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INTERSECTING RATIONAL BEATTY SEQUENCES

Jamie Simpson
Dept. of Mathematics and Statistics, Curtin University, Perth, WA, Australia
Jamie.Simpson@curtin.edu.au

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Abstract

A rational Beatty sequence has the form $\{\lfloor pi/q + b\rfloor : i \in \mathbb{Z}\}$ where $p > q > 0$ and $\gcd(p, q) = 1$. We call p/q the *modulus* of the sequence and b the *offset*. Morikawa gave a condition on the moduli of two Beatty sequences such that they would be disjoint for a suitable choice of offsets. Holzman and Fraenkel showed that the sequence formed by the intersection of two Beatty sequences with moduli p_1/q_1 and p_2/q_2, $q_2 \le q_1$, could have as many as q_2+3 distinct consecutive differences. In this note we show that if the moduli satisfy the Morikawa condition but the sequences do intersect then the consecutive differences take on at most three different values.

1. Introduction

A Beatty sequence has the form $\{\lfloor i\alpha + \beta\rfloor : i \in \mathbb{Z}\}$. We call α the *modulus* and β the *offset* of the sequence. The sequences were named for Samuel Beatty [1] who asked for a proof that two Beatty sequences, with offsets equal to 0 and moduli α_1 and α_2, partition the positive integers if both moduli are irrational and $1/\alpha_1 + 1/\alpha_2 = 1$. A proof of this pleasing result appeared in [2]. A Beatty sequence is *rational* or *irrational* according to whether its modulus is rational or irrational. Covering properties of irrational Beatty sequences are now well understood. See, for instance, [8] and its bibliography. This is not so for coverings by collections of rational Beatty sequences which are the subject of this paper. We write $S(p/q, b)$, where $\gcd(p, q) = 1$, for the Beatty Sequence $\{\lfloor pi/q + b\rfloor : i \in \mathbb{Z}\}$. We will assume throughout that b here is an integer – this involves no loss of generality by a result in [9]. The following famous conjecture is due to Aviezri Fraenkel [4].

Conjecture 1. If the collection of Beatty sequences $\{S(p_i/q_i, b_i) : i = 1, \dots, t\}$ partitions the integers with $t > 2$ then $\{p_1/q_1, \dots, p_t/q_t\} = \{(2^t - 1)/2^{t-i} : 1 \le i \le t\}$.

This conjecture has generated a considerable literature. The strongest result to date is by Bark and Varjú [3] who showed that any counterexample must have $t > 7$. See also the surveys [8], and Section F14 of [6].

A Beatty sequence may be regarded as an approximation to an arithmetic progression in that its consecutive differences take two values ($\lfloor \alpha \rfloor$ and $\lceil \alpha \rceil$) rather than one. The intersection properties of arithmetic progressions are given by the Chinese Remainder Theorem which we give, in a long-winded way, here.

Theorem 2. (Chinese Remainder Theorem) *Let a_1, a_2, b_1, b_2 be integers with a_1 and a_2 positive.*
(a) There exist integers b_1 and b_2 such that $S(a_1, b_1)$ and $S(a_2, b_2)$ are disjoint if and only if $\gcd(a_1, a_2) > 1$*.*
(b) If $\gcd(a_1, a_2) = 1$*, then the intersection of $S(a_1, b_1)$ and $S(a_2, b_2)$ is an arithmetic progression with common difference a_1a_2.*
(c) If $\gcd(a_1, a_2) > 1$*, and $S(a_1, b_1)$ and $S(a_2, b_2)$ do intersect, then their intersection is an arithmetic progression with common difference* $\operatorname{lcm}(a_1a_2)$*.*

The situation for Beatty sequences is more complicated. Instead of part (a) we have the following result of Ryozu Morikawa [7], [10].

Theorem 3. (Japanese Remainder Theorem) *With $p = (p_1, p_2)$, $q = (q_1, q_2)$, $q_1 = u_1q$ and $q_2 = u_2q$, there exist numbers b_1 and b_2 such that $S(p_1/q_1, b_1)$ and $S(p_2/q_2, b_2)$ are disjoint if and only if there exist positive integers x and y such that*

$$xu_1 + yu_2 = p - 2u_1u_2(q - 1). \tag{1}$$

When this is so we say that p_1/q_1 and p_2/q_2 satisfy the *Morikawa condition*.

Definition 4. If $a_1, \ldots, a_n$ is an increasing sequence of integers then we say that the differences $a_{i+1} - a_i$ are the *gap sizes* of the sequence. If S is a set of residues modulo p, whose members have been reduced modulo p to integers in the interval $[0, p-1]$, and labeled $g_1 \le g_2 \le \cdots \le g_n$, then the set of *gap sizes* of S is $\{g_{i+1} - g_i : i = 1, \ldots, n-1\} \cup \{p + g_1 - g_n\}$.

