27n) q^n = 4 \sum_{n \geq 0} r_3(3n) q^n - 3 \sum_{n \geq 0} r_3(n) q^{3n}. \tag{8}$$

Theorem 2. *We have*

$$\sum_{n\geq 0} t_3(27n+3)q^n = 4\sum_{n\geq 0} t_3(3n)q^n - 3\sum_{n\geq 0} t_3(n)q^{3n+1}, \tag{9}$$

$$\sum_{n\geq 0} t_3(27n+12)q^n = 3\sum_{n\geq 0} t_3(3n+1)q^n, \tag{10}$$

$$\sum_{n\geq 0} t_3(27n+21)q^n = 5\sum_{n\geq 0} t_3(3n+2)q^n. \tag{11}$$

On p. 310 of his second notebook, S. Ramanujan [9] recorded the following two beautiful theta function identities. If $\phi(q)$, $\psi(q)$, and $f(-q)$ are as defined in (1)–(3), then

$$\frac{\varphi^3(q^{1/3})}{\varphi(q)} = \frac{\varphi^3(q)}{\varphi(q^3)} + 6q^{1/3}\frac{f^3(q^3)}{f(q)} + 12q^{2/3}\frac{f^3(-q^6)}{f(-q^2)} \tag{12}$$

and

$$\frac{\psi^3(q^{1/3})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{1/3}\frac{f^3(-q^3)}{f(-q)} + 3q^{2/3}\frac{f^3(-q^6)}{f(-q^2)}. \tag{13}$$

The first proofs of (12) and (13) were given by B. C. Berndt [6, p. 185, Entry 33] by using Ramanujan's modular equations and a method of parameterizations. N. D. Baruah and J. Bora [3] gave alternative proofs by using other theta function identities of Ramanujan.

The purpose of this paper is to prove Theorem 1 and Theorem 2 by using (12), (13). In the next section, we present some simple properties of theta functions which will be used in the subsequent sections. In Section 3, we prove Theorem 1 and in Section 4, we prove Theorem 2.

2. Preliminary Results

In this section, we state some results which will be used to derive our results related to $r_3(n)$ and $t_3(n)$.

Lemma 3. [5, p. 39, Entries 24(ii)–(iv)] *If* $\chi(q) := (-q;q^2)_\infty$, *then*

$$f^3(-q) = \varphi^2(-q)\psi(q),$$

$$\chi(q)\chi(-q) = \chi(-q^2),$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \left(\frac{\varphi(q)}{\psi(-q)}\right)^{1/3} = \frac{\varphi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)}. \tag{14}$$

Lemma 4. [4, Lemma 2.9] *We have*

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \quad \psi(q) = \frac{f_2^2}{f_1} \quad \varphi(-q) = \frac{f_1^2}{f_2}, \quad \psi(-q) = \frac{f_1 f_4}{f_2},$$
$$f(q) = \frac{f_2^3}{f_1 f_4}, \quad \chi(q) = \frac{f_2^2}{f_1 f_4}, \quad \text{and} \quad \chi(-q) = \frac{f_1}{f_2}, \tag{15}$$

where $f_n := f(-q^n)$, and this notation will be used throughout the sequel.

Lemma 5. [5, p. 49, Corollaries (i) and (ii)] *We have*

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}), \tag{16}$$
$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{17}$$

Lemma 6. [5, p. 51, Example (v)] *We have*

$$f(q, q^5) = \psi(-q^3)\chi(q). \tag{18}$$

Lemma 7. [5, p. 350, Eq. (2.3)] *We have*

$$f(q, q^2) = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{19}$$

Lemma 8. [3] *We have*

$$1 + \frac{\chi^9(-q^3)}{q\chi^3(-q)} = \frac{\psi^4(q)}{q\psi^4(q^3)}. \tag{20}$$

Lemma 9. [2, Eq. (53)] *We have*

$$\varphi^4(q) - \varphi^4(q^3) = 8q\varphi^2(-q^6)\frac{\chi^2(q)\chi(-q^2)\psi(-q^3)\psi(q^6)}{\chi(-q)}. \tag{21}$$

