EULER-TYPE IDENTITIES FOR INTEGER COMPOSITIONS VIA ZIG-ZAG GRAPHS

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Abstract

This paper is devoted to a systematic study of combinatorial identities which assert the equality of different sets of compositions, or ordered partitions, of integers. The proofs are based on properties of zig-zag graphs - the graphical representations of compositions introduced by Percy A. MacMahon in his classic book *Combinatory Analysis*. In particular it is demonstrated, by means of general theorems, that the conjugate composition enjoys the same essential status as the conjugate partition in the proofs of such identities.

1. Introduction

A composition of a positive integer n is a representation of n as a sum of positive integers in which the order of the summands is taken into account. The summands are called parts of the composition. A partition of n is a representation of n as a sum of positive integers without regard to order. Compositions of n will be written as vectors with positive-integer entries which sum to n. For example 4 has eight compositions namely

$$(4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1),$$

and five partitions namely

$$(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).$$

It is well known that there are 2^{n-1} unrestricted compositions of n and $\binom{n-1}{k-1}$ compositions of n into k parts. However, no such simple formulas exist for the number of partitions of n (see for example [3, Ch. 5]).

Compositions and partitions may also be exhibited graphically by means of arrays of nodes. In the graph of a partition $(\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geq \cdots \geq \lambda_k$, also called a Ferrers

graph, the nodes are left-justified with λ_i nodes in the ith row, from top to bottom. For example, the graph of $(5, 3, 2, 2, 1)$ is

$$
\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & & \\
\bullet & \bullet & & & \\
\bullet & \bullet & & & \\
\bullet & & & & \\
\end{array}
$$
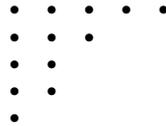
The graph of a composition (also called a zig-zag graph), as defined by MacMahon in [8, Sec. IV, Ch. 1, p. 153], resembles the partition Ferrers graph except that the first node of each part is aligned with the last part of its predecessor. Thus the zig-zag graph of the composition $(5, 3, 1, 2, 2)$ is

$$
\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & & \\
& & & & \bullet & \bullet & \bullet \\
& & & & & & \bullet \\
& & & & & \bullet & \bullet \\
& & & & & & \bullet & \bullet \\
\end{array}
\tag{1}
$$

One goal of this paper is to explore the role of zig-zag graphs in proving identities which assert the equality of different sets of restricted compositions. In particular we highlight the significance of the conjugate composition. The conjugate of a composition (or partition) is obtained by reading its graph by columns, from left to right. For example, the graph (1) gives the conjugate of the composition $(5, 3, 1, 2, 2)$ as $(1, 1, 1, 1, 2, 1, 3, 2, 1)$.

Several proofs of partition identities have been resolved by transformations of Ferrers graphs leading to interesting bijections between sets of partitions (see, for example, [3, 4]). A good example is the famous ODD=DISTINCT partition identity of Euler:

Euler's Theorem. *The number of partitions of n into odd parts equals the number of partitions of n in which no part is repeated.*

Bijective proofs of this theorem which rely solely on Ferrers graphs can be found in Igor Pak's survey [9] (and references therein). Agarwal [1] discovered an analogue of the theorem for color compositions, which is rather too complicated to state here. Suffice it to say that his proofs are independent of zig-zag graphs.

Inspired by Euler's Theorem, Andrew Sills recently published a bijective proof of the following theorem ([10, Theorem 1]):

Theorem 1.0 (Sills) *The number of compositions of n into odd parts equals the number of compositions of $n + 1$ into parts greater than 1.*

Sills' proof demonstrates a nice application of the MacMahon conjugate of a composition that is akin to the essential role of the conjugate partition.

In the same spirit one may give a zig-zag graph proof of the assertion:

Theorem 1.1 *The number of compositions of n into parts $> m$ ($m \neq 1$) equals the number of compositions of $n - 2m$ into 1's and 2's with no consecutive 2's.*

The following type of argument is familiar to partition theorists. Without loss of generality consider the case $m = 2$. The zig-zag graph of a composition into parts > 2, say $(3, 4, 3, 5)$, is shown in the first diagram below.

