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A COMBINATORIAL PROOF OF TWO EQUIVALENT IDENTITIES BY FREE 2-MOTZKIN PATHS

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Abstract

We present a combinatorial proof of the equivalence between a formula of MacMahon and a formula of Gould using free 2-Motzkin paths, and give a generalization for a formula asked by Gould as well.

1. Introduction

More than a century ago, MacMahon [4] derived a well-known formula as follows:

$$\sum_{k=0}^{n}\binom{n}{k}^3=\sum_{0\le k\le \frac{n}{2}}\binom{n}{2k}\binom{2k}{k}\binom{n+k}{k}2^{n-2k}, \tag{1}$$

and further obtained that

$$\sum_{k=0}^{n}\binom{n}{k}^3x^ky^{n-k}=\sum_{0\le k\le \frac{n}{2}}\binom{n}{2k}\binom{2k}{k}\binom{n+k}{k}x^ky^k(x+y)^{n-2k}. \tag{2}$$

Recently, Gould [2] found that

$$\sum_{k=0}^{n}\binom{n}{k}^3=\sum_{0\le k\le \frac{n}{2}}\binom{n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k} \tag{3}$$

by using the Carlitz's formula [1]

$$\sum_{k=0}^{n}\binom{n}{k}^3=[x^n](1-x^2)^nP_n\left(\frac{1+x}{1-x}\right), \tag{4}$$

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where $[x^n]f(x)$ means the coefficient of x^n in the series expansion of $f(x)$, and $P_n(x)$ is the Legendre polynomial defined by

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}}.$$

In deriving the equivalence between the right-hand sides of formulas (1) and (3), Gould [2] explored the identity

$$\sum_{k\leq j\leq \frac{n}{2}} \binom{n}{2j}\binom{2j}{j}\binom{j}{k}2^{n-2j} = \binom{n}{k}\binom{2n-2k}{n}, \tag{5}$$

and asked for an extension analogous to (2) of MacMahon.

In this note, we aim to prove the equivalence of the formulas (1) and (3) combinatorially in terms of free 2-Motzkin paths and meanwhile give a generalization of the identity (5).

2. Free 2-Motzkin paths

We first introduce some definitions about a special kind of lattice path. A free 2-Motzkin path of length n is a lattice path starting at $(0,0)$, ending at $(n,0)$, with possible up steps $(1,1)$, down steps $(1,-1)$ and level steps $(1,0)$, where the level steps can be either of two kinds: straight or wavy. Let $\mathcal{M}(n)$ be the set of free 2-Motzkin paths of length n. More generally, let $\mathcal{M}(n-k,-k)$ be the set of lattice paths starting at $(0,0)$, ending at $(n-k,-k)$, with the same possible steps as free 2-Motzkin paths. It is easy to see that the set of free 2-Moztkin paths is just the set $\mathcal{M}(n,0)$.

Proposition 2.1. *The number of free 2-Motzkin paths in $\mathcal{M}(n)$ with a total of k up and straight level steps equals $\binom{n}{k}^2$.*

Proof. For a free 2-Motzkin path in $\mathcal{M}(n)$ with a total of k up and straight level steps, if there are i $(0\leq i\leq k)$ up steps, then there are $k-i$ straight level steps. We can generate such a path by first choosing i positions from the n positions to place the i up steps and another i positions from the $n-i$ positions that are left to place the i down steps in $\binom{n}{i}\binom{n-i}{i}$ ways; then choosing $k-i$ positions from the remaining $n-2i$ positions for the straight level steps, and putting the wavy level steps on the positions that are left. Thus we have $\binom{n}{i}\binom{n-i}{i}\binom{n-2i}{k-i}$ choices to generate such a path with i up steps and $k-i$ straight level steps. Since

$$\binom{n}{i}\binom{n-i}{i}\binom{n-2i}{k-i} = \binom{n}{k}\binom{k}{i}\binom{n-k}{n-k-i},$$

summing up all possible i and using Vandermonde's convolution formula [3]

$$\sum_{i=0}^{k}\binom{k}{i}\binom{n-k}{n-k-i}=\binom{n}{k},$$

the proposition is followed. □

Proposition 2.2. *The number of paths in* $\mathcal{M}(n-k,-k)$ $(k\geq 0)$ *equals* $\binom{2n-2k}{n}$.