Lemma 10. [1] *If*

$$a(q) = \varphi(-q^3), \ b(q) = \frac{f_1 f_6^2}{f_2 f_3},$$

then

$$a^3(q) - 8qb^3(q) = \frac{\varphi^4(-q)}{\varphi(-q^3)}. \tag{22}$$

Lemma 11. *If*

$$P(q) = \frac{\varphi(q^3)}{\chi(q)} \quad \text{and} \quad Q(q) = \psi(-q^3),$$

then

$$P^3(q) - qQ^3(q) = \frac{\psi^4(-q)}{\psi(-q^3)}. \tag{23}$$

Proof. Employing (14), we have

$$\begin{aligned} P^3(q) - qQ^3(q) &= \frac{\varphi^3(q^3)}{\chi^3(q)} - q\psi^3(-q^3) \\ &= \frac{\chi^9(q^3)\psi^3(-q^3)}{\chi^3(q)} - q\psi^3(-q^3) \\ &= q\psi^3(-q^3)\left(\frac{\chi^9(q^3)}{q\chi^3(q)} - 1\right). \end{aligned} \tag{24}$$

Employing (20), with q replaced by $-q$, in (24), we easily arrive at (23). □

3. Proof of Theorem 1

Replacing q by q^3 in (12), we find that

$$\begin{aligned} \varphi^3(q) &= \frac{\varphi^4(q^3)}{\varphi(q^9)} + 6q\frac{f^3(q^9)\varphi(q^3)}{f(q^3)} + 12q^2\frac{f^3(-q^{18})\varphi(q^3)}{f(-q^6)} \\ &= A(q^3) + 6qB(q^3) + 12q^2C(q^3), \end{aligned} \tag{25}$$

where

$$A(q) = \frac{\varphi^4(q)}{\varphi(q^3)}, \; B(q) = \frac{f^3(q^3)\varphi(q)}{f(q)}, \text{ and } C(q) = \frac{f^3(-q^6)\varphi(q)}{f(-q^2)}.$$

Since

$$\varphi^3(q) = \sum_{n\geq 0} r_3(n)q^n,$$

we readily derive from (25) that

$$\sum_{n\geq 0} r_3(3n)q^n = A(q) = \frac{\varphi^4(q)}{\varphi(q^3)}, \tag{26}$$

$$\sum_{n\geq 0} r_3(3n+1)q^n = 6B(q) = 6\frac{f^3(q^3)\varphi(q)}{f(q)} = 6\frac{f_6^9 f_2^2}{f_1 f_3^3 f_4 f_{12}^3}, \tag{27}$$

$$\sum_{n\geq 0} r_3(3n+2)q^n = 12C(q) = 12\frac{f^3(-q^6)\varphi(q)}{f(-q^2)} = 12\frac{f_2^4 f_6^3}{f_1^2 f_4^2}. \tag{28}$$

Using (25), (16), and (18) in (26), we find that

$$\begin{aligned}\sum_{n\geq 0} r_3(3n)q^n &= \frac{\varphi^4(q)}{\varphi(q^3)} = \frac{\varphi^3(q)}{\varphi(q^3)}\ \varphi(q) \\ &= \left(\frac{\varphi^3(q^3)}{\varphi(q^9)} + 6q\frac{f^3(q^9)}{f(q^3)} + 12q^2\frac{f^3(-q^{18}}{f(-q^6)}\right)\left(\varphi(q^9) + 2qf(q^3, q^{15})\right) \\ &= \left(\frac{\varphi^3(q^3)}{\varphi(q^9)} + 6q\frac{f^3(q^9)}{f(q^3)} + 12q^2\frac{f^3(-q^{18}}{f(-q^6)}\right)\left(\varphi(q^9) + 2q\chi(q^3)\psi(-q^9)\right).\end{aligned} \tag{29}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} from both sides of (29), we obtain

$$\sum_{n\geq 0} r_3(9n)q^n = \varphi^3(q) + 24q\chi(q)\psi(-q^3)\frac{f^3(-q^6)}{f(-q^2)}, \tag{30}$$

$$\sum_{n\geq 0} r_3(9n+3)q^n = 2\chi(q)\psi(-q^3)\frac{\varphi^3(q)}{\varphi(q^3)} + 6\varphi(q^3)\frac{f^3(q^3)}{f(q)},$$

and

$$\sum_{n\geq 0} r_3(9n+6)q^n = 12\chi(q)\psi(-q^3)\frac{f^3(q^3)}{f(q)} + 12\varphi(q^3)\frac{f^3(-q^6)}{f(-q^2)},$$

respectively.