Notice that such graph always has at least 2 nodes before the first stack of vertical nodes, and at least 2 nodes after the last stack. Also the large sizes (≥ 3) of the parts insures that each stack contains exactly two vertical nodes, with at least a node between successive pairs of vertical nodes. Thus on deleting the first 2 nodes, and the last 2 nodes (marked), we find that the conjugate of the remaining graph is a composition of the second type, $(2, 1, 1, 2, 1, 2, 1, 1)$. This completes the proof.

The proofs of the following theorems require more knowledge of the structural properties of zigzag graphs, and so are put off until Section 3. Throughout this paper the letter m denotes a positive integer > 1.

Theorem 1.2 *The following sets of compositions are equinumerous:*
   (i) Compositions of n using only the parts 1 and m.
   (ii) Compositions of $n + 1$ into parts $\equiv 1$ (mod m).
   (iii) Compositions of $n + m$ into parts greater than $m - 1$.

Note that Theorem 1.0 is the special case $m = 2$ of (ii) $\iff$ (iii) in Theorem 1.2.

Example. When $n = 8$ and $m = 4$, there are 7 compositions of each type namely:
(i) $(4, 4), (1, 1, 1, 1, 4), (1, 1, 1, 4, 1), (1, 1, 4, 1, 1), (1, 4, 1, 1, 1), (4, 1, 1, 1, 1),$
        $(1, 1, 1, 1, 1, 1, 1, 1)$;
(ii) $(9), (1, 1, 1, 1, 5), (1, 1, 1, 5, 1), (1, 1, 5, 1, 1), (1, 5, 1, 1, 1), (5, 1, 1, 1, 1),$
        $(1, 1, 1, 1, 1, 1, 1, 1, 1)$;
(iii) $(4, 4, 4), (8, 4), (7, 5), (6, 6), (5, 7), (4, 8), (12)$.

Theorem 1.3 *The number of compositions of n into parts $\equiv m$ (mod $m + 1$) equals the number of compositions of n without 1's into parts $\equiv 1$ (mod m).*

The zigzag graph possesses a rich combinatorial structure providing several equivalent paths to the conjugate composition. The latter are outlined in Section 2.

We close this section by listing four sets of compositions that are counted by the Fibonacci numbers F_n ($F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, n > 1$), as a base case for Theorem 1.2 ($m = 2$). The actual enumeration proofs can be found, for example, in [2, 5, 6, 7].

Corollary 1.4 (Fibonacci family) *The following sets of compositions are counted by F_n:*

  (i) *Compositions of n into odd parts.*

 (ii) *Compositions of $n+1$ without 1's.*

(iii) *Compositions of $n+1$ into 1's and 2's such that the first and last parts equal 1.*

(iv) *Compositions of $n-1$ into 1's and 2's.*

General enumeration questions are briefly discussed in Section 4.

2. Equivalent Structures and the Conjugate Composition

We present two combinatorial structures that return the same conjugate compositions as the MacMahon zig-zag graphs, followed by a short-cut method for finding the conjugate of a composition.

The different but equivalent approaches to the conjugate composition will be represented by the acronyms **ZG**, **LG**, **SB** and **DD**, where **ZG** stands for **The Zig-Zag Graph**, and the others are explained below. The conjugate of a composition C will be denoted by C'.

LG: The Line graph (also originally due to MacMahon [8, Sec. IV, Ch. 1, p. 151])

The number n is depicted as a line divided into n equal segments and separated by $n-1$ spaces. A composition $C = (c_1, \ldots, c_k)$ then corresponds to a choice of $k-1$ from the $n-1$ possible spaces, indicated with nodes, such that a node is placed after c_1 segments, and the next node is placed after a further c_2 segments, and so forth. The conjugate C' is obtained by placing nodes on the other $n-k$ spaces. For instance the line graph of the composition $(5, 3, 1, 2, 2)$ is

$$\text{— — — — —•— — —•—•— —•— —} \, ,$$
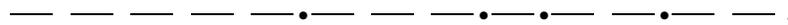
from which we deduce that $C' = (1, 1, 1, 1, 2, 1, 3, 2, 1)$ in agreement with the **ZG** method. It follows that C' has $n - k + 1$ parts.