Proof. If there are i $(k\leq i\leq n-k)$ down steps in a path of $\mathcal{M}(n-k,-k)$, then there are $i-k$ up steps and $n-2i$ level steps. Such a path can be constructed as follows: first we could arrange those up and down steps in $\binom{2i-k}{i}$ ways, and then the remaining $n-2i$ level steps can be repeatedly put in the $2i-k+1$ positions in between the $2i-k$ determined steps in $\binom{2i-k+1+n-2i-1}{n-2i}=\binom{n-k}{n-2i}$ ways. Since a level step can be straight or wavy, the number of such paths with i down steps is $\binom{2i-k}{i}\binom{n-k}{n-2i}2^{n-2i}$.

Because of $\binom{2i-k}{i}\binom{n-k}{n-2i}2^{n-2i}=\binom{n-k}{i}\binom{n-k-i}{n-2i}2^{n-2i}$, it remains to show that

$$\sum_{i=k}^{n-k}\binom{n-k}{i}\binom{n-k-i}{n-2i}2^{n-2i}=\binom{2n-2k}{n}.$$

Evaluating in two different ways in the expansion on t of the expression $(1+t)^{2n-2k}$, we have

$$(1+t)^{2n-2k}=\sum_{n\geq 0}\binom{2n-2k}{n}t^n;$$

on the other hand,

$$(1+t)^{2n-2k}=(1+2t+t^2)^{n-k}=\sum_{i\geq 0}\binom{n-k}{i}t^{2i}(1+2t)^{n-k-i}$$

$$=\sum_{i\geq 0}\binom{n-k}{i}t^{2i}\sum_{j\geq 0}\binom{n-k-i}{j}2^jt^j=\sum_{n\geq 0}\sum_{i\geq 0}\binom{n-k}{i}\binom{n-k-i}{n-2i}2^{n-2i}t^n.$$

Comparison of the coefficient of t^n in the above leads to the desired result. □

3. The Combinatorial Proof

A colored free 2-Motzkin path is a free 2-Motzkin path with some steps (up, down or level) being colored, say yellow. Define $\mathcal{CM}(n)$ be the set of colored free 2-Motzkin paths of length n in which the number of yellow steps equals the sum of the number of up steps and straight level steps. If there are k up and straight level steps in a path of $\mathcal{CM}(n)$, then we have $\binom{n}{k}$ choices to color the yellow steps. From Proposition 2.1, we get

$$|\mathcal{CM}(n)| = \sum_{k=0}^{n} \binom{n}{k}^3. \tag{6}$$

The proof of formula (3) is based on a different enumeration order for the counting of the set $\mathcal{CM}(n)$.

Theorem 3.1. *We have*

$$|\mathcal{CM}(n)| = \sum_{0 \le k \le \frac{n}{2}} \binom{n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}. \tag{7}$$

Proof. Suppose that there are k up steps and i straight level steps in a colored free 2-Motzkin path of $\mathcal{CM}(n)$. We can construct such a path by first choosing k positions to place the k up steps, and then choosing k down steps from the remaining $n-k$ steps. After this we choose i positions for the straight level steps and the wavy level steps are placed on the positions that are left. Finally, we pick $i+k$ steps from the n steps to color them yellow. Thus the total number of such paths is

$$\binom{n}{k}\binom{n-k}{k}\binom{n-2k}{i}\binom{n}{k+i}.$$

Summing up all the possible i and k, and using the formula

$$\sum_{0 \le i \le n-2k} \binom{n-2k}{i}\binom{n}{k+i} = \binom{2n-2k}{n-k},$$

we obtain that the number of colored free 2-Motzkin paths of length n equals

$$\sum_{0 \le k \le \frac{n}{2}} \binom{n}{k}\binom{n-k}{k} \sum_{0 \le i \le n-2k} \binom{n-2k}{i}\binom{n}{k+i} = \sum_{0 \le k \le \frac{n}{2}} \binom{n}{k}\binom{n-k}{k}\binom{2n-2k}{n-k}.$$