Instead of part (b) of Theorem 2 we have the following result of Fraenkel and Holzman [5].

Theorem 5. *If $S(p_1/q_1, b_1)$ and $S(p_2/q_2, b_2)$ are Beatty sequences whose moduli do not satisfy the Morikawa condition, and $q_1 \ge q_2 \ge 2$, then the intersection of the two sequences has at most $q_2 + 3$ distinct gap sizes.*

The bound here is best possible. In this paper we obtain an analogy of part (c) of Theorem 2 by giving a precise description of the intersection of two Beatty sequences whose moduli satisfy the Morikawa condition. In particular, it follows that in this case the intersection has at most three gap sizes.

2. Results

Notation 6. Throughout this section we will use the following notation. We will be considering Beatty sequences $S(p_1/q_1, b_1)$ and $S(p_2/q_2, b_2)$. We assume, without loss of generality, that $q_1 \le q_2$. We put $p = \gcd(p_1, p_2)$, $p_1 = mp$, and $p_2 = np$. This implies

$$\gcd(m, q_1) = \gcd(n, q_2) = 1.$$

We set $\overline{q}_1$ and $\overline{q}_2$ to be the least non-negative residues satisfying $q_1\overline{q}_1 \equiv -1 \pmod p$ and $q_2\overline{q}_2 \equiv -1 \pmod p$ respectively. Similarly, $\overline{q}_m$ and $\overline{q}_n$ are the least non-negative residues satisfying $q_1\overline{q}_m \equiv -1 \pmod m$ and $q_2\overline{q}_n \equiv -1 \pmod n$, respectively. We set $k_1 = (q_1\overline{q}_1 + 1)/p$ and $k_2 = (q_2\overline{q}_2 + 1)/p$.

The argument proceeds in three steps. In Theorem 11 we obtain an expression for the intersection of $S(p/q_1, b_1)$ and $S(p/q_2, b_2)$. This is used in Theorem 15 to obtain an expression for the intersection of $S(pm/q_1, b_1)$ and $S(p/q_2, b_2)$, and that result is used in Theorem 16 to obtain an expression for the intersection of $S(pm/q_1, b_1)$ and $S(pn/q_2, b_2)$.

Definition 7. Let b, n, p, d be positive integers with $n \le p$, $\gcd(p, d) = 1$ and $S = \{id + b \mod p : i = 0, \dots, n-1\}$. Reduce each member of S to an integer in $[0, p-1]$ and label them $g_1, \dots, g_n$, such that $g_1 \le g_2 \le \cdots \le g_n$. We say that this sequence is a *modular arithmetic progression modulo* p with *additive difference* d.

The following is easily derived from the usual Three Gap Theorem, see [11].

Theorem 8 (Three Gap Theorem). *The set of gap sizes of a modular arithmetic progression has cardinality at most* 3*, and if the cardinality equals* 3 *then the largest member of the set equals the sum of the other two.*

The following corollary follows immediately from the preceding theorem and Definition 7.

Corollary 9. *If* $g_1, \dots, g_n$ *is a modular arithmetic progression modulo* p*, then the set of gap sizes in the doubly infinite increasing sequence with range* $\{g_i + jp : 1 \le i \le n, j \in \mathbb{Z}\}$ *has cardinality at most* 3*, and if the cardinality equals* 3 *then the largest member of the set equals the sum of the other two.*

The Beatty sequence $S = S(p/q, b)$ has period p in the sense that $a \in S$ if and only if $a + p \in S$, and so is characterised by a set of residues modulo p. The following is Theorem 3 of [10].

Theorem 10. *The sequence* $S(p/q, b)$ *with* $\gcd(p, q) = 1$ *coincides with the set of integers congruent modulo* p *to a member of* $\{i\overline{q} + b : 0 \le i \le q-1\}$*, where* $q\overline{q} \equiv -1 \pmod p$*.*

Thus the set of residues in a Beatty sequence forms a modular arithmetic progression, and the Beatty sequence itself fulfils the conditions of Corollary 9. In fact the Beatty sequence has at most two gap sizes. These are $\lfloor p/q \rfloor$ and $\lceil p/q \rceil$ (which are equal when $q = 1$).

Theorem 11. *Let p, q_1 and q_2 be positive integers with $\gcd(p, q_1) = \gcd(p, q_2) = 1$ such that p/q_1 and p/q_2 satisfy the Morikawa condition. If b_1, b_2 are integers such that $S(p/q_1, b_1)$ and $S(p/q_2, b_2)$ intersect then the intersection is the set of residues*

$$\{a\overline{q}_1 + iG_1\overline{q}_1 + b_1 : 0 \leq i \leq t - 1\} \tag{2}$$

modulo p for some positive integer t where G_1 is the smallest gap size in $\{-q_1 i\overline{q}_2 - q_1 b_2 \pmod p) : i = 0, \ldots, q_2 - 1\}$ if $t > 2$, and the second or third smallest gap size if $t = 2$, and a is a non-negative integer satisfying

$$a + G_1(t - 1) < q_1. \tag{3}$$

The ideas of the following proof are illustrated in the accompanying figure.