Next, employing (14), we rewrite (30) as

$$\sum_{n\geq 0} r_3(9n)q^n = \varphi^3(q) + 24q\frac{f^3(-q^6)\psi(-q^3)}{\psi(-q)}. \tag{31}$$

Now, replacing q by $-q$ in (17), and then using (19), we obtain

$$\begin{aligned}\psi(-q) &= f(-q^3, q^6) - q\psi(-q^9) \\ &= \frac{\varphi(q^9)}{\chi(q^3)} - q\psi(-q^9) \\ &= P(q^3) - qQ(q^3),\end{aligned}$$

where $P(q)$ and $Q(q)$ are as defined in Lemma 11. Therefore,

$$\frac{1}{\psi(-q)} = \frac{1}{P(q^3) - qQ(q^3)} = \frac{P^2(q^3) + qP(q^3)Q(q^3) + q^2Q^2(q^3)}{P^3(q^3) - q^3Q^3(q^3)}. \tag{32}$$

Employing (25) and (32) in (31), we find that

$$\begin{aligned}\sum_{n\geq 0} r_3(9n)q^n &= A(q^3)+6qB(q^3)+12q^2C(q^3)+24qf^3(-q^6)\psi(-q^3)\\ &\quad\times\left(\frac{P^2(q^3)+qP(q^3)Q(q^3)+q^2Q^2(q^3)}{P^3(q^3)-q^3Q^3(q^3)}\right)\\ &= A(q^3)+24q^3\frac{f^3(-q^6)\psi(-q^3)Q^2(q^3)}{P^3(q^3)-q^3Q^3(q^3)}\\ &\quad+q\left(6B(q^3)+24\frac{f^3(-q^6)\psi(-q^3)P^2(q^3)}{P^3(q^3)-q^3Q^3(q^3)}\right)\\ &\quad+q^2\left(12C(q^3)+24\frac{f^3(-q^6)\psi(-q^3)P(q^3)Q(q^3)}{P^3(q^3)-q^3Q^3(q^3)}\right). \qquad (33)\end{aligned}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} from both sides of (33), we obtain

$$\sum_{n\geq 0} r_3(27n)q^n = A(q)+24q\frac{f^3(-q^2)\psi(-q)Q^2(q)}{P^3(q)-qQ^3(q)}, \qquad (34)$$

$$\sum_{n\geq 0} r_3(27n+9)q^n = 6B(q)+24\frac{f^3(-q^2)\psi(-q)P^2(q)}{P^3(q)-qQ^3(q)}, \qquad (35)$$

and

$$\sum_{n\geq 0} r_3(27n+18)q^n = 12C(q)+24\frac{f^3(-q^2)\psi(-q)P(q)Q(q)}{P^3(q)-qQ^3(q)}, \qquad (36)$$

respectively.

Employing Lemma 11 and Lemma 15 in (35), we find that

$$\begin{aligned}\sum_{n\geq 0} r_3(27n+9)q^n &= 6\frac{f^3(q^3)\varphi(q)}{f(q)}+24\frac{f^3(-q^2)\psi(-q^3)\varphi^2(q^3)}{\psi^3(-q)\chi^2(q)}\\ &= 6\frac{f_6^9f_2^2}{f_1f_3^2f_4f_{12}^3}+24\frac{f_6^9f_2^2}{f_1f_3^2f_4f_{12}^3}=30\frac{f_6^9f_2^2}{f_1f_3^2f_4f_{12}^3}. \qquad (37)\end{aligned}$$

Using (27) in (37), we readily arrive at (6).

Next, employing Lemma 11, Lemma 15, and (28) in (36), we obtain

$$\begin{aligned}\sum_{n\geq 0} r_3(27n+18)q^n &= 12C(q)+24\frac{f^3(-q^2)\psi(-q)P(q)Q(q)}{P^3(q)-qQ^3(q)}\\ &= 12\frac{f^3(-q^6)\varphi(q)}{f(-q^2)}+24\frac{f^3(-q^2)\psi(-q)\psi^2(-q^3)\varphi(q^3)}{\chi(q)\psi^4(-q)}\\ &= 36\frac{f_2^4f_6^3}{f_1^2f_4^2}=3\sum_{n\geq 0} r_3(3n+2)q^n,\end{aligned}$$

to complete the proof of (7).