The proof is completed by noticing that $\binom{n}{k}\binom{n-k}{k} = \binom{n}{2k}\binom{2k}{k}$. □

For formula (1), we introduce another combinatorial structure which is motivated by observing that the number of the elements in the set $\mathcal{CM}(n)$ can also be evaluated as follows:

Proposition 3.2. *We have*

$$|\mathcal{CM}(n)| = \sum_{0 \le k \le \frac{n}{2}} \binom{n}{k}^2 \binom{2n-2k}{n}. \tag{8}$$

Proof. If there are k up steps and i straight level steps in a colored free 2-Motzkin path in $\mathcal{CM}(n)$, then we could construct such a path by first choosing k positions for the steps with yellow color. After this, we pick k positions to place the k up steps, k other positions to place the k down steps, and i positions for the straight level steps. Finally, we choose another i steps to color them yellow. Since we have colored k steps initially, the choice of the remaining i yellow steps equals $\binom{n-k}{i}/\frac{(k+i)!}{k!i!}$. Thus the total number of such colored paths is

$$\binom{n}{k}\binom{n}{k}\binom{n-k}{k}\binom{n-2k}{i}\binom{n-k}{i}\Big/\frac{(k+i)!}{k!i!}.$$

Summing up all the possible i, we have

$$\sum_{0\le i\le n-2k}\binom{n-k}{k}\binom{n-2k}{i}\binom{n-k}{i}\Big/\frac{(k+i)!}{k!i!}$$
$$=\sum_{0\le i\le n-2k}\binom{n-k}{k+i}\binom{n-k}{n-k-i}=\binom{2n-2k}{n}.$$

Thus, the number of colored free 2-Motzkin paths of length n is given by

$$\sum_{0\le k\le \frac{n}{2}}\binom{n}{k}^2\binom{2n-2k}{n},$$

which completes the proof. □

The above proposition can be regarded as a combinatorial explanation for the identity

$$\binom{n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}=\binom{n}{k}^2\binom{2n-2k}{n}.$$

Now we are in the position to introduce the desired structure in the process of the proof of (1). A double colored free 2-Motzkin path is a free 2-Motzkin path such that some steps (up, down or level) can be colored by two colors, say yellow and green. Define $\mathcal{DM}(n)$ be the set of double colored free 2-Motzkin paths of length n such that the up steps can be colored by yellow or green and other steps can be colored only by green, moreover the number of yellow steps equals the number of green steps.

Theorem 3.3. *We have*

$$|\mathcal{DM}(n)|=\sum_{0\le k\le \frac{n}{2}}\binom{n}{k}^2\binom{2n-2k}{n}.$$

Proof. Suppose that there are k yellow up steps in a path of $\mathcal{DM}(n)$. We can construct a double colored free 2-Motzkin path by first choosing k positions for the up steps with yellow color, and the remaining $n-k$ steps can be regarded as a free 2-Motzkin path in $\mathcal{M}(n-k,-k)$. From Proposition 2.2, we see that the number of such paths is $\binom{2n-2k}{n}$. Finally it suffices to choose k steps to color them green. Therefore, we produce a double colored free 2-Motzkin path with k up steps. Summing up all the possible k, we derive that the total number of paths in $\mathcal{DM}(n)$ is

$$\sum_{0\le k\le \frac{n}{2}} \binom{n}{k}\binom{2n-2k}{n}\binom{n}{k},$$

and this completes the proof. □

From Proposition 3.2 and Theorem 3.3, we see that $|\mathcal{CM}(n)| = |\mathcal{DM}(n)|$. The equivalence between formulas (3) and (1) is established by showing the following:

Theorem 3.4. *We have*

$$|\mathcal{DM}(n)| = \sum_{0\le k\le \frac{n}{2}} \binom{n}{2k}\binom{2k}{k}\binom{n+k}{k}2^{n-2k}.$$