Proof. Without loss of generality suppose $b_1 = 0$. Let B_1 be the set of residues modulo p in $S(p/q_1, b_1)$ and B_2 be the set in $S(p/q_2, b_2)$. Theorem 10 implies that $B_1 \equiv \{i\overline{q}_1 : i = 0, \ldots, q_1 - 1\} \pmod p$ and $B_2 \equiv \{i\overline{q}_2 + b_2 : i = 0 \ldots q_2 - 1\} \pmod p$. Let

$$\begin{aligned} B_1^* &\equiv \{-q_1 i\overline{q}_1 : 0 \leq i \leq q_1 - 1\} \pmod p \\ &\equiv \{0, \ldots, q_1 - 1\} \pmod p, \end{aligned}$$

and

$$B_2^* \equiv \{-q_1 i\overline{q}_2 - q_1 b_2 \bmod p : 0 \leq i \leq q_2 - 1\} \pmod p.$$

Clearly

$$-q_1(B_1 \cap B_2) \equiv B_1^* \cap B_2^* \pmod p \tag{4}$$

so $|B_1 \cap B_2| = |B_1^* \cap B_2^*|$. If $|B_1 \cap B_2| = 1$ then we have nothing to prove, and if $|B_1 \cap B_2| = 2$ then a simpler version of the proof applies (but note the comments at the end of the proof), so we assume $|B_1 \cap B_2| \geq 3$. Consider the set of gaps in the modular arithmetic progression B_2^*. By Theorem 8 there are at most 3 gap sizes. We will assume there are 3 (if there are less, then a simpler version of the proof applies) and that the gaps are $G_1 < G_2 < G_3$. Since the moduli of the Beatty sequences satisfy the Morikawa condition there will be some value for b_2 which makes $|B_1 \cap B_2|$ empty, and therefore $B_1^* \cap B_2^* = \{0, \ldots, q_1 - 1\} \cap B_2^*$ empty, which implies $G_3 > q_1$. Now consider a different value of b_2 for which the sequences do intersect (note that this doesn't change the gap sizes of B_2^*) and let the intersection be the sequence

$$0 \leq a_1 < a_2 < \cdots < a_t \leq q_1 - 1 \tag{5}$$

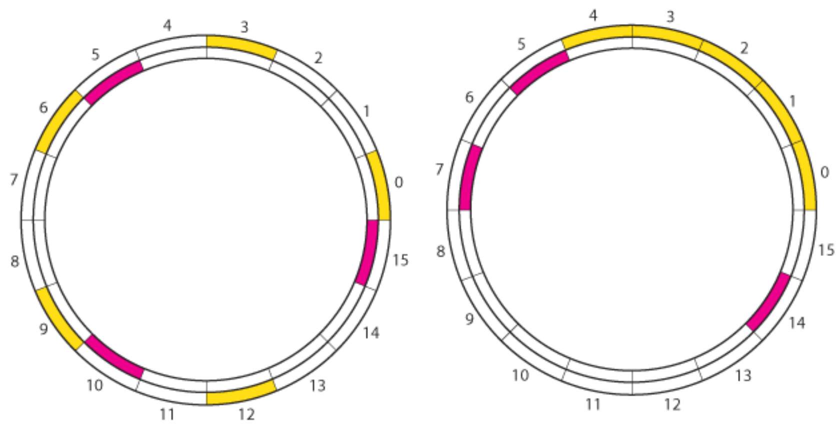

Figure 1: The diagram on the left shows the set of residues modulo 16 of $B_1 = S(16/5, 0)$ in the outer ring and $B_2 = S(16/3, 5)$ in the inner ring. The diagram on the right shows $B_1^* \equiv -5B_1 \pmod{16}$ and $B_2^* \equiv -5B_2 \pmod{16}$. The Beatty sequences do not intersect. Making a suitable change to the offset of B_1 will cause the outer rings to rotate so that they do.

where, by the assumption above, $t \geq 3$. We claim that the only gap size in this sequence is G_1. Clearly no gap can equal G_3 since

$$G_3 > q_1 > a_t - a_1. \tag{6}$$

Suppose G_2 occurs in the sequence. Then, since $t \geq 3$, there is an adjacent gap of size at least G_1. This implies $a_t - a_1 \geq G_1 + G_2$, but $G_1 + G_2 = G_3$ by Theorem 8 and we get a contradiction as in (6). Thus all gaps equal G_1 and

$$B_1^* \cap B_2^* = \{a + iG_1 : 0 \leq i \leq t-1\}$$

for some integer a. By (5) we have

$$a + (t-1)G_1 \leq q_1 - 1. \tag{7}$$

Therefore from (4)

$$\begin{aligned} B_1 \cap B_2 &\equiv \overline{q}_1\{a + iG_1 : 0 \leq i \leq t-1\} \pmod{p} \\ &\equiv \{a\overline{q}_1 + iG_1\overline{q}_1, 0 \leq i \leq t-1\} \pmod{p} \end{aligned}$$

which is (2). If there are only two elements in the intersection we cannot conclude that $a_2 - a_1 < G_2$. This observation leads to the anomalous case in the theorem. $\square$

This completes the analysis of the case when two Beatty sequences with the same numerator in their moduli intersect. Before progressing to the more general case we prove the following theorem and its corollary.