Now, from (34), Lemma 3, and (21), we have

$$\begin{aligned}\sum_{n\geq 0} r_3(27n)q^n &= \frac{\varphi^4(q)}{\varphi(q^3)} + 24q\frac{f^3(-q^2)\psi^3(-q^3)}{\psi^3(-q)}\\ &= \frac{\varphi^4(q)}{\varphi(q^3)} + 24q\frac{\chi^2(q)\chi(-q^2)\varphi^2(-q^6)\psi(-q^3)\psi(q^6)}{\varphi(q^3)\chi(-q)}\\ &= \frac{\varphi^4(q)}{\varphi(q^3)} + 3\frac{\varphi^4(q)-\varphi^4(q^3)}{\varphi(q^3)} = 4\frac{\varphi^4(q)}{\varphi(q^3)} - 3\varphi^3(q^3).\end{aligned}$$

Employing (26) and (4) in the above, we arrive at (8) to finish the proof.

4. Proof of Theorem 2

Replacing q by q^3 in (13), we find that

$$\begin{aligned}\psi^3(q) &= \frac{\psi^4(q^3)}{\psi(q^9)} + 3q\frac{f^3(-q^9)\psi(q^3)}{f(-q^3)} + 3q^2\frac{f^3(-q^{18})\psi(q^3)}{f(-q^6)}\\ &= L(q^3) + 3qM(q^3) + 3q^2N(q^3), \qquad (38)\end{aligned}$$

where

$$L(q) = \frac{\psi^4(q)}{\psi(q^3)},\ M(q) = \frac{f^3(-q^3)\psi(q)}{f(-q)},\ \text{and } N(q) = \frac{f^3(-q^6)\psi(q)}{f(-q^2)}.$$

Since $\psi^3(q) = \sum_{n\geq 0} t_3(n)q^n$, we deduce from (38) that

$$\sum_{n\geq 0} t_3(3n)q^n = L(q) = \frac{\psi^4(q)}{\psi(q^3)}, \qquad (39)$$

$$\sum_{n\geq 0} t_3(3n+1)q^n = 3M(q) = 3\frac{f^3(-q^3)\psi(q)}{f(-q)} = 3\frac{f_3^3 f_2^2}{f_1^2}, \qquad (40)$$

$$\sum_{n\geq 0} t_3(3n+2)q^n = 3N(q) = 3\frac{f^3(-q^6)\psi(q)}{f(-q^2)} = 3\frac{f_6^3 f_2}{f_1}, \qquad (41)$$

where we have also used Lemma 4.

Employing (13), with q replaced by $-q$, (17), and (19) in (39), we find that

$$\begin{aligned}\sum_{n\geq 0} t_3(3n)q^n &= \left(\frac{\psi^3(q^3)}{\psi(q^9)} + 3q\frac{f^3(-q^9)}{f(-q^3)} + 3q^2\frac{f^3(-q^{18})}{f(-q^6)}\right)\left(f(q^3,q^6)+q\psi(q^9)\right)\\ &= \left(\frac{\psi^3(q^3)}{\psi(q^9)} + 3q\frac{f^3(-q^9)}{f(-q^3)} + 3q^2\frac{f^3(-q^{18})}{f(-q^6)}\right)\left(\frac{\varphi(-q^9)}{\chi(-q^3)} + q\psi(q^9)\right). \qquad (42)\end{aligned}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} from both sides of (42), we obtain

$$\sum_{n\geq 0} t_3(9n)q^n = \frac{\psi^3(q)\varphi(-q^3)}{\psi(q^3)\chi(-q)} + 3q\frac{f^3(-q^6)\psi(q^3)}{f(-q^2)}, \tag{43}$$

$$\sum_{n\geq 0} t_3(9n+3)q^n = \psi^3(q) + 3\frac{\chi(-q^3)f^4(-q^3)}{\varphi(-q)}, \tag{44}$$

and

$$\sum_{n\geq 0} t_3(9n+6)q^n = 3\left(\frac{f^3(-q^3)}{f(-q)}\psi(q^3) + \frac{f^3(-q^6)}{f(-q^2)}\frac{\varphi(-q^3)}{\chi(-q)}\right) = 6\frac{f_6^2 f_3^2}{f_1}, \tag{45}$$

respectively, where in the last equality we used the identities in (15).

