SOME RESULTS ON BALANCING, COBALANCING, (a, b)-TYPE BALANCING AND, (a, b)-TYPE COBALANCING NUMBERS

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Received: 4/16/12, Revised: 12/2/12, Accepted: 4/7/13, Published: 4/10/13

Abstract

In this paper, we present new results on balancing, cobalancing, (a, b)-type balancing and (a, b)-type cobalancing numbers as well as establish some new identities.

1. Introduction and Notation

A positive integer n is called by Behera et al. a balancing number [1], if there exists a positive integer r, which is called the balancer of n, such that:

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r) \,. \tag{1}$$

Panda [4] sets $n = 1$ as the first balancing number and $r = 0$ as its corresponding balancer. Panda et al. [5] define cobalancing numbers as the solutions to the diophantine equation:

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r) \,, \tag{2}$$

where r is the cobalancer of n.

[1]Supported by L'IFORCE Laboratory
[2]Supported by Laboratory of Algebra

Throughout this paper, we denote by B_m, R_m, b_m and r_m, the m^{th} balancing number, the m^{th} balancer, the m^{th} cobalancing number and the m^{th} cobalancer, respectively. These numbers have already been extensively investigated in several papers.

2. Background

The present work is strongly connected to the theory of diophantine equations and more specifically, to the integer solutions of the following equation in two variables:

$$x^2 - 2y^2 = u^2 - 2v^2, \tag{3}$$

where u and v are integers. Note that for $u = \pm 1$ and $v = 0$, Equation (3) is Pell's equation. It is well known, that the form $x^2 - 2y^2$ is irreducible over the field $\mathbb{Q}$ of rational numbers, but in the extension field $\mathbb{Q}(\sqrt{2})$ it can be factored as a product of linear factors $(x + y\sqrt{2})(x - y\sqrt{2})$. Using the norm concept for the extension field $\mathbb{Q}(\sqrt{2})$, Equation (3) which has $\xi = u + v\sqrt{2}$ as solution, can be written in the form:

$$N(x + y\sqrt{2}) = N(\xi). \tag{4}$$

It is easily checked that the set of all numbers of the form $x + y\sqrt{2}$, where x and y are integers, form a ring, which is denoted $\mathbb{Z}[\sqrt{2}]$. The subset of units of $\mathbb{Z}[\sqrt{2}]$, which we denote $\mathcal{U}$ forms a group. It is easy to show that $\alpha \in \mathcal{U}$ if and only if $N(\alpha) = \pm 1$ [2]. Applying Dirichlet's Theorem of units via subtle calculations, we can show that $\mathcal{U} = \{\pm \left(1 + \sqrt{2}\right)^m,\ m \in \mathbb{Z}\}$. Since

$$N\left(\left(1 + \sqrt{2}\right)^m\right) = N\left(\left(1 + \sqrt{2}\right)\right)^m = (-1)^m, \tag{5}$$

we obtain

$$N(\alpha) = +1 \Leftrightarrow \alpha = \left(1 + \sqrt{2}\right)^{2m},\ m \in \mathbb{Z}, \tag{6}$$

and

$$N(\alpha) = -1 \Leftrightarrow \alpha = \left(1 + \sqrt{2}\right)^{2m+1},\ m \in \mathbb{Z}. \tag{7}$$

For any $\alpha \in \mathcal{U}$ with $N(\alpha) = 1$, Equation (4) becomes

$$N(x + y\sqrt{2}) = N(\alpha\xi).$$

Thus, all integral solutions of Equation (3) have take the form:

$$x + y\sqrt{2} = \xi\left(1 + \sqrt{2}\right)^{2m},\ m \in \mathbb{Z}. \tag{8}$$