Theorem 12. *Let m and t be positive integers, and a_1, a_2, b_1 and b_2 be integers in the interval $[0, t-1]$. If*

$$\{a_1 i + b_1 : i = 0, \ldots, t-1\} \equiv \{a_2 i + b_2 : i = 0, \ldots, t-1\} \pmod{m} \tag{8}$$

and

$$(t+1)\gcd(a_1, a_2, m) < m \tag{9}$$

then either $a_1 \equiv a_2 \pmod{m}$ and $b_1 \equiv b_2 \pmod{m}$, or $a_1 \equiv -a_2 \pmod{m}$ and $b_2 \equiv b_1 + a_1(t-1) \pmod{m}$.

Proof. Let $S = \{a_1 i + b_1 : i = 0, \ldots, t-1\}$. We first show that neither $a_1 t + b_1$ nor $a_1(t+1) + b_1$ is congruent modulo m to a member of S. Suppose otherwise. If $a_1 t + b_1$ is congruent modulo m to a member of S, then

$$\begin{aligned} & a_1 t + b_1 \equiv a_1 i + b_1 \pmod{m} \text{ for some } i \in [0, t-1] \\ \Rightarrow \quad & a_1(t-i) \equiv 0 \pmod{m}, \end{aligned}$$

which implies $i = 0$ else the members of S would not be distinct. Hence m divides $a_1 t$ but m does not divide $a_1 i$ for any $i \in [1, t-1]$. Hence t divides m and m/t divides a_1. Thus $a_1 = Am/t$ for some integer A where $\gcd(A, m) = 1$ and

$$S \equiv \{(Am/t)i + b_1 : i = 0, \ldots, t-1\} \pmod{m}.$$

Clearly

$$\{(Am/t)i + b_1 : i = 0, \ldots, t-1\} \equiv \{(m/t)i + b_1 : i = 0, \ldots, t-1\} \pmod{m}$$

so $S \equiv \{mi/t + b_1 : i = 0, \ldots, t-1\} \pmod{m}$. It follows that m/t divides a_2. In fact we have

$$m | \gcd(a_1, m)t \text{ and } m | \gcd(a_2, m)t,$$

and thus m divides $\gcd(a_1, a_2, m)t$ which implies $t\gcd(a_1, a_2, m) \geq m$, contradicting (9). We conclude $a_1 t + b_1$ is not congruent modulo m to any member of S.

Now suppose $a_1(t+1) + b_1$ is congruent modulo m to a member of S. As above this leads to

$$a_1(t+1-i) \equiv 0 \pmod{m}$$

for some i in $[0, t-1]$. In order for the members of S to be distinct this implies $i = 0$ or $i = 1$. If $i = 1$ we get $a_1 t \equiv 0 \pmod{m}$ which is impossible as in the previous case. Using similar reasoning to the previous case we see that if $i = 0$ which implies that m divides $a_1(t+1)$, $t+1$ divides m, and

$$S \cup \{a_1 t + b_1\} \equiv \{mi/(t+1) + b_1 : i = 0, \ldots, t\} \pmod{m}.$$

Then $m/(t+1)$ divides a_2 which leads to

$$m | \gcd(a_1, a_2, m)(t+1),$$

implying that $(t+1)\gcd(a_1, a_2, m) \geq m$, again contradicting (9). We conclude that $a_1(t+1) + b_1$ is not congruent modulo m to any member of S.

By similar reasoning we conclude that neither $a_2 t + b_2$ nor $a_2(t+1) + b_2$ is congruent modulo m to a member of S.