SB: Subsets and Bit-Encoding

As a consequence of the **LG** representation, there is a bijection between compositions of n into k parts and $(k-1)$-subsets of $\{1, \ldots, n-1\}$ via partial sums (see also [12]) given by

$$C = (c_1, \ldots, c_k) \mapsto \{c_1, c_1 + c_2, \ldots, c_1 + c_2 + \cdots + c_{k-1}\} = L. \qquad (2)$$

Thus C' is the composition corresponding to the complementary set $\{1, \ldots, n-1\} \backslash L$.

It is sometimes necessary to express compositions as bit strings. The procedure for such *bit-encoding* consists of converting the set L into a unique bit string $B = (b_1, \ldots, b_{n-1}) \in \{0, 1\}^{n-1}$ such that

$$b_i = \begin{cases} 1 & \text{if } i \in L \\ 0 & \text{if } i \notin L. \end{cases}$$

The complementary bit string B' (obtained from B by changing 1's to 0's and 0's to 1's) is then the bit encoding of C'.

DD: Direct Detection of Conjugates

We deduce from the **ZG** and **LG** representations a rule for the conjugate of a composition by mere inspection. A sequence of x consecutive equal parts $c, \ldots, c$ will be abbreviated as c^x. First notice that

$$(c^v)' = \begin{cases} (v) & \text{if } c = 1; \\ (1, 2^{v-2}, 1) & \text{if } c = 2; \\ (1^{c-1}, 2, 1^{c-2}.2, 1^{c-2}, \ldots, 2, 1^{c-2}, 2, 1^{c-1}) & \text{if } c > 2, \end{cases} \tag{3}$$

where, in the last case, 2's appear $v - 1$ times.

Before giving the rule for a general composition $C = (c_1, c_2, \ldots, c_k)$, we define the *interior* of a part $c_i > 1$ as

$$interior(c_i) = \begin{cases} c_i - 2 & \text{if } 1 < i < k \\ c_i - 1 & \text{otherwise.} \end{cases} \tag{4}$$

The interiors of big (> 1) parts of C translate into contiguous 1's in C'. The second case of (3) shows that an intermediate part $c_i = 2$ has a zero interior. The occurrence of such 2's in C account for the presence of consecutive big parts in C'.

In general, subject to (4) and reversal of the order of parts, the conjugate of the composition $C = (c_1, c_2, \ldots)$ is given by the following rule.

(i) If C has the form $C = (1^u, a, 1^v, b, 1^w, \ldots)$, $0 < u, 0 \leq v, w, \ldots, 1 < a, b, \ldots,$, then

$$C' = ((u+1), 1^{-1+a-1}, (1+v+1), 1^{-1+b-1}, (1+w+1), \ldots)$$
$$= ((u+1), 1^{a-2}, (v+2), 1^{b-2}, (w+2), \ldots),$$

where some of the 1^* may be NULL which happens if any of $a, b, \ldots$ is 2.

(ii) Similarly if C has the form $C = (a, 1^u, b, 1^v, \ldots)$, then

$$C' = (1^{a-1}, (v+2), 1^{b-2}, (w+2), \ldots), \; 0 \leq v, w, \ldots, 1 < a, b, \ldots,$$

For example, $(1, 3, 4, 1^3, 2, 1^2, 6)'$ is given by

$$((1+1), 1^{3-2}, (1+1), 1^{4-2}, (1+1^3+1), (1+1^2+1), 1^{6-1}) = (2, 1, 2, 1^2, 5, 4, 1^5).$$

The **SB** approach provides a conducive platform for detailed arguments. The **DD** method works well in a variety of contexts since it often gives a general form of the conjugate composition explicitly.

3. Proofs of Main Theorems

3.1. Proof of Theorem 1.2

The theorem will be proved according to the bijection scheme: $(ii) \iff (iii) \iff (i)$. Let the corresponding sets be denoted by

$$(i) : V_n\{1, m\}, \ (ii) : C_1(n+1, m), \ (iii) : E_m(n+m). \tag{5}$$

Lemma 3.1 *Let C be a composition of an integer $n > 0$ into parts $\equiv 1$ (mod m). Then the conjugate composition $C' = (c'_1, c'_2, \dots)$ satisfies $c'_j = 1$ for all $j \not\equiv 1$ (mod m). Moreover, if C' has k' parts, then $k' \equiv 1$ (mod m).*

Proof. The composition C has the form

$$C = (mt_1 + 1, mt_2 + 1, mt_3 + 1, \dots, mt_k + 1), \ k, t_1, \dots, t_k \geq 0.$$

The second part of the lemma is immediate since

$$k' = n - k + 1 = \sum_{i=1}^{k}(mt_i + 1) - k + 1 \equiv 1 \ (\text{mod } m).$$

We give two proofs of the first part using the **SB** and **DD** approaches.