Now, replacing q by $-q$ in (16), we have

$$\varphi(-q) = a(q^3) - 2qb(q^3),$$

where $a(q)$ and $b(q)$ are as defined in Lemma 10.
Therefore, we have

$$\frac{1}{\varphi(-q)} = \frac{1}{a(q^3) - 2qb(q^3)} = \frac{a^2(q^3) + 2qa(q^3)b(q^3) + 4q^2b^2(q^3)}{a^3(q^3) - 8q^3b(q^3)}. \tag{46}$$

Employing (38) and (46) in (44), we find that

$$\begin{aligned}\sum_{n\geq 0} t_3(9n+3)q^n &= L(q^3) + 3qM(q^3) + 3q^2N(q^3) + 3\chi(-q^3)f^4(-q^3)\\ &\qquad \times \frac{a^2(q^3) + 2qa(q^3)b(q^3) + 4q^2b^2(q^3)}{a^3(q^3) - 8q^3b(q^3))}\\ &= \left(L(q^3) + 3\frac{a^2(q^3)\chi(-q^3)f^4(-q^3)}{a^3(q^3) - 8q^3b(q^3)}\right)\\ &\quad + 3q\left(M(q^3) + 2\frac{a(q^3)b(b^3)\chi(-q^3)f^4(-q^3)}{a^3(q^3) - 8q^3b(q^3)}\right)\\ &\quad + 3q^2\left(N(q^3) + 4\frac{b^2(q^3)\chi(-q^3)f^4(-q^3)}{a^3(q^3) - 8q^3b(q^3)}\right).\end{aligned} \tag{47}$$

Comparing the terms involving q^{3n} on both sides of the above identity, and then applying Lemma 10, we deduce that

$$\begin{aligned}\sum_{n\geq 0} t_3(27n+3)q^n &= L(q) + 3\frac{a^2(q)\chi(-q)f^4(-q)}{a^3(q) - 8q^3(q)})\\ &= \frac{\psi^4(q)}{\psi(q^3)} + 3\frac{\varphi^3(-q^3)\chi(-q)f^4(-q)}{\varphi^4(-q)}.\end{aligned}$$

Employing (14), the above can be rewritten as

$$\begin{aligned}\sum_{n\geq 0} t_3(27n+3)q^n &= \frac{\psi^4(q)}{\psi(q^3)} + 3\frac{\varphi^3(-q^3)}{\chi^3(-q)}\\ &= \frac{\psi^4(q)}{\psi(q^3)} + 3\frac{\chi^9(-q^3)\psi^3(q^3)}{\chi^3(-q)}. \qquad (48)\end{aligned}$$

Using (20) in (48), we obtain

$$\sum_{n\geq 0} t_3(27n+3)q^n = \frac{\psi^4(q)}{\psi(q^3)} + 3\frac{\psi^4(q) - q\psi^4(q^3)}{\psi(q^3)} = 4\frac{\psi^4(q)}{\psi(q^3)} - 3q\psi^3(q^3),$$

to arrive at (9), with further aids from (39) and (5).

Next, comparing the terms involving q^{3n+1} on both sides of (47), and then applying Lemma 10 and (15), we find that

$$\begin{aligned}\sum_{n\geq 0} t_3(27n+12)q^n &= 3M(q) + 6\frac{a(q)b(q)\chi(-q)f^4(-q)}{a^3(q) - 8qb^3(q)}\\ &= 9\frac{f_3^3 f_2^2}{f_1^2}. \qquad (49)\end{aligned}$$

The identity (10) now follows from (49) and (40).

Finally, comparing the terms involving q^{3n+2} on both sides of (47), and then applying Lemma 10 and (15), we obtain

$$\begin{aligned}\sum_{n\geq 0} t_3(27n+21)q^n &= 3N(q) + 12\frac{b^2(q)\chi(-q)f^4(-q)}{a^3(q) - 8q^3(q)}\\ &= 15\frac{f_6^3 f_2}{f_1},\end{aligned}$$

and then with the aid of (41), the identity (11) follows easily.

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