Now consider the set $S' = \{a_2 i + b_2 : i = 1, \ldots, t\}$ modulo m. That is, S' is formed by adding a_2 to each member of S. Note that $|S \cap S'| = t - 1$ since $a_2(t+1) + b_2$ is not congruent modulo m to any member of S. Hence

$$S \cap S' = \{a_1 i + b_1 : i = 0, \ldots, t-1, i \neq j\}, \tag{10}$$

for some $j \in [0, t-1]$. We will show that j equals 0 or $t - 1$. Suppose, for the sake of contradiction, that $0 < j < t - 1$. Since $j > 0$, $a_1(j-1) + b_1$ belongs to $S \cap S'$. Then,

$$a_1(j-1) + b_1 \equiv a_1(t-1) + b_1 + a_2 \pmod{m}, \tag{11}$$

since if $t-1$ were replaced by k with $0 \leq k < t-1$, then $a_1 j + b_1$ would be congruent modulo m to $a_1(k+1) + b_1 + a_2$ and so belong to S'. From (11) we therefore get

$$a_2 \equiv (j-t)a_1 \pmod{m}.$$

Now from our assumption that $j < t-1$ we have $a_1(j+1)+b_1$ congruent to a member of S, and by (11) congruent to $a_1(t+1) + b_1 + a_2$ modulo m. So $a_1(t+1) + b_1 + a_2$ is congruent modulo m to a member of S', which implies that $a_1(t+1) + b_1$ is congruent to a member of S which we showed earlier to be impossible. We conclude that $j = 0$ or $j = t - 1$.

If $j = 0$ then (10) gives

$$S' \cap S \equiv \{a_1 i + b_1 : i = 1, \ldots, t-1\} \pmod{m},$$

so that no member is congruent modulo m to b_1, and $a_1 + b_1 \equiv a_1 k + b_1 + a_2$ for some k in $\{0, \ldots, t-1\}$. We must have $k = 0$, else $S' \cap S$ would include an element congruent to $a_1(k-1) + b_1 + a_2 \equiv b_1 \pmod{m}$. Hence we get $a_1 \equiv a_2 \pmod{m}$, and from (8) we see that $b_1 = b_2$.

Similarly, if $j = t - 1$ then (10) gives $a_1(t-2) + b_1 \equiv a_1 k + b_1 + a_2 \pmod{m}$ for some k in $\{0, \ldots, t-1\}$, and this k must equal $t - 1$ else $a_1(t-1) + b_1$ would be congruent modulo m to a member of S' and we get $a_1 \equiv -a_2 \pmod{m}$. In this case (8) then gives

$$\begin{aligned}
& \{a_1 i + b_1 : 0 \leq i \leq t-1\} \\
\equiv\ & \{a_1 i + b_2 : 0 \leq i \leq t-1\} \pmod{m} \\
\equiv\ & \{a_1(t-1-i) + b_2 - a_1(t-1) : 0 \leq i \leq t-1\} \pmod{m} \\
\equiv\ & \{a_1 j + b_2 - a_1(t-1) : 0 \leq j \leq t-1\} \pmod{m},
\end{aligned}$$

which implies

$$b_1 \equiv b_2 - a_1(t-1) \pmod{m},$$

as required. ☐

Corollary 13. *Using Notation 6 we have either*

$$a_2\overline{q}_2 + b_2 \equiv a\overline{q}_1 + b_1 \pmod{p}$$

or

$$a_2\overline{q}_2 + b_2 \equiv \overline{q}_1(a + (q_2 - 1)G_1) + b_1 \pmod{p}.$$

Proof. Let H be the smallest gap size in $\{-q_2 i\overline{q}_1 - q_2 b_1 \mod p : 0 \le i \le q_1 - 1\}$. By swapping the roles of $S(p/q_1, b_1)$ and $S(p/q_2, b_2)$ in Theorem 11 we can rewrite (2) as follows. The set of residues modulo p in the intersection of $S(p/q_1, b_1)$ and $S(p/q_2, b_2)$ is

$$\{a_2\overline{q}_2 + iH\overline{q}_2 + b_2 : 0 \le i \le t - 1\},$$

where a_2 satisfies $0 \le a_2 + H(t-1) \le q_2$. Note that we don't need a different t value as the size of the intersection doesn't change. We thus have, using the notation of the theorem,

$$\{a\overline{q}_1 + iG_1\overline{q}_1 + b_1 : 0 \le i \le t - 1\} \equiv \{a_2\overline{q}_2 + iH\overline{q}_2 + b_2 : 0 \le i \le t - 1\} \tag{12}$$

modulo p. We will show that either $H\overline{q}_2 \equiv G_1\overline{q}_1$ or $H\overline{q}_2 \equiv -G_1\overline{q}_1$ modulo p. This is immediate if $t = 1$ or $t = 2$, so we assume $t \ge 3$. In Notation 6 we assumed that $q_1 \le q_2$. Since the moduli of our Beatty sequences satisfy the Morikawa condition (so the sequences would be disjoint for suitable offsets) we must have $q_1 + q_2 \le p$, and so $q_1 \le p/2$. Then from (7) $(t - 1)G_1 < q_1 \le p/2$, so that, for $t \ge 3$,

$$G_1(t + 1) < p. \tag{13}$$

We now apply the theorem with (12) in the role of (8). Since $\overline{q}_1$ and $\overline{q}_2$ are relatively prime to p,