First Proof. The sequence of partial sums L has the form

$$L = (m\ell_1 + 1, m\ell_2 + 2, m\ell_3 + 3, \dots, m\ell_{k-1} + k - 1), \text{ where } \ell_v = \sum_{i=1}^{v} t_i.$$

Notice that each pair of consecutive terms of L is separated by N (missing) elements of $\{1, \dots, n-1\}$ with N a multiple of m. Thus the bit encoding of C must have 0's appearing only in strings of lengths divisible by m. Correspondingly the bit encoding B' of the conjugate C' must have 1's appearing only in strings of lengths divisible by m. Therefore, if B' contains the string

$$\dots, 1, 0, \dots 0, \underbrace{1, 1, 1, \dots, 1}_{m \ ones} \dots$$

in which the first and second 1's are indexed by p and q, $p < q$, then the decoding process first gives the segment of partial sums (identify corresponding entries):

$$(\dots, a_{m\ell}, a_{m\ell+1}, a_{m\ell+2}, \dots, a_{m(\ell+1)}, \dots) = (\dots, p, q, q + 1, \dots, q + m - 1, \dots).$$

This in turn translates into

$$(\dots, c'_{m\ell}, c'_{m\ell+1}, c'_{m\ell+2}, \dots, c'_{m(\ell+1)}, \dots) = (\dots, 1, q - p, 1, \dots, 1, \dots)$$

as a substring of C', as desired.

Second Proof. Without loss of generality, C may also be expressed as

$$C = (mt_1 + 1, 1^u, mt_2 + 1, 1^v, mt_3 + 1, 1^w, \dots,), \ t_i \geq 1, \ u, v, w, \dots, \geq 0.$$

The conjugate then takes the form

$$C' = (1^{mt_1}, (u + 2), 1^{mt_2 - 1}, (v + 2), 1^{mt_3 - 1}, (w + 2), \dots,),$$

or, equivalently,

$$C' = (1^{m\ell_1}, (u+2), 1^{m\ell_2+m-1}, (v+2), 1^{m\ell_3+m-1}, (w+2), \ldots,), \ \ell_i \geq 0,$$

that is, $C' = (1^{m\ell_1}, (u+2), 1^{m-1}, 1^{m\ell_2}, (v+2), 1^{m-1}, 1^{m\ell_3}, (w+2), \ldots,), \ \ell_i \geq 0$, which shows that the assertion is true.   □

We are ready to prove the theorem.

Proof of Theorem 1.2

$(ii) \iff (iii)$: We establish a bijection

$$\theta : \ C_1(n+1, m) \longrightarrow E_m(n+m). \tag{6}$$

If $C = (c_1, \ldots, c_k) \in C_1(n+1, m)$, then by Lemma 3.1 the conjugate C' has the form

$$C' = (c'_1, 1, \ldots, 1, c'_{m+1}, 1, \ldots, 1, \ldots, c'_{mt+1}, 1, \ldots, 1, z'), \tag{7}$$

where $t \geq 0$, and the c'_{mj+1} and z' are positive integers.

Now define

$$\theta(C) = (f_0, f_1, \ldots, f_t, z' + m - 1), \tag{8}$$

where

$$f_j = c'_{mj+1} + \sum_{i=1}^{m-1} 1, \ j = 0, 1, \ldots t.$$

Conversely, given $E = (f_1, \ldots, f_k) \in E_m(n+m)$, define $\theta^{-1}(E) = C$ such that

$$C' = (f_1 - m + 1, 1, \ldots, 1, \ldots, f_{k-1} - m + 1, 1, \ldots, 1, f_k - m + 1)$$

in which $m - 1$ ones follow each $f_i - m + 1$, $i = 1, \ldots, k - 1$.