Proof. Suppose that there are k up steps in a path of $\mathcal{DM}(n)$, and such a path can be constructed by first establishing a free 2-Motzkin path with k up steps, and then coloring its steps. If there are i yellow up steps, then the total number of colorings is

$$\sum_{0\le i\le k} \binom{k}{i}\binom{n}{i} = \binom{n+k}{n},$$

and the number of free 2-Motzkin path of length n with k up steps equals

$$\binom{n}{k}\binom{n-k}{k}2^{n-2k}.$$

This theorem follows by summing up all possible k. □

In order to give a generalization of the formula (5), we will use the weighted path as an underlying set. A weighted colored free 2-Motzkin path is a path in which every step is endowed with a weight. The weight of a path is the product of the weights of the steps and the weight of a set of paths means the sum of the weights of the paths in it. For our purpose, we define the weight of the paths in $\mathcal{CM}(n)$ as follows: the up steps and straight level steps are given the weight x, whereas the down steps and wavy level steps are given the weight y. We present an extension of the formula (5) as follows.

Theorem 3.5. *The following holds:*

$$\sum_{k\le j\le \frac{n}{2}} \binom{n}{2j}\binom{2j}{j}\binom{j}{k} x^j y^j (x+y)^{n-2j} = \sum_{j=k}^{n-k} \binom{n}{k}\binom{n-k}{j}\binom{n-k}{n-j} x^j y^{n-j}. \quad (9)$$

Proof. By our weight assignment, the left-hand side of the formula (9) is the sum of the weighted colored free 2-Motzkin paths of length n with k yellow up steps.

Meanwhile, we can generate such a path by the following procedure: i) we choose k positions to put the k yellow up steps in $\binom{n}{k}$ ways; ii) if there are j down steps in such a colored free 2-Motzkin paths, then we have $j-k$ uncolored up steps, and we choose j positions from the remaining $n-k$ positions to place the down steps; iii) we pick $j-k$ positions from the $n-k-j$ positions that are left to place the uncolored up steps, and iv) the remaining $n-2j$ positions can be put by a straight level step with weight x or a wavy level step with weight y. Thus we have

$$\sum_{j=k}^{n-k} \binom{n}{k}\binom{n-k}{j}\binom{n-k-j}{j-k} x^j y^j (x+y)^{n-2j}$$

weighted paths of this kind. By expanding the expression $\left[1+(x+y)t+xyt^2\right]^{n-k}$, we see that

$$\begin{aligned}\left[1+(x+y)t+xyt^2\right]^{n-k} &= \sum_{j=0}^{n-k} \binom{n-k}{j} x^j y^j t^{2j} \left[1+(x+y)t\right]^{n-k-j} \\ &= \sum_{j=0}^{n-k} \binom{n-k}{j} x^j y^j t^{2j} \sum_{i=0}^{n-k-j} \binom{n-k-j}{i} (x+y)^i t^i.\end{aligned}$$

Also, we have

$$\begin{aligned}\left[1+(x+y)t+xyt^2\right]^{n-k} &= (1+xt)^{n-k}(1+yt)^{n-k} \\ &= \sum_{j=0}^{n-k} \binom{n-k}{j} x^j t^j \sum_{i=0}^{n-k} \binom{n-k}{i} y^i t^i.\end{aligned}$$

Equating the coefficient of t^n in the above two expansions, we have

$$\sum_{k\le j\le \frac{n}{2}} \binom{n-k}{j}\binom{n-k}{n-j} x^j y^{n-j} = \sum_{j=k}^{n-k} \binom{n-k}{j}\binom{n-k-j}{n-2j} x^j y^j (x+y)^{n-2j}.$$

This implies that weight of such kind of paths also equals

$$\binom{n}{k}\sum_{j=k}^{n-k} \binom{n-k}{j}\binom{n-k}{n-j} x^j y^{n-j},$$

and identity (9) follows. □

We see that the identity (9) is reduced to the identity (5) by setting $x = 1, y = 1$, and using the convolution

$$\sum_{j=k}^{n-k} \binom{n-k}{j} \binom{n-k}{n-j} = \binom{2n-2k}{n}.$$

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