$$\gcd(G_1\overline{q}_1, H\overline{q}_2, p) = \gcd(G_1, H, p) \le G_1.$$

So with (13) we have $\gcd(G_1\overline{q}_1, H\overline{q}_2, p)(t + 1) < p$, which plays the role of (9). We conclude that either $H\overline{q}_2 \equiv G_1\overline{q}_1$ and

$$a_2\overline{q}_2 + b_2 \equiv a\overline{q}_1 + b_1 \pmod{p},$$

or $H\overline{q}_2 \equiv -G_1\overline{q}_1$ modulo p and

$$\begin{aligned} a_2\overline{q}_2 + b_2 &\equiv a_1\overline{q}_1 + b_1 + G_1\overline{q}_1(t - 1) \pmod{p} \\ &\equiv \overline{q}_1(a_1 + G_1(t - 1)) + b_1 \pmod{p}, \end{aligned}$$

as required. ☐

We now analyse the intersection $S(p_1/q_1, b_1) \cap S(p/q_2, b_2)$.

Lemma 14. *The set of residues modulo pm in $S(pm/q_1, b)$ is*

$$\{i(\overline{q}_1 + p\overline{q}_m k_1) + b_1 : 0 \le j \le q_1 - 1\}.$$

Proof. By Theorem 10 the set of residues modulo pm in $S(pm/q_1, b_1)$ is $\{i\overline{Q} + b_1 : 0 \le i \le q_1 - 1\}$, where $\overline{Q}$ is the least non-negative residue modulo pm satisfying $q_1\overline{Q} \equiv -1 \pmod{pm}$. Using Notation 6 we then have $\overline{Q} \equiv \overline{q}_1 \pmod{p}$ so that $\overline{Q} = \overline{q}_1 + lp$ for some integer l. Then

$$\begin{aligned}\overline{Q}q_1 &= (\overline{q}_1 + lp)q_1 \\ &= -1 + k_1 p + lpq_1.\end{aligned}$$

But $\overline{Q}q_1 \equiv -1 \pmod{pm}$ so $k_1 + lq_1 \equiv 0 \pmod{m}$, which implies that $l \equiv \overline{q}_m k_1 \pmod{m}$, and the result follows. □

Theorem 15. *We use Notation 6, recalling that $p_1 = pm$. If p_1/q_1 and p/q_2 satisfy the Morikawa condition then $S(p_1/q_1, b_1) \cap S(p/q_2, b_2)$ equals*

$$\{(a + iG_1)(\overline{q}_1 + p\overline{q}_m k_1) + b_1 + \mu mp : 0 \le i \le t - 1, \mu \in \mathbb{Z}\},$$

where a, G_1 and t have the same meaning as in Theorem 11.

Proof. We write S_1, S_2, and S_m for $S(p/q_1, b_1)$, $S(p/q_2, b_2)$, and $S(pm/q_1, b_1)$ respectively. Since $S_m \subseteq S_1$ we have

$$S_m \cap S_2 = (S_1 \cap S_2) \cap S_m.$$

By Theorem 11

$$S_1 \cap S_2 \equiv \{a\overline{q}_1 + iG_1\overline{q}_1 + b_1 : 0 \le i \le t - 1\} \pmod{p}$$

where a and t are positive integers satisfying $0 < a + G_1(t-1) \le q_1$. By Lemma 14

$$S_m \equiv \{j(\overline{q}_1 + p\overline{q}_m k_1) + b_1 : 0 \le j \le q_1 - 1\} \pmod{pm}.$$

Suppose $x \in S_2 \cap S_m$. Since $x \in S_1 \cap S_2$ we have

$$x = a\overline{q}_1 + i_1 G_1 \overline{q}_1 + b_1 + lp,$$

for some i_1 in $\{0, \ldots, t-1\}$ and $l \in \mathbb{Z}$. Then, since $x \in S_m$,

$$a\overline{q}_1 + i_1 G_1 \overline{q}_1 + b_1 + lp = j_1(\overline{q}_1 + p\overline{q}_m k_1) + b_1 + \mu pm \tag{14}$$

for some $\mu \in \mathbb{Z}$ and j_1 in $\{0, \ldots, q_1\}$. It follows that

$$(a + i_1 G_1)\overline{q}_1 \equiv j_1 \overline{q}_1 \pmod{p}.$$

Since $0 < a + G_1(t-1) \le q_1 < p$ and $\gcd(\overline{q}_1, p) = 1$, we have $a + i_1 G_1 = j_1$. Then (14) gives

$$l = (a + i_1 G_1)\overline{q}_m k_1 + \mu m.$$

The implications here can be reversed, so that $S_m \cap S_2$ equals

$$\{(a + iG_1)(\overline{q}_1 + p\overline{q}_m k_1) + b_1 + \mu mp : 0 \le i \le t-1, \mu \in \mathbb{Z}\},$$

as required. □

Now we obtain our main result.