So θ is a bijection.   ◇

$(iii) \iff (i)$: Let $E \in E_m(n+m)$. We associate E uniquely with $V \in V_n\{1, m\}$, a composition of n using only the parts 1 and m. Then the conjugate E' has the form

$$E' = (1^{a+1}, 2, 1^b, 2, 1^c, \ldots, 2, 1^{z+1}), \ a, b, c, \ldots, z \geq m - 2 \geq 0 \tag{9}$$

That is, E' is a composition into 1's and 2's such that
   (i) both the first $m - 1$ parts and the last $m - 1$ parts are 1's; and
   (ii) every successive pair of 2's is separated by $1^v, v \geq m - 2$, since the interior of each part of $E \geq m - 2$.
There is a bijection

$$\gamma : E_m(n+m) \to V_n\{1, m\} \tag{10}$$

such that $E \mapsto V$.

If m is even with $m = 2$, then V is given at once by (9): $E' = V$. For $m = 2i > 2$, we obtain V from E' by replacing each string $1^{i-1}, 2, 1^{i-1}$ with $2i$, and then deleting the first i ones and the last i ones. This operation is well-defined since (9) implies

$$E' = (1^{a-i+2}, 1^{i-1}, 2, 1^{i-1}, 1^{b-2i+2}, 1^{i-1}, 2, 1^{i-1}, \ldots, 2, 1^{i-1}, 1^{z-i+2}),$$

which is finally transformed into the form

$$(1^{a-2i+2}, 2i, 1^{b-2i+2}, 2i, \ldots, 2i, 1^{z-2i+2}) = V.$$

The condition $a, b, c, \ldots, z \geq m - 2$ implies that each exponent of 1 is nonnegative.

If m is odd, $m = 2i + 1 > 1$, then V is obtained, as in the even case, with a slight counter-balancing adjustment. In E' replace each string $1^i, 2, 1^{i-1}$, with $2i + 1$, and then delete the first i ones and the last $i + 1$ ones. In this case (9) becomes

$$E' = (1^{a-i+1}, 1^i, 2, 1^{i-1}, 1^{b-2i+1}, 1^i, 2, 1^{i-1}, \ldots, 2, 1^{i-1}, 1^{z-i+2}),$$

which finally gives

$$(1^{a-2i+1}, (2i+1), 1^{b-2i+1}, (2i+1), \ldots, (2i+1), 1^{z-2i+1}) = V.$$

It can be verified that the same V is obtained if we first replace the string $1^{i-1}, 2, 1^i$, followed by deleting the first $i + 1$ ones and the last i ones. $\qquad \square$

Illustration. Let $n = 13$, $m = 3$ and consider a composition of 14 into parts $\equiv 1 \pmod 3$, say $C = (1, 7, 1, 1, 4) \in C_1(14, 3)$. Then the conjugate is $C' = (2, 1, 1, 1, 1, 1, 4, 1, 1, 1)$. Thus

$\theta(C) = (2 + 1 + 1, 1 + 1 + 1, 4 + 1 + 1, 1 + 2) = (4, 3, 6, 3) \in E_3(16)$.

Furthermore, the conjugate of $E = (4, 3, 6, 3)$ is

$E' = (1, 1, 1, 2, 1, 2, 1, 1, 1, 1, 2, 1, 1) \to (1, 1, (1, 2), (1, 2), 1, 1, 1, (1, 2), 1, 1)$.

Now replace the $(1, 2)$'s with 3's, and delete the first 1 and the last two 1's, to obtain:

$\gamma(E) = (1, 3, 3, 1, 1, 1, 3) \in V_{13}\{1, 3\}$.

See Figure 1; the nodes enclosed in a box form a (new) part of an image.

