ARITHMETIC PROGRESSIONS IN THE POLYGONAL NUMBERS

Kenneth A. Brown
Dept. of Mathematics, University of South Carolina, Columbia, South Carolina
brownka5@mailbox.sc.edu

Scott M. Dunn
Dept. of Mathematics, University of South Carolina, Columbia, South Carolina
scottmdunn@gmail.com

Joshua Harrington
Dept. of Mathematics, University of South Carolina, Columbia, South Carolina
harrinjs@mailbox.sc.edu

Received: 3/1/12, Accepted: 7/1/12, Published: 7/13/12

Abstract

In this paper, we investigate arithmetic progressions in the polygonal numbers with a fixed number of sides. We first show that four-term arithmetic progressions cannot exist. We then describe explicitly how to find all three-term arithmetic progressions. Finally, we show that not only are there infinitely many three-term arithmetic progressions, but that there are infinitely many three-term arithmetic progressions starting with an arbitrary polygonal number. Special attention is paid to the case of squares and triangular numbers.

1. Introduction

We recall that an arithmetic progression with a common difference d is a sequence of numbers, finite or infinite, such that the difference of any two consecutive terms is a constant d. Throughout this paper, let s be a fixed integer with $s \geq 3$. We will use the notation $P_s(n)$ to represent the n-th s-gonal number – that is, the number of points that are needed to create a regular s-gon with each side being of length $n-1$. See Figure 1. This number is given by $P_s(n) = (s/2-1)n^2 - (s/2-2)n$.

We will show that four-term arithmetic progressions with common difference $d \neq 0$ in the polygonal numbers do not exist. We then show that not only are there infinitely many three-term arithmetic progressions with common difference $d > 0$, but that there are infinitely many such progressions starting with an arbitrary

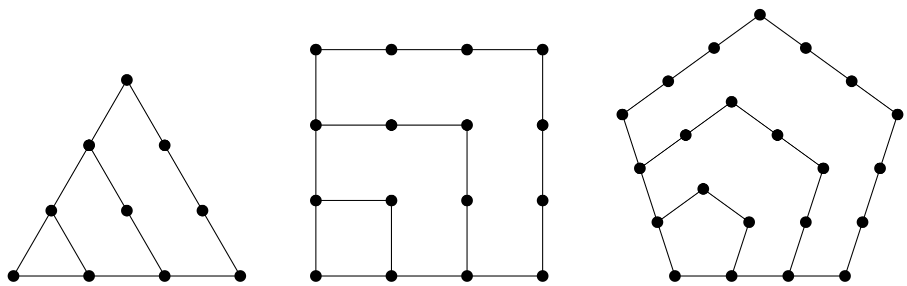

Figure 1: Examples of Polygonal Numbers for $s = 3, 4, 5$ and $n = 1, 2, 3, 4$.

polygonal number. Finally, we describe explicitly how to find all such three-term arithmetic progressions with common difference $d > 0$.

A natural question that arises from the results in this paper is to consider arithmetic progressions in the polyhedral numbers and then to generalize this to analyzing arithmetic progressions in the figurative numbers.

For a more general problem one could ask about arithmetic progressions in the sequence $f(n)$ for positive integers n and an arbitrary integer polynomial $f(x)$. This appears to be far from trivial, however. In particular, we note that if $f(x) = x^3$, then finding three-term arithmetic progressions with common difference $d \neq 0$ in $\{f(n) : n \in \mathbb{Z}^+\}$ would amount to solving the Diophantine equation $A^3 + C^3 = 2B^3$ in positive integers $A < B < C$. That there are no such three-term arithmetic progressions for third-powers follows then from [4, Theorem 3, p. 126]. On the other hand, if instead $f(x) = x^3 - x$, then the numbers $f(1) = 0$, $f(4) = 60$ and $f(5) = 120$ form a three-term arithmetic progression with common difference $d = 60$.

2. Four-Term Arithmetic Progressions

We first show that four-term arithmetic progressions with a common difference $d \neq 0$ cannot occur in the polygonal numbers. To do this, we will reference the following result from [4, pages 21–22] and [5, page 75]:

Theorem. (Mordell 1969; Sierpiński 1964) *There cannot be four squares in arithmetic progression with common difference $d \neq 0$.*

Using this, we have the following result.