Theorem 16. *We use Notation 6 recalling that $p_1 = pm$ and $p_2 = pn$. If p_1/q_1 and p_2/q_2 satisfy the Morikawa condition then the intersection of the Beatty sequences $S_m = S(pm/q_1, b_1)$ and $S_n = S(pn/q_2, b_2)$ is the modular arithmetic progression given below, where a, G_1 and t have the same meaning as in Theorem 11:*

$$\{a(\overline{q}_1 + pk_1\overline{q}_m) + b_1 + \mu_0 mp + i(G_1(\overline{q}_1 + pk_1\overline{q}_m) + \mu_1 mp) : 0 \le i \le t-1\}$$

modulo mnp, and one of the following cases holds.
`Case 1` *In this case μ_0 is the least non-negative residue satisfying*

$$\mu_0 m \equiv \frac{a_2\overline{q}_2 + b_2 - a_1\overline{q}_1 - b_1}{p} + a_2k_2\overline{q}_n - a_1k_1\overline{q}_m \pmod{n},$$

and μ_1 is the least non-negative residue satisfying

$$\mu_1 m \equiv \frac{G_2\overline{q}_2 - G_1\overline{q}_1}{p} + G_2k_2\overline{q}_n - G_1k_1\overline{q}_m \pmod{n}.$$

`Case 2` *In this case μ_0 is the least non-negative residue satisfying*

$$\mu_0 m \equiv \frac{a_1\overline{q}_1 + b_1 - a_2\overline{q}_2 - b_2}{p} + G_2(\overline{q}_2 + pk_2\overline{q}_n) - a_1k_1\overline{q}_m + a_2k_2\overline{q}_n \pmod{n},$$

and μ_1 is the least non-negative residue satisfying

$$\mu_1 m \equiv -\frac{G_1\overline{q}_1 + G_2\overline{q}_2}{p} - G_1k_1\overline{q}_m - G_2k_2\overline{q}_n \pmod{n}.$$

Proof. By Theorem 15 $S_m \cap S_2$ equals

$$\{(a_1 + iG_1)(\overline{q}_1 + pk_1\overline{q}_m) + b_1 + \mu mp : 0 \le i \le t-1, \mu \in \mathbb{Z}\}, \tag{15}$$

and $S_n \cap S_1$ equals

$$\{(a_2 + iG_2)(\overline{q}_2 + pk_2\overline{q}_n) + b_2 + \nu np : 0 \le i \le t-1, \nu \in \mathbb{Z}\}. \tag{16}$$

Since $S_m \subseteq S_1$ and $S_n \subseteq S_2$ we can obtain $S_m \cap S_n$ by evaluating $(S_m \cap S_2) \cap (S_n \cap S_1)$. Suppose $x \in S_m \cap S_n$. Then there exist integers $i_1, i_2 \in \{1, \dots, t\}$ and $\mu, \nu \in \mathbb{Z}$ such that

$$\begin{aligned} x &= i_1 G_1(\overline{q}_1 + pk_1\overline{q}_m) + a_1(\overline{q}_1 + pk_1\overline{q}_m) + b_1 + \mu mp \\ &= i_2 G_2(\overline{q}_2 + pk_2\overline{q}_n) + a_2(\overline{q}_2 + pk_2\overline{q}_n) + b_2 + \nu np. \end{aligned} \tag{17}$$

Considering this modulo p we get

$$i_1 G_1 \overline{q}_1 + a_1 \overline{q}_1 + b_1 \equiv i_2 G_2 \overline{q}_2 + a_2 \overline{q}_2 + b_2 \pmod{p}. \tag{18}$$

This is the congruence considered in Corollary 13. We therefore have either $i_2 = i_1$ or $i_2 = p - i_1$. We suppose the first of these holds and return to the other case at the end of the proof. Then (17) gives

$$\begin{aligned} &i_1(G_1(\overline{q}_1 + pk_1\overline{q}_m) - G_2(\overline{q}_2 + pk_2\overline{q}_n)) + a_1(\overline{q}_1 + pk_1\overline{q}_m) \\ &- a_2(\overline{q}_2 + pk_2\overline{q}_n) + b_1 - b_2 = \nu np - \mu mp \end{aligned} \tag{19}$$

for some integers μ and ν. Using Corollary 13 again we may divide through by p getting

$$\begin{aligned} &i_1\left(\frac{G_1\overline{q}_1 - G_2\overline{q}_2}{p} + G_1 k_1 \overline{q}_m - G_2 k_2 \overline{q}_n\right) + \frac{a_1\overline{q}_1 + b_1 - a_2\overline{q}_2 - b_2}{p} \\ &+ a_1 k_1 \overline{q}_m - a_2 k_2 \overline{q}_n = \nu n - \mu m. \end{aligned}$$