3. Preliminary Results

From (1) we have

$$r^2 + (2n+1)\,r - n\,(n-1) = 0. \tag{9}$$

The discriminant Δ of Equation (9) with respect to r is $\Delta = 8n^2 + 1$. Then

$$r = \frac{-\,(2n+1) + \sqrt{8n^2+1}}{2}\cdot \tag{10}$$

Since r is a positive integer, $8n^2+1$ is a perfect square, i.e., $8n^2+1 = u^2$, with u odd. Therefore

$$2n^2 = \left(\frac{u-1}{2}\right)\left(\frac{u+1}{2}\right)\cdot \tag{11}$$

Letting $A = \dfrac{u-1}{2}$, we get from (10) and (11)

$$r = A - n, \tag{12}$$

and

$$n^2 = \frac{A\,(A+1)}{2} = 1 + \cdots + A. \tag{13}$$

Consequently, n^2 is a triangle number (see also [1]).

`Case 1.` If A is even, then from (13) we have $n^2 = \frac{A}{2}\,(A+1)$. Letting $a = \frac{A}{2}$, we get

$$n^2 = a\,(2a+1)\,. \tag{14}$$

Since a and $2a+1$ are coprime, they are both necessarily perfect squares. Hence, from (12) and (14), we get

$$\begin{aligned} a &= d^2, \\ r &= 2d^2 - n, \\ n &= d\sqrt{2d^2+1}. \end{aligned} \tag{15}$$

`Case 2.` If A is odd, we obtain from (13) that $n^2 = \left(\frac{A+1}{2}\right) A\cdot$ Letting $a = \frac{A+1}{2}$, we get

$$n^2 = a\,(2a-1)\,. \tag{16}$$

Since a and $2a-1$ are coprime, they are necessarily both perfect squares. Hence, from (12) and (16), we get

$$\begin{aligned} a &= d^2, \\ r &= 2d^2 - n - 1, \\ n &= d\sqrt{2d^2-1}. \end{aligned} \tag{17}$$

Now we are in a position to formulate our result as follows:

Theorem 1. *Let n be a positive integer. The number n is a balancing number if and only if there exists a proper divisor d of n (except for $n = 1$) for which $2d^2 + 1$ or $2d^2 - 1$ is a perfect square. The pair (n, r) of each balancing with its cobalancer is then explicitly given by*

$$(n,r) = \begin{cases} \left(d\sqrt{2d^2+1}, \quad 2d^2 - n\right) & \text{if} \quad 2d^2+1 \text{ is a perfect square,} \\ \left(d\sqrt{2d^2-1}, \quad 2d^2 - n - 1\right) & \text{if} \quad 2d^2-1 \text{ is a perfect square.} \end{cases}$$

Table 1 summarizes the 10 first balancing numbers based on Theorem 1.

d	$2d^2-1$	$2d^2+1$	n	r
1	1		$1\sqrt{1} = 1$	0
2		9	$2\sqrt{9} = 6$	2
5	49		$5\sqrt{49} = 35$	14
12		289	$12\sqrt{289} = 204$	84
29	1681		$29\sqrt{1681} = 1189$	492
70		9801	$70\sqrt{9801} = 6930$	2870
169	57121		$169\sqrt{57121} = 40391$	16730
408		332929	$408\sqrt{332929} = 235416$	97512
985	1940449		$985\sqrt{1940449} = 1372105$	568344
2378		11309769	$2378\sqrt{11309769} = 7997214$	3312554

Table 1.

Remark 1. Theorem 1 proves that no prime number could be a balancing number. This result was also obtained by Panda et al., who showed that $B_m = P_m Q_m$, where P_m and Q_m are the m^{th} Pell number and the m^{th} associated Pell number respectively [6].