Theorem 1. *Let s be a fixed integer with $s \geq 3$ Then there cannot be four s-gonal numbers in arithmetic progression with common difference $d \neq 0$.*

Proof. Let s be a fixed integer with $s \geq 3$. By way of contradiction, suppose that there is a four-term arithmetic progression with common difference $d \neq 0$ in the s-gonal numbers. Then there exists positive integers n, a, b, and c satisfying

$$P_s(a) - P_s(n) = P_s(b) - P_s(a) = P_s(c) - P_s(b) = d \neq 0.$$

First, we consider any two adjacent terms in the arithmetic progression, say $P_s(n)$ and $P_s(a)$. We have that

$$P_s(a) = (s/2-1)a^2 - (s/2-2)a \quad \text{and} \quad P_s(n) = (s/2-1)n^2 - (s/2-2)n.$$

By assumption, we also have that $d = P_s(a) - P_s(n)$, so that

$$2d = 2P_s(a) - 2P_s(n) = a^2(s-2) - a(s-4) - n^2(s-2) + n(s-4).$$

We now consider $(2a(s-2)-(s-4))^2$ and $(2n(s-2)-(s-4))^2$. We see that

$$\begin{aligned}(2a(s-2)&-(s-4))^2 - (2n(s-2)-(s-4))^2\\ &= 4(s-2)\left(a^2(s-2) - a(s-4) - n^2(s-2) + n(s-4)\right)\\ &= 8(s-2)d.\end{aligned}$$

Similarly, we have that

$$\begin{aligned} &(2b(s-2)-(s-4))^2 - (2a(s-2)-(s-4))^2 = 8(s-2)d\\ \text{and}\ &(2c(s-2)-(s-4))^2 - (2b(s-2)-(s-4))^2 = 8(s-2)d.\end{aligned}$$

This contradicts Sierpiński's and Mordell's theorem, completing the proof. □

3. Three-Term Arithmetic Progressions

In order to examine three-term arithmetic progressions with common difference d in the polygonal numbers, we first prove a short lemma. To simplify the proof of the lemma, we will momentarily consider P_s to be a continuous function from $\mathbb{C}$ into $\mathbb{C}$.

Lemma 1. *Let s be a fixed integer with $s \geq 3$ and n, a, and b be complex numbers. Then $P_s(n)$, $P_s(a)$, and $P_s(b)$ satisfy $P_s(a) - P_s(n) = P_s(b) - P_s(a)$ if and only if*

$$N = 2(s-2)n - (s-4), \quad A = 2(s-2)a - (s-4), \quad \textit{and} \quad B = 2(s-2)b - (s-4)$$

satisfy the equation $B^2 - 2A^2 = -N^2$.

Proof. Let s be a fixed integer with $s \geq 3$. Suppose that n, a, and b are complex numbers such that $P_s(a) - P_s(n) = P_s(b) - P_s(a)$. It follows that

$$\begin{aligned}\left(\frac{s}{2}-1\right)a^2 - \left(\frac{s}{2}-2\right)a - \left(\frac{s}{2}-1\right)n^2 + \left(\frac{s}{2}-2\right)n&\\ = \left(\frac{s}{2}-1\right)b^2 - \left(\frac{s}{2}-2\right)b - \left(\frac{s}{2}-1\right)a^2 + \left(\frac{s}{2}-2\right)a.&\quad (1)\end{aligned}$$

Multiplying both sides of (1) by $8(s-2)$ and rearranging, we obtain that (1) is equivalent to $B^2 - 2A^2 = -N^2$, where N, A, and B are as defined in the statement of the lemma.

Since these steps work in reverse, the converse is immediate. This proves the lemma. $\square$

We get an immediate consequence of Lemma 1 if we revert to viewing P_s as a function from $\mathbb{N}$ into $\mathbb{N}$. If $P_s(n)$, $P_s(a)$, and $P_s(b)$ form a three-term arithmetic progression for positive integers n, a, and b with $n \le a \le b$, then $B^2 - 2A^2 = -N^2$ is satisfied for N, A, and B as given in Lemma 1. Conversely, every positive integer solution N, A, and B to $B^2 - 2A^2 = -N^2$ where

$$n = \frac{N + (s-4)}{2(s-2)}, \quad a = \frac{A + (s-4)}{2(s-2)}, \quad \text{and} \quad b = \frac{B + (s-4)}{2(s-2)}$$

are positive integers with $n \le a \le b$ gives us that $P_s(n)$, $P_s(a)$, and $P_s(b)$ form a three-term arithmetic progression in the s-gonal numbers.