Since m and n are relatively prime, we can find μ and ν satisfying this for any i_1. Consider the case $i_1 = 0$, and let μ_0 and ν_0 be the unique values of $\mu \in \{0, \dots, n-1\}$ and $\nu \in \{0, \dots, m-1\}$ satisfying,

$$\nu_0 n - \mu_0 m \equiv \frac{a_1\overline{q}_1 + b_1 - a_2\overline{q}_2 - b_2}{p} + a_1 k_1 \overline{q}_m - a_2 k_2 \overline{q}_n \pmod{mn}, \tag{20}$$

and let μ_1 and ν_1 be the unique values of $\mu \in \{0, \dots, n-1\}$ and $\nu \in \{0, \dots, m-1\}$ satisfying,

$$\nu_1 n - \mu_1 m \equiv \frac{G_1\overline{q}_1 - G_2\overline{q}_2}{p} + G_1 k_1 \overline{q}_m - G_2 k_2 \overline{q}_n \pmod{mn}. \tag{21}$$

We see that (19) will be satisfied for any $i_1 \in \{0, \dots, t-1\}$ if we set

$$\begin{aligned} \mu &= \mu_0 + i_1\mu_1 + sn \\ \nu &= \nu_0 + i_1\nu_1 + sm, \end{aligned}$$

where s is any integer. Substituting in (15) and replacing i_1 with i we get,

$$\begin{aligned} S_m \cap S_n = \{&(a_1 + iG_1)(\overline{q}_1 + pk_1\overline{q}_m) + b_1 + (\mu_0 + i\mu_1 + sn)mp \\ &: 0 \le i \le t-1, s \in \mathbb{Z}\} \\ = \{&a_1(\overline{q}_1 + pk_1\overline{q}_m) + b_1 + \mu_0 mp + i(G_1(\overline{q}_1 + pk_1\overline{q}_m) + \mu_1 mp) \\ &+ smnp : 0 \le i \le t-1, s \in \mathbb{Z}\}, \end{aligned}$$

as required.

If we put $i_2 = p - i_1$ instead of $i_2 = i_1$ as a consequence of (18) then (17) gives the values of μ_0 and μ_1 in Case 2.

$\square$

Corollary 17. *If p_1/q_1 and p_2/q_2 satisfy the Morikawa condition then the intersection of $S_1 = S(pm/q_1, b_1)$ and $S_2 = S(pn/q_2, b_2)$ contains at most 3 distinct gap sizes.*

Proof. Immediate from Theorem 16 and Corollary 9. $\square$

We end with an example. Consider the pair of Beatty Sequences $S(737/10, 0)$, and $S(469/15, 2)$. This gives $p = 67$, $m = 11$, $n = 7$, $q_1 = 10$, $q_2 = 15$, $b_1 = 0$, $b_2 = 2$, $\overline{q}_1 = 20$, $\overline{q}_2 = 58$, $\overline{q}_m = 1$, $\overline{q}_n = 6$, $k_1 = 3$, $k_2 = 13$, $G_1 = 2$, $G_2 = 3$, and $t = 4$. By Theorem 11 the intersection of $S(67/10, 0)$ and $S(67/15, 2)$ is $\{20a_1 + 40i : 0 \le i \le t-1\} \pmod{67}$. With $a_1 = 3$ this gives the intersection $\{6, 33, 46, 6\}$. We similarly have $a_2 = 1$. Now consider Theorem 16. We have μ_0 being the least non-negative residue satisfying

$$\begin{aligned}
\mu_0 m &\equiv \frac{a_2\overline{q}_2 + b_2 - a_1\overline{q}_1 - b_1}{p} + a_2k_2\overline{q}_n - a_1k_1\overline{q}_m \pmod{n} \\
\implies 11\mu_0 &\equiv 69 \pmod{7} \\
\implies \mu_0 &\equiv 5 \pmod{7},
\end{aligned}$$

and μ_1 is the least non-negative residue satisfying

$$\begin{aligned}
\mu_1 m &\equiv \frac{G_2\overline{q}_2 - G_1\overline{q}_1}{p} + G_2k_2\overline{q}_n - G_1k_1\overline{q}_m \pmod{n} \\
\implies 11\mu_1 &\equiv 230 \pmod{7} \\
\implies \mu_1 &\equiv 5 \pmod{7}.
\end{aligned}$$

Then $S_m \cap S_n$ is

$$\begin{aligned}
&\{a_1(\overline{q}_1 + pk_1\overline{q}_m) + b_1 + \mu_0 mp + i(G_1(\overline{q}_1 + pk_1\overline{q}_m) \\
&\quad + \mu_1 mp) : 0 \le i \le t-1\} \pmod{pmn} \\
&\equiv \{4384 + 4127i : 0 \le i \le 3\} \pmod{5159} \\
&\equiv \{1252, 2284, 3316, 4384\} \pmod{5159}.
\end{aligned}$$

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