A SHORT PROOF OF A RESULT OF GICA AND LUCA

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Received: 10/13/12, Accepted: 1/8/13, Published: 1/25/13

Abstract
We provide a short proof of a generalization of a recent result of Gica and Luca on the diophantine equation $2^x = y^2 + z^2(x^2 - 2)$.

– Dedicated to the memory of Preechar Polasub.

1. Introduction

In a recent paper, Gica and Luca [2], in connection with a result of Lee [3] on the class number one problem for quadratic fields of the shape $\mathbb{Q}(\sqrt{n^2 \pm 2})$, were led to consider the diophantine equation

$$2^x = y^2 + z^2(x^2 - 2), \tag{1}$$

where x, y and z are positive integers. This equation has many known solutions, including $(x, y, z) = (3, 1, 1), (5, 3, 1), (7, 9, 1)$, and infinite families of solutions obtained by choosing, say,

$$x = 2^\alpha + 1,\ y = 2^{2^{\alpha-1}-\alpha}\left(2^\alpha - 1\right),\ z = 2^{2^{\alpha-1}-\alpha}$$

or

$$x = 2^\alpha - 1,\ y = 2^{2^{\alpha-1}-\alpha-1}\left(2^\alpha + 1\right),\ z = 2^{2^{\alpha-1}-\alpha-1}.$$

Gica and Luca prove that there are only the three known solutions with $z = 1$. Their argument relies fundamentally upon lower bounds for linear forms in p-adic logarithms. Our goal in this short note is to give a quick proof of a stronger result, which immediately generalizes to partially resolve a conjecture of Gica and Luca on equation (1). Apparently, this strengthening does not follow from the techniques of [2].

Suppose we have a solution to (1). Then, working 2-adically, we readily obtain that x is odd, say $x = 2x_0 + 1$. It follows that

$$0 < \sqrt{2} - \frac{y}{2^{x_0}} = \frac{z^2(x^2-2)}{2^{x_0}\left(2^{x_0+1/2} + y\right)} < \frac{z^2(x_0^2 + x_0)}{2^{2x_0-3/2}}.$$

On the other hand, from Corollary 1.6 of Bauer and Bennett [1], either $x_0 \in \{3,7,8\}$, or
$$\sqrt{2} - \frac{y}{2^{x_0}} > 2^{-1.48 x_0}.$$
In the latter case, if $z=1$, we thus have
$$2^{0.52 x_0} < 2^{1.5}\left(x_0^2 + x_0\right)$$
and so, using calculus, $x_0 \leq 19$. A quick check yields the main result of [2].

The same argument with a little more work, applying Corollary 1.6 of [1] to either
$$\sqrt{2} - \frac{y}{2^{x_0}} \quad \text{or} \quad \sqrt{2} - \frac{zx}{2^{x_0}},$$
implies the following :

Theorem. For any solution to equation (1) in positive integers x, y and z, we either have $(x,y,z) \in \{(3,1,1),(5,3,1),(7,9,1),(9,14,2),(13,3,7)\}$ or may conclude that
$$\min\{y,z\} > 2^{x/8}.$$

Gica and Luca conjecture that equation (1) has only the solutions $(x,y,z) = (3,1,1),(5,3,1),(7,9,1)$ and $(13,3,7)$ in odd integers x,y and z with x^2-2 prime. The above result confirms this in case either y or z are "small". Indeed, an almost immediate consequence of this theorem (together with routine computation) is that (1) has only the following solutions with $z < 10^8$:

$(x,y,z) = (3,1,1),(5,3,1),(7,9,1),(9,14,2),(11,12,4),(13,3,7),(15,136,8),$
$(17,240,16),(21,1324,28),(23,2496,64),(25,5568,64),(27,11456,64),$
$(31,33792,1024),(31,29502,1154),(33,71300,1796),(33,63488,2048),$
$(37,318048,5152),(39,741152,544),(39,376832,16384),(45,1934080,124672),$
$(45,655360,131072),(47,8639880,173048),(51,23078252,813316),$
$(51,7995392,917504),(55,189629256,151672),(63,2181038080,33554432),$
$(63,2309764600,31307768),(63,2043658240,35667968),(65,4490035200,62947328),$
$(65,4227858432,67108864),(65,3949423632,71012336),(69,24270711104,16066112).$

References

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