Remark 3.2 *The condition required for closing the apparent cycle in Figure 1 is that θ acts on a composition without 1's into parts $\equiv 1 \pmod m$ (see the proof of Theorem 1.3 below).*

3.2. Proof of Theorem 1.3

Let $C_1(n + 1, m)_{>1}$ denote the set of compositions of $n + 1$ without 1's into parts $\equiv 1 \pmod m$, and let $C \in C_1(n + 1, m)_{>1}$. Then the conjugate C' is a composition into 1's and 2's such that the first m parts and the last m parts are 1's (see Theorem
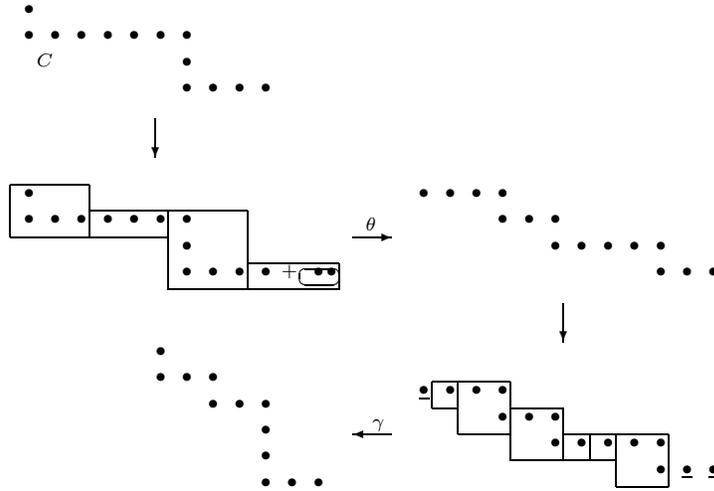
Figure 1: Bijections: $C = (1, 7, 1, 1, 4) \xrightarrow{\theta} (4, 3, 6, 3) \xrightarrow{\gamma} (1, 3, 3, 1, 1, 1, 3)$

1.1), and since each part of C is also congruent to 1 (mod m), it follows from (7) that C' assumes the form

$$C' = (1^{\ell m}, 2, 1, \ldots, 1, 2, 1, \ldots, 1, \ldots, 2, 1^{tm}, 1),\ \ell, t > 0, \qquad (11)$$

where each 2 is indexed by $j \equiv 1$ (mod m).

On applying θ, and deleting the last part from $\theta(C)$, we obtain a bijection, say β, from $C_1(n+1, m)_{>1}$ to the set of compositions of n into the parts m and $m+1$ in which the first part is m. Hence the bijection we set out to prove is given by the composition of maps, $\rho\beta$, where ρ is a specialization of the bijection asserted by Lemma 3.3 below.   □

Lemma 3.3 *The number of compositions of n into parts $\equiv r$ (mod m) equals the number of compositions of n into the parts r and m in which the first part is r.*

Proof. Let S denote a composition of n into the parts r and m in which the first part is r, and define F as the composition obtained from S by replacing each string of the form $r, m, \ldots, m$ with the sum of its parts, where $m, \ldots, m$ is a (possibly empty) maximal string of m's. This operation is reversible since every positive integer N satisfying $N \equiv r$ (mod m), has a partition of the type $N = r + \ell m$, $\ell \geq 0$. Therefore F is uniquely a composition of n into parts $\equiv r$ (mod m). Thus there is bijection between the two types of compositions.   □

4. A Remark on Enumeration

Enumeration functions for compositions generally have considerably simpler formulas than partition functions. Perhaps this is the reason for the scarce attention received by identities for compositions vis-a-vis the conjugate composition. This fact may be illustrated with the three enumeration functions associated with Theorem 1.2. Using the notations defined in (5) we obtain

$$\sum_{n\geq 0} |V_n\{1,m\}|q^n = \sum_{k\geq 0} (q+q^m)^k = \frac{1}{1-q-q^m},$$

$$\sum_{n\geq 0} |C_1(n,m)|q^n = \sum_{k\geq 0} (q+q^{1+m}+q^{1+2m}+\dots)^k = \frac{1-q^m}{1-q-q^m},$$

and

$$\sum_{n\geq 0} |E_m(n)|q^n = \sum_{k\geq 0} (q^m+q^{m+1}+\dots)^k = \frac{1-q}{1-q-q^m},$$

from which it is a routine exercise to show that

$$[q^n]\frac{1}{1-q-q^m} = [q^{n+1}]\frac{1-q^m}{1-q-q^m} = [q^{n+m}]\frac{1-q}{1-q-q^m} = \sum_{j\geq 0} \binom{n-(m-1)j}{j}.$$

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