4. An Explicit Formula for Balancing Numbers and Some New Identities

A quick glance at Table 1 seems to indicate that the balancing numbers are alternatively odd and even (see also [8]), while the balancer numbers are even. In the present section we prove this indication in a more explicit form. Indeed, from (15) and (17), we have both,

$$\left(\frac{n}{d}\right)^2 - 2d^2 = 1, \tag{18}$$

and

$$\left(\frac{n}{d}\right)^2 - 2d^2 = -1. \tag{19}$$

Letting $x = \frac{n}{d}$ and $y = d$, Equations (18) and (19) become the Pell equations

$$x^2 - 2y^2 = 1, \tag{20}$$

and

$$x^2 - 2y^2 = -1, \tag{21}$$

respectively. According to (6) and (7), all the solutions to Equations (20) and (21) are given by

$$\begin{aligned} x + \sqrt{2}y &= \left(1+\sqrt{2}\right)^{2m} \\ &= \sum_{i=0}^{2m} \binom{2m}{i} 2^{i/2} \\ &= \left(\sum_{i=0}^{m} \binom{2m}{2i} 2^i\right) + \sqrt{2}\left(\sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^i\right), \end{aligned} \tag{22}$$

and

$$\begin{aligned} x + \sqrt{2}y &= \left(1+\sqrt{2}\right)^{m} \\ &= \sum_{i=0}^{m} \binom{m}{i} 2^{i/2} \\ &= \left(\sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m}{2i} 2^i\right) + \sqrt{2}\left(\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2i+1} 2^i\right), \end{aligned}$$

respectively, with m a positive integer.
Substituting x by $\dfrac{n}{d}$ and d by y, we get after identification

$$B_{2m-1} = n = yx = \left(\sum_{i=0}^{m-1} \binom{2m-1}{2i+1} 2^i\right)\left(\sum_{i=0}^{m-1} \binom{2m-1}{2i} 2^i\right),$$

and

$$B_{2m} = n = yx = \left(\sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^i\right)\left(\sum_{i=0}^{m} \binom{2m}{2i} 2^i\right).$$

for $m \geq 1$.

Since both $\sum_{i=0}^{m-1} \binom{2m-1}{2i+1} 2^i$ and $\sum_{i=0}^{m-1} \binom{2m-1}{2i} 2^i$ are odd, the balancing numbers of the subsequence $\{B_{2m-1}\}_{m\geq 1}$ are odd as well. Similarly, since $\sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^i$ is even, the balancing numbers of the subsequence $\{B_{2m}\}_{m\geq 1}$ are even. Hence, according to Theorem 1, we have proved the following theorem.

Theorem 2. *For any positive integer $m \geq 1$, (B_{2m}, R_{2m}) is an even-even pair and (B_{2m-1}, R_{2m-1}) is an odd-even pair and we have*

$$\begin{aligned} B_{2m-1} &= \left(\sum_{i=0}^{m-1} \binom{2m-1}{2i+1} 2^i\right) \left(\sum_{i=0}^{m-1} \binom{2m-1}{2i} 2^i\right), \\ R_{2m-1} &= 2\left(\sum_{i=0}^{m-1} \binom{2m-1}{2i+1} 2^i\right)^2 - B_{2m-1} - 1, \end{aligned}$$

and

$$\begin{aligned} B_{2m} &= \left(\sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^i\right) \left(\sum_{i=0}^{m} \binom{2m}{2i} 2^i\right), \\ R_{2m} &= 2\left(\sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^i\right)^2 - B_{2m}. \end{aligned}$$

Now let us rewrite Equation (9) as $(2(r+n)+1)^2 - 2(2n)^2 = 1$. Letting $x = 2(r+n)+1$ and $y = 2n$, we find Pell's equation (20) again. By identification, according to (22), we get

$$\begin{aligned} n &= \frac{y}{2} \\ &= \frac{1}{2}\sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^i \\ &= m + \sum_{i=0}^{m-2} \binom{2m}{2i+3} 2^i \\ &= \sum_{i=-1}^{m-2} \binom{2m}{2i+3} 2^i, \end{aligned} \tag{23}$$

and since $x = 2r + y + 1$, we get

$$\begin{aligned} r &= \frac{x-y-1}{2} \\ &= -n + \frac{x-1}{2} \\ &= -n + \frac{1}{2}\left(-1 + \sum_{i=0}^{m} \binom{2m}{2i} 2^i\right) \\ &= -n + \sum_{i=1}^{m} \binom{2m}{2i} 2^{i-1} \\ &= -n + \sum_{i=0}^{m-1} \binom{2m}{2i+2} 2^i. \end{aligned} \tag{24}$$

We have thus proved, via the above discussion, the following theorem.