We now show that there are infinitely many three-term arithmetic progressions with common difference $d > 0$ starting at a given polygonal number, which is our second theorem. The proof of this theorem uses some basic algebraic number theory as detailed in [2] or [3].

Theorem 2. *Let s be a fixed integer with $s \ge 3$. Let n be an arbitrary positive integer. Then there exist infinitely many integers $d > 0$ such that there is a three-term arithmetic progressions with a common difference d in the s-gonal numbers beginning with $P_s(n)$.*

Proof. Let s be a fixed integer with $s \ge 3$. Let n be an arbitrary positive integer. Let $N = 2(s-2)n - (s-4)$ as in Lemma 1.

Suppose that X and Y are positive integers satisfying

$$X^2 - 2Y^2 = -1, \quad X \equiv 1 \pmod{2(s-2)}, \quad \text{and} \quad Y \equiv 1 \pmod{2(s-2)}. \tag{2}$$

Notice that $X = 1$ and $Y = 1$ satisfy (2), so such X and Y exist.

Observe that by multiplying the equation in (2) by N^2 we have

$$(NX)^2 - 2(NY)^2 = -N^2.$$

Our goal is to apply Lemma 1.

Now let

$$a = \frac{NY + (s-4)}{2(s-2)} = nY + \frac{(1-Y)(s-4)}{2(s-2)} \tag{3}$$

and

$$b = \frac{NX + (s-4)}{2(s-2)} = nX + \frac{(1-X)(s-4)}{2(s-2)}. \tag{4}$$

Since $s \geq 3$ and $Y \equiv 1 \pmod{2(s-2)}$, we may write $Y = 1 + 2(s-2)k$ for some integer $k \geq 0$. Thus, from (3),

$$a = n(1 + 2(s-2)k) - (s-4)k = n + (2n(s-2) - (s-4))k \geq n + 2k > 0.$$

Hence, a is a positive integer.

Similarly from (4), b is also a positive integer.

As already noted, we have $(NX)^2 - 2(NY)^2 = -N^2$. Observe that if $X > Y > 1$ then $n < a < b$. Thus, by the comments after Lemma 1, $P_s(n)$, $P_s(a)$, and $P_s(b)$ would form a three-term arithmetic progression with common difference $d > 0$ in the s-gonal numbers. Therefore it suffices to show that there are infinitely many positive integers X and Y satisfying (2) with $X > Y > 1$.

The solutions to $X^2 - 2Y^2 = -1$ with X and Y being positive integers are given by $X + Y\sqrt{2} = \left(1 + \sqrt{2}\right)^m$ where m is an odd positive integer. We know that

$$G = \left((\mathbb{Z}/(2(s-2))\mathbb{Z})\left[\sqrt{2}\right]\right)^{\times}$$

is a finite group and $1 + \sqrt{2}$ is an element of G. Letting m be 1 plus any even multiple of the order of $1 + \sqrt{2}$ in G gives us that $\left(1 + \sqrt{2}\right)^m$ is equivalent to $1 + \sqrt{2}$ in G. Thus, for any such m, $X + Y\sqrt{2} = \left(1 + \sqrt{2}\right)^m$ satisfies $X \equiv 1 \pmod{2(s-2)}$ and $Y \equiv 1 \pmod{2(s-2)}$. With the exception of $X = Y = 1$, we have that $X > Y > 1$. This guarantees that $P_s(a) - P_s(n) = P_s(b) - P_s(a) = d$ with $d > 0$.

This completes the proof of the theorem. $\square$

4. Concluding Remarks

We conclude with a few comments on Theorem 2. The special cases of $s = 3$ and $s = 4$, corresponding to the triangular numbers and squares respectively, provide interesting examples. For $s = 3$, integers X and Y satisfying the equation in (2) give solutions a and b to (1) given by (3) and (4). In other words,

$$a = nY + \frac{Y-1}{2} \quad \text{and} \quad b = nX + \frac{X-1}{2}.$$

It is easy to show, however, that for every integral solution X and Y to the equation in (2), both X and Y are odd. Thus every integral solution to the equation in (2) gives us integral solutions a and b to (1).

With $s = 4$, the integral solutions X and Y to (2) give solutions a and b to (1) by $a = nY$ and $b = nX$. Again, we have integral solutions a and b to (1) for every integral solution X and Y to (2).