Theorem 3. *For $m \geq 1$, the balancing number B_m and its balancer number R_m are given by*

$$B_m = \sum_{i=-1}^{m-2} \binom{2m}{2i+3} 2^i \quad \text{and} \quad R_m = -B_m + \sum_{i=0}^{m-1} \binom{2m}{2i+2} 2^i.$$

The following identities on binomial coefficients are a direct consequence of both Theorem 2 and Theorem 3.

Corollary 2. *For $m \geq 1$, we have*

$$\sum_{i=-1}^{2m-2} \binom{4m}{2i+3} 2^i = \left(\sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^i \right) \left(\sum_{i=0}^{m} \binom{2m}{2i} 2^i \right),$$

$$\sum_{i=-1}^{2m-3} \binom{4m-2}{2i+3} 2^i = \left(\sum_{i=0}^{m-1} \binom{2m-1}{2i+1} 2^i \right) \left(\sum_{i=0}^{m-1} \binom{2m-1}{2i} 2^i \right),$$

$$\sum_{i=0}^{2m-1} \binom{4m}{2i+2} 2^i = 2 \left(\sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^i \right)^2 = \left(\sum_{i=0}^{m} \binom{2m}{2i} 2^i \right)^2 - 1,$$

$$\sum_{i=0}^{2m-2} \binom{4m-2}{2i+2} 2^i = 2 \left(\sum_{i=0}^{m-1} \binom{2m-1}{2i+1} 2^i \right)^2 - 1 = \left(\sum_{i=0}^{m-1} \binom{2m-1}{2i} 2^i \right)^2.$$

Remark 2. In [8], Ray establishes an other interesting formula for B_m using the generating function $g(z) = \dfrac{z}{1-6z+z^2}$. He gets

$$B_m = \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \binom{m-i-1}{i} 6^{m-2i-1}.$$

From this Remark and Theorem 3, we obtain the new identity in the following Corollary.

Corollary 3. *For $m \geq 1$, we have*

$$\sum_{i=-1}^{m-2} \binom{2m}{2i+3} 2^i = \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \binom{m-i-1}{i} 6^{m-2i-1}.$$

5. An Explicit Formula for Cobalancing Numbers

From (2), we have

$$r^2 + (2n+1)\,r - n\,(n+1) = 0, \tag{25}$$

which, when solved for r gives

$$r = \frac{-\,(2n+1) + \sqrt{8n^2+8n+1}}{2}. \tag{26}$$

Since r is positive, $8n^2+8n+1$ is a perfect square, i.e.,

$$8n^2+8n+1 = u^2, \text{ with } u \text{ odd.} \tag{27}$$

Therefore,

$$2n\,(n+1) = \left(\frac{u-1}{2}\right)\left(\frac{u+1}{2}\right). \tag{28}$$

Letting $A = \dfrac{u-1}{2}$, we get from (26) and (28)

$$r = A - n,$$

and

$$n\,(n+1) = \frac{A\,(A+1)}{2} = 1 + \cdots + A. \tag{29}$$

Consequently, $n\,(n+1)$ is a triangle number (see also [8]).

Letting $x = 2\,(n-r)+1$ and $y = 2r$, Equation (25) leads again to the above Pell's equation (20). It follows from (23) and (24), that

$$r = \frac{y}{2} = B_m,$$

and

$$\begin{aligned} n &= \frac{x+y-1}{2} \\ &= \frac{x-y-1}{2} + y \\ &= R_m + 2r. \end{aligned}$$

The above discussion proves the following theorem.

Theorem 4. *For $m \geq 1$, the cobalacing number b_m and its cobalancer r_m are given by: $b_m = 2B_{m-1} + R_{m-1}$ and $r_m = B_{m-1}$, with $B_0 = R_0 = 0$.*

m	$b_m = 2B_{m-1} + R_{m-1}$	$r_m = B_{m-1}$
1	0	0
2	2	1
3	14	6
4	84	35
5	492	204
6	2870	1189
7	16730	6930
8	97512	40391
9	568344	235416
10	3312554	1372105

Table 2.

Table 2 summarizes the 10 first cobalancing numbers with there cobalancers, based on Table 1 and Theorem 4.
The following corollary is a direct consequence of Theorem 3 and Theorem 4.

Corollary 4. *For $m \geq 1$, we have*

$$b_{m+1} = \sum_{i=1}^{2m} \binom{2m}{i} 2^{\left\lfloor \frac{i-2}{2} \right\rfloor} \quad \textit{and} \quad r_{m+1} = \sum_{i=-1}^{m-2} \binom{2m}{2i+3} 2^i.$$

An immediate consequence of Theorems 2 and 4 is the following (see also [5]).

Corollary 5. *Every cobalancing number is even. Thus, no odd prime number could be a cobalancing number.*

6. New Formulas for (a, b)-Type Balancing and (a, b)-Type Cobalancing Numbers

Panda [7] defines sequence balancing and sequence cobalancing numbers as follows:

Definition 1. Let $\{u_n\}_{n\geq 1}$ be a sequence of real numbers. The number u_n is called a sequence balancing number if there exists a natural number r such that

$$u_1 + u_2 + \cdots + u_{n-1} = u_{n+1} + u_{n+2} + \cdots + u_{n+r}.$$

Similarly, the number u_n is called a sequence cobalancing number if

$$u_1 + u_2 + \cdots + u_n = u_{n+1} + u_{n+2} + \cdots + u_{n+r},$$

for some natural number r.

Kovács et al. [3] extend the concept of balancing numbers to arithmetic progressions as follows:

Definition 2. Let a, b be nonnegative coprime integers. If for some positive integers n and r, we have

$$(a+b)+\cdots+(a(n-1)+b)=(a(n+1)+b)+\cdots+(a(n+r)+b), \tag{30}$$

then we say that $an+b$ is an (a,b)-type balancing number.

Similarly, $an+b$ is an (a,b)-type cobalancing number if

$$(a+b)+\cdots+(an+b)=(a(n+1)+b)+\cdots+(a(n+r)+b), \tag{31}$$

for some natural number r.

Let $B_m^{(a,b)}$, $R_m^{(a,b)}$, $b_m^{(a,b)}$ and $r_m^{(a,b)}$ denote the m^{th} (a,b)-type balancing number, the m^{th} (a,b)-type cobalancing number, the m^{th} (a,b)-type balancer and the m^{th} (a,b)-type cobalancer, respectively.

6.0.1. (a,b)-Type Balancing Numbers

From (30), we have

$$an\,(n-1)+2b\,(n-1)-2arn-ar\,(r+1)-2br=0,$$

which, via straightforward calculations, is equivalent to

$$(2a\,(n-r-1)+a+2b)^2-2\,(a\,(2r+1))^2=(a+2b)^2-2a^2. \tag{32}$$

Letting $x=2a\,(n-r-1)+a+2b$, $y=a\,(2r+1)$, $u=a+2b$ and $v=a$, Equation (25) becomes:

$$x^2-2y^2=u^2-2v^2, \tag{33}$$

which has from (8), the integral solutions in the form:

$$x+y\sqrt{2}=\left(u+v\sqrt{2}\right)\left(1+\sqrt{2}\right)^{2m},\ m\geq 0. \tag{34}$$

From (22), we obtain

$$\begin{aligned}x+y\sqrt{2}=&\left(u\sum_{i=0}^{m}\binom{2m}{2i}2^i+2v\sum_{i=0}^{m-1}\binom{2m}{2i+1}2^i\right)\\&+\sqrt{2}\left(v\sum_{i=0}^{m}\binom{2m}{2i}2^i+u\sum_{i=0}^{m-1}\binom{2m}{2i+1}2^i\right).\end{aligned}$$

After identification, we get

$$2a\left(n-r-1\right)+a+2b=\left(a+2b\right)\sum_{i=0}^{m}\binom{2m}{2i}2^{i}+2a\sum_{i=0}^{m-1}\binom{2m}{2i+1}2^{i},$$

and

$$a\left(2r+1\right)=a\sum_{i=0}^{m}\binom{2m}{2i}2^{i}+\left(a+2b\right)\sum_{i=0}^{m-1}\binom{2m}{2i+1}2^{i}.$$

Therefore

$$n=1+r+\frac{a+2b}{a}\sum_{i=0}^{m-1}\binom{2m}{2i+2}2^{i}+\sum_{i=0}^{m-1}\binom{2m}{2i+1}2^{i},$$

and

$$r=\sum_{i=0}^{m-1}\binom{2m}{2i+2}2^{i}+\frac{a+2b}{a}\sum_{i=-1}^{m-2}\binom{2m}{2i+3}2^{i}.$$

From Theorem 3 and Theorem 4, we obtain

$$\begin{aligned} n &= 1+r+\frac{a+2b}{a}\left(B_m+R_m\right)+2B_m \\ &= 1+r+\frac{a+2b}{a}\left(b_{m+1}-r_{m+1}\right)+2r_{m+1} \\ &= 1+r+\frac{a-2b}{a}\ r_{m+1}+\frac{a+2b}{a}\ b_{m+1} \\ &= 1+r+r_{m+1}+b_{m+1}+\frac{2b}{a}\left(b_{m+1}-r_{m+1}\right), \end{aligned}$$

and

$$\begin{aligned} r &= B_m+R_m+\frac{a+2b}{a}\ B_m \\ &= b_{m+1}-r_{m+1}+\frac{a+2b}{a}\ r_{m+1} \\ &= b_{m+1}+\frac{2b}{a}\ r_{m+1}. \end{aligned}$$

Since n and r are positive integers and a and b are coprime, $2b_{m+1}$ and $2r_{m+1}$ should be both divisible by a. This discussion proves the following theorem.

Theorem 5. *Let* $\left(b_{\varphi(m)}^{/a},\ r_{\varphi(m)}^{/a}\right)$ *denote the* m^{th} *pair of cobalacing number and its cobalancer such that* $2b_{\varphi(m)}^{/a}$ *and* $2r_{\varphi(m)}^{/a}$ *are both divisible by* a*. Then we have*

$$B_m^{(a,b)}=1+r_{\varphi(m)}^{/a}+\frac{2\left(a+b\right)}{a}\ b_{\varphi(m)}^{/a},$$

and

$$R_m^{(a,b)} = b_{\varphi(m)}^{/a} + \frac{2b}{a}\ r_{\varphi(m)}^{/a}.$$

Example 1. Let $a = 9$. The first pair $\left(b_{\varphi(1)}^{/9}, r_{\varphi(1)}^{/9}\right)$ of cobalancing number and its cobalancer both divisible by 9 is $(b_1, r_1) = (0, 0)$. Hence

$$B_1^{(9,b)} = 1 \ \text{ and } \ R_1^{(9,b)} = 0.$$

According to Corollary 4 and using Maple, the second pair $\left(b_{\varphi(2)}^{/9}, r_{\varphi(2)}^{/9}\right)$ of cobalancing number and its cobalancer both divisible by 9 is $(b_{13}, r_{13}) = (655869060, 271669860)$. Thus

$$B_2^{(9,b)} \quad = \quad 1 + r_{13} + \frac{2\,(9+b)}{9}\ b_{13} = 1583407981 + 145748680\ b,$$

and

$$R_2^{(9,b)} \quad = \quad b_{13} + \frac{2b}{9}\ r_{13} = 655869060 + 60371080\ b.$$