For both $s = 3$ and $s = 4$, every integral solution to the equation in (2) gives an integral solution to (1). However, for each $s \geq 5$ this is no longer the case. Take

$s = 5$ and $n = 1$, for example, and consider arithmetic progressions with common difference d in the pentagonal numbers starting with $P_5(1) = 1$. Here $X = 7$ and $Y = 5$ is the first non-trivial solution to the equation in (2), but this does not give an integral solution to (1). In fact, the first non-trivial solution to the equation in (2) that does give an integral solution to (1) is $X = 1393$ and $Y = 985$, which gives us the three-term arithmetic progression $P_5(1) = 1$, $P_5(821) = 1010651$, and $P_5(1161) = 2021301$ with the common difference of 1010650.

Furthermore, not every integral solution to (1) is given by a solution to (2). Take the case of $s = 3$ and $n = 3$. Here $P_3(3) = 6$, $P_3(8) = 36$, and $P_3(11) = 66$ is an arithmetic progression with common difference of 30, but $a = 8$ and $b = 11$ are not given by a solution to (2). This choice of a and b do arise, however, from the discussion after the proof of Lemma 1 with $A = 17$ and $B = 23$, as illustrated next.

We wish to find all three-term arithmetic progressions beginning with $P_3(3)$ as discussed after Lemma 1. Thus we want to find all positive solutions A and B to the Pell equation $B^2 - 2A^2 = -N^2$, where $N = 7$.

For every divisor δ of $N = 7$, we have the associated Pell equation

$$X^2 - 2Y^2 = -\left(\frac{N}{\delta}\right)^2,$$

where $B = \delta X$, $A = \delta Y$, and we wish for X and Y to be relatively prime. In our case $N = 7$, so we consider the two equations

$$X^2 - 2Y^2 = -1 \tag{5}$$

$$\text{and } X^2 - 2Y^2 = -49. \tag{6}$$

In the case of (5), we have $\delta = 7$; and in the case of (6), we have $\delta = 1$.

Equation (5) as previously noted has the solution $X = 1$ and $Y = 1$. All other solutions $X + Y\sqrt{2}$ are given by $\left(1 + \sqrt{2}\right)^m$ for any odd positive integer m. Again, we note that the solution $X = 1$ and $Y = 1$ gives the trivial arithmetic progression with common difference $d = 0$.

The solutions X and Y, with X and Y relatively prime, for equation (6) are given by $X = |U|$ and $Y = |V|$ where $U + V\sqrt{2} = (1 + 5\sqrt{2})(1 + \sqrt{2})^r$, where r is an even integer, possibly negative. Since $\delta = 1$ in this case, if we set $r = 2$, we obtain the values $A = 17$ and $B = 23$ as previously mentioned.

We note that in general there may not be any relatively prime solutions to a Pell equation, as in the case of $X^2 - 2Y^2 = -9$.

We can explicitly describe in a finite number of steps all three-term arithmetic progressions in the s-gonal numbers for any fixed $s \geq 3$ beginning with $P_s(n)$ for any positive integer n. One method for achieving this is through the use of continued fractions, as presented in [1, pages 423-527]. Of special importance to note is that the algorithm that is presented in [1] terminates in a finite number of steps, giving a description of all solutions to a Pell equation in terms of certain constructed general solutions to the Pell equation.

Acknowledgments The authors wish to thank the anonymous referee for their comments which improved the readability of this paper. Additionally, the authors wish to thank Dr. Jerrold R. Griggs of the University of South Carolina for posing the original problem which dealt with arithmetic progressions in the triangular numbers, and also Dr. Michael Filaseta of the University of South Carolina, for his many suggestions and guidance.

References

[1] G. Chrystal, *Algebra: an Elementary Text-Book for the Higher Classes of Secondary Schools and for Colleges*, seventh ed., vol. II, Chelsea Publishing Company, New York, NY, 1964.

[2] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, second ed., Springer Verlag, New York, NY, 1972.

[3] D. A. Marcus, *Number Fields*, Springer Verlag, New York, NY, 1977.

[4] L. J. Mordell, *Diophantine Equations*, Pure and Applied Mathematics, vol. 30, Academic Press, London, 1969.

[5] W. Sierpiński, *Elementary Theory of Numbers*, second ed., North-Holland Mathematical Library, vol. 31, North-Holland, Amsterdam, 1988.