6.1. (a, b)-Type Cobalancing Numbers

From (31), we have

$$an\,(n+1) + 2bn - 2arn - ar\,(r+1) - 2br = 0,$$

which, via straightforward calculations, is equivalent to

$$\left(a\,(2n - 2r + 1) + 2b\right)^2 - 2\,(2ar)^2 = (a + 2b)^2\,. \tag{35}$$

Then, from (8) and (22), we obtain

$$a\,(2n - 2r + 1) + 2b = (a + 2b)\sum_{i=0}^{m}\binom{2m}{2i}2^i,$$

i.e.,

$$\begin{aligned} n \quad &= \quad r + \frac{a+2b}{2a}\sum_{i=1}^{m}\binom{2m}{2i}2^i \\ &= \quad r + \frac{a+2b}{a}\sum_{i=0}^{m-1}\binom{2m}{2i+2}2^i, \end{aligned}$$

and

$$\begin{aligned} r &= \frac{a+2b}{2a}\sum_{i=0}^{m-1}\binom{2m}{2i+1}2^i \\ &= \frac{a+2b}{a}\sum_{i=-1}^{m-2}\binom{2m}{2i+3}2^i. \end{aligned}$$

Hence, from Corollary 4, Theorem 3 and Theorem 4, we get

$$r = \frac{a+2b}{a}\, r_{m+1},$$

and

$$\begin{aligned} n &= r + \frac{a+2b}{a}\,(R_m + B_m) \\ &= \frac{a+2b}{a}\, r_{m+1} + \frac{a+2b}{a}\,(b_{m+1} - r_{m+1}) \\ &= \frac{a+2b}{a}\, b_{m+1}. \end{aligned}$$

Since n and r are positive integers and a and b are coprime, then $2b_{m+1}$ and $2r_{m+1}$ should be both divisible by a. Hence we have proved the following theorem.

Theorem 6. *Let* $\left(b^{/a}_{\varphi(m)}, r^{/a}_{\varphi(m)}\right)$ *denote the* m^{th} *pair of cobalacing number and its cobalancer such that* $2b^{/a}_{\varphi(m)}$ *and* $2r^{/a}_{\varphi(m)}$ *are both divisible by* a*. Then we have*

$$b_m^{(a,b)} = \frac{a+2b}{a}\, b^{/a}_{\varphi(m)} \quad \text{and} \quad r_m^{(a,b)} = \frac{a+2b}{a}\, r^{/a}_{\varphi(m)}.$$

Example 2. For $a = 9$, we have $b_1^{(9,b)} = r_1^{(9,b)} = 0$, and according to Example 1, we get

$$b_2^{(9,b)} = \frac{9+2b}{9}\, b_{13} = 72874340\,(9+2b)\,,$$

and

$$r_2^{(9,b)} = \frac{9+2b}{9}\, r_{13} = 30185540\,(9+2b)\,.$$

Acknowledgements. The authors would like to thank the anonymous referee for his valuable comments which have improved the quality of the manuscript. We also thank Professor Hannoun Noureddine for his help in improving the English.

References

[1] A. Behera and G. K. Panda, *On the square roots of triangular numbers*, Fibonacci Quart. 37 (1999), 98–105.

[2] Z. I. Borevich and I. R. Shafarevich, Number Theory, Academic Press, 1966.

[3] T. Kovács, K. Liptai, P. Olajos, *On (a,b)-balancing numbers*, Publ. Math. Debrecen (accepted).

[4] G. K. Panda, *Some fascinating properties of balancing numbers*, to appear in "Applications of Fibonacci Numbers" Vol. 10, Kluwer Academic Pub., 2006.

[5] G. K. Panda and P. K. Ray, *Cobalancing numbers and cobalancers*, Int. J. Math. Math. Sciences, 8 (2005), 1189–1200.

[6] G. K. Panda and P. K. Ray, *Some links of balancing and cobalancing numbers with Pell and associated Pell numbers*, Bull. Inst. Math. Acad. Sin. (N.S.) 6(1), (2011), 41–72.

[7] G. K. Panda, *Sequence balancing and cobalancing numbers*, Fibonacci Quart. 45 (2007), 265–271.

[8] P. K. Ray, Balancing and cobalancing numbers, Ph. D. Thesis, Submitted to National Institute of Technology, Rourkela, India, August